Chapter 1

Introduction

The need to study various mathematical techniques grows out of the requirement to understand, solve and/or evaluate the different types of equations that arise from the mathematical modeling of phenomena in the natural, social, and engineering sciences. The first equations written down for a particular system involve a number of parameters appearing in the equations relating variable quantities. What is generally required, at this point, is to convert these equations into structurally similar equations, but having the important property that all of the new parameters and variables have no physical dimensions. One of the purposes of this chapter is to provide some indication as to how to do this and illustrate the basic method by means of examples.

The final section briefly introduces the concept of nonlinearity and the important role that it plays in the sciences. The bibliography gives a short listing of books and related materials on the main issues raised in this chapter.

1.1 Mathematical Modeling

Suppose there is a system that you want to understand. This understanding may include the correlation of its past behavior with the current state of the system, along with the ability to predict its future evolution. An important tool for obtaining such information is to construct a mathematical model for the relevant phenomena related to the system. The particular mathematical formalism used will depend on the nature of the system. It should be pointed out that the same system can have various mathematical representations with the one selected dependent upon what questions need to be answered relative to the system. The actual equations occurring in the construction of a particular mathematical model will depend on what broad principles can be applied to the system to restrict its dynamical behavior. Examples of such requirements include the conservation laws (energy, momentum, charge, mass, etc.). However, in general, we define mathematical modeling and the associated model as follows:

Definition *Mathematical modeling* is the process of constructing appropriate equations to represent the relevant aspects of a system. This activity includes the various methods needed to analyze these equations, the obtaining of exact and approximate solutions, possible numerical evaluations, and simulations.

Definition A *mathematical model* is the set of equations arising from the modeling process, along with any restrictions or auxiliary conditions that must be imposed on the solutions to the equations.

While there cannot be a general prescription which, if followed, leads to a successful mathematical model for a given system, the following is a minimal list of issues and steps that clearly must be included in any such effort:

(i) The underlying science should be as fully understood as possible.

(ii) As many simplifying assumptions, as needed, should be made to reduce the complexities of the system to a form where suitable mathematical equations can be derived, while at the same time assuring that all relevant behavior and properties of the system are captured by the equation.

(iii) The equations of the mathematical model should be checked for both consistency within a given equation and consistency among the various equations.

(iv) The mathematical analysis of the model equations should begin with a detailed study of the qualitative properties of its possible solution behaviors. Possible methods to be used include examination of trajectories in the phase-space of the dynamical variables of the system, and the application of dimensional analysis and scaling to reduce the number of free parameters.

(v) If possible, special, but exact solutions should be found and their properties examined.

(vi) If exact solutions cannot be obtained, then analytical approximations to these solutions should be constructed by use of appropriate mathematical techniques.

(vii) It is often of great value to apply numerical methods to generate numerical solutions. A deep understanding of the system may sometimes be obtained using this technique.

(viii) Finally, all of the above elements should be combined to form a grand synthesis that hopefully will allow a full and deep understanding of those aspects of the system which were of original interest.

Finally, it should be realized that these steps may have to be iterated several times before a suitable mathematical model is reached. Also, it must be understood that difference systems may lead to similar, if not identical mathematical models. Consequently, an understanding of the dynamics of one system may provide additional insight into the mechanisms of other (mathematically) related processes.

The bibliography to the chapter provides a listing of books that treat mathematical modeling in detail, including general questions on philosophical aspects of the general modeling process. Several present a variety of examples as to how this procedure is to be applied to various areas of the sciences.

1.2 Mathematical versus Physical Equations

Mathematical models in the sciences generally are expressed in terms of differential equations [1]. In the standard case, the variables and parameters appearing in these equations have magnitudes determined by the system of "units" used for their expression [2]. However, the system of units used has no absolute significance and other units can be applied to the same problem. Often, the nature of the units used is dependent on what questions the modeler has in mind when considering a given problem. That is, different questions can lead the modeler to use different sets of units to characterize the same problem. For example, consider a mechanical system composed of two subsystems, each of which are electrically charged. It is always possible to study the full system using the basic set of standard units $(M, L, T) \equiv$ (mass, length, time). But, it will prove to be more useful, for this case, if an additional physical unit is added to the standard units, i.e., a unit of electrical charge, Q. Thus, the more appropriate units are now (M, L, T, Q) [2].

Assuming that the original modeling equations are differential equations, then they will take the general form

$$\frac{\partial u}{\partial t} = f(x, t, u, p), \tag{1.2.1}$$

$$\begin{cases} u^T \equiv (u_1, u_2, \ldots, u_N), \\ x^T \equiv (x_1, x_2, \ldots, x_M), \\ p^T \equiv (p_1, p_2, \ldots, p_K), \end{cases} \tag{1.2.2}$$

and u represents the N-components of the dependent variable; x, the M-components of independent the "space" variable; t is the time variable; and p represents the K-parameters which appear in the modeling equations. In general, Eq. (1.2.1) is a short-hand notation for N-coupled differential equations and they comprise the mathematical model.

Equation (1.2.1) is a *physical equation* in the sense that, in general, all of the variables and parameters, (u, x, t, p), are expressed in terms of a given set of physical units such as mass, length, time, etc. Now what is most desirable is to have a set of equations for which all the new variables and parameters are dimensionless. This new or derived set of equations are the mathematical equations and they possess the important property of not having the values of their variables and parameters depend on any particular system of units [2].

1.3 Dimensionless Variables and Characteristic Scales

Given a set of "physical" differential equations, the primary issue is how does one construct from them the related "mathematical equations"? As we outline below, the method for their determination is relatively direct, but does not lead, in general, to a unique set of mathematical equations. However, this non-uniqueness can often be used by the investigator to study different aspects of the same system.

Consider, for example, a system characterized by k parameters, $p = (p_1, p_2, \ldots, p_k)$. All possible length scales will take the form

$$L_i = \prod_{j=1}^{k} (p_j)^{\ell_j}, \qquad i = 1, 2, \ldots, I; \tag{1.3.1}$$

where the ℓ_i can be explicitly calculated and there are I scales. Note that it is unlikely that value of I will be known prior to the actual determination of the set of length scales $\{L_i : 1, 2, \ldots, I\}$. Similar forms also hold for the other scales involving mass, time, dependent variables, etc. [2]; for example,

$$M_r = \prod_{j=1}^{k} (p_j)^{v_j}, \qquad r = 1, 2, \ldots, R; \tag{1.3.2}$$

$$T_s = \prod_{j=1}^{k} (p_j)^{w_j}, \qquad s = 1, 2, \ldots, S; \tag{1.3.3}$$

where the $\{v_j, w_j : j = 1, 2, \ldots, K\}$ are calculable.

To form a dimensionless or mathematical set of equations, we select a member from each different type of scale, i.e.,

$$\begin{cases} L : \{L_1, L_2, \ldots, L_I\}, \\ M : \{M_1, M_2, \ldots, M_R\}, \\ T : \{T_1, T_2, \ldots, T_S\}, \\ u^* : \{u_1^*, u_2^*, \ldots, u_Q^*\}, \\ \qquad \text{etc.,} \end{cases} \tag{1.3.4}$$

where (I, R, T, Q) are positive integers and the scales for the dependent variables are indicated by u^*, and define new dimensionless variables $\bar{u}, \bar{x}, \bar{t}$, etc., as follows,

$$u = u^* \bar{u}, \quad x = L\bar{x}, \quad t = T\bar{t}, \ \text{etc.} \tag{1.3.5}$$

Substitution of these quantities into Eq. (1.2.1) and simplifying, gives an equation having the structure

$$\frac{d\bar{u}}{d\bar{t}} = F(\bar{x}, \bar{t}, \bar{u}, \lambda), \tag{1.3.6}$$

where the function F is determined from a knowledge of f, and the new dimensionless set of parameters $\{\lambda_j : 1, 2, \ldots, \bar{K}\}$ is usually fewer in number than the original set of k-parameters, i.e., $\bar{K} \leq K$. Since Eq. (1.3.6) is now expressed only in terms of dimensionless variables and parameters, it is the required mathematical equation. This is the equation that can now be studied using all available tools of applied mathematics.

Finally, an issue indicated above should be reiterated. Suppose we have a problem for which only the scales (L, M, T, u^*) arise. Then it follows, from the results stated in Eq. (1.3.4), that there are $IQRS$ possible combinations of scalings that can be constructed and used to transform the physical equations to mathematical equations. Thus, the selection of a particular set of scales should be based on both an understanding of the dynamics of the system and what questions are being asked. In few cases does this nonuniqueness cause difficulties. A very useful way to proceed is to construct the mathematical equation by beginning with generic, *a priori* unspecified, scales (T_1, L_1, etc.) that are later calculated to make the coefficients of the various terms in this equation reflect their magnitude [2].

In the next section, we illustrate the general method by applying it to a number of mathematical models for a variety of systems.

1.4 Construction of Mathematical Equations

1.4.1 *Decay Equation*

The following first-order, linear differential equation plays a fundamental role in the study of systems that can transform their state by decaying [3],

$$\frac{dc}{dt} = -\lambda c, \qquad c(0) = c_0 > 0. \tag{1.4.1}$$

If we measure c in terms of particle density, then the various quantities have the dimensions,

$$\begin{cases} [c] \equiv \# = \text{particles per unit volume}, \\ [t] \equiv T = \text{time}, \\ [\lambda] \equiv T^{-1}, \end{cases} \tag{1.4.2}$$

where the square bracket $[\cdots]$ around a symbol denotes its units. Now for this system, only two units occur, namely, # and T. Also, only two parameters are at our disposal: the measure of the decay rate, given by λ and the initial concentration, c_0. Thus, the scales for time and concentration are

$$T_1 = \frac{1}{\lambda}, \qquad c^* = c_0, \tag{1.4.3}$$

and the corresponding dimensionless quantities are

$$c \to \bar{c} = \frac{c}{c_0}, \qquad t \to \bar{t} = \frac{t}{T_1} = \lambda t. \tag{1.4.4}$$

Substitution of these results into Eq. (1.4.1) gives the mathematical equation

$$\frac{d\bar{c}}{d\bar{t}} = -\bar{c}, \qquad \bar{c}(0) = 1. \tag{1.4.5}$$

Observe that the ODE for the dimensionless quantities only involves a single, definite value of the initial condition and no free parameters occur in Eq. (1.4.5).

In summary, the initial physical equation was characterized by one parameter, λ, and one initial value for the concentration, and the initial value, c_0, could take on a range of values, i.e.,

$$c_0 > 0. \tag{1.4.6}$$

However, the mathematical equation had no free parameters and only one initial value needed to be considered.

1.4.2 *Logistic Equation*

The logistic equation provides a model for population growth of a single species, but with self-interaction that places a limit on the ultimate size of the population [4]. Let $P(t)$ be population number, then the modeling "physical" equation is

$$\frac{dP}{dt} = \lambda_1 P - \lambda_2 P^2, \qquad P(0) = P_0 > 0, \tag{1.4.7}$$

where λ_1 is a parameter related to the birthrate at low values of P and λ_2 is related to the strength of the self-interaction in the population. The units of the various quantities for this system are

$$[P] = \#, \quad [P_0] = \#, \quad [\lambda] = \frac{1}{T}, \quad [\lambda_2] = \frac{1}{\#T}, \quad [t] = T. \tag{1.4.8}$$

From these, two characteristic times, T_1 and T_2, can be constructed, as well as one characteristic population value, P^*. They are given by

$$T_1 = \frac{1}{\lambda_1}, \qquad T_2 = \frac{1}{P_0\lambda_2}, \qquad P^* = \frac{\lambda_1}{\lambda_2}. \tag{1.4.9}$$

The following dimensionless variables can now be constructed

$$t \to \bar{t} = \frac{t}{T_1} = \lambda_1 t, \qquad P \to \bar{P} = \frac{P}{P^*} = \left(\frac{\lambda_2}{\lambda_1}\right) P, \tag{1.4.10}$$

and with their substitution into Eq. (1.4.7), the logistic equation becomes

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1-\bar{P}), \qquad \bar{P}(0) = \frac{P(0)}{P*} > 0. \tag{1.4.11}$$

Note that the mathematical equation contains no arbitrary parameters, while the original physical equation had two parameters, λ_1 and λ_2. However, both the physical and mathematical equations have to satisfy a single *initial condition*.

A more detailed study of the logistic equation [4] allows the following general conclusions to be reached with regard to the qualitative properties of its solutions:

(i) For any positive initial condition, $\bar{P}(0) > 0$, all solutions monotonically approach the value one, i.e.,

$$\operatorname*{Lim}_{t\to\infty} \bar{P}(t) = 1, \qquad \bar{P}(0) > 0. \tag{1.4.12}$$

(ii) Starting from an initial value $\bar{P}(0) > 0$, then the time, T_∞, (in dimensionless units) to "almost" reach the population value, $\bar{P}(\infty) = 1$, is approximately given by the expression

$$T_\infty \simeq \frac{\lambda_1}{P_0\lambda_2}\,. \tag{1.4.13}$$

1.4.3 *The Fisher Equation*

Many interesting and important phenomena in the sciences can be modeled by the nonlinear Fisher partial differential equation [5]. This equation takes the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda_1 u - \lambda_2 u^2, \tag{1.4.14}$$

where all three parameters $(D, \lambda_1, \lambda_2)$ are non-negative. If the unit of u is particle density, then

$$[u] = \#, \quad [D] = \frac{L^2}{T}, \quad [\lambda_1] = \frac{1}{T}, \quad [\lambda_2] = \frac{1}{\#T}\,. \tag{1.4.15}$$

The corresponding scales are given by the expressions

$$u^* = \frac{\lambda_1}{\lambda_2}, \quad T_1 = \frac{1}{\lambda_1}, \quad L_1 = \left(\frac{D}{\lambda_1}\right)^{1/2}. \tag{1.4.16}$$

With the dimensionless variables defined by

$$\bar{u} = \frac{u}{u^*} = \left(\frac{\lambda_2}{\lambda_1}\right) u, \quad \bar{t} = \frac{t}{T_1} = \lambda_1 t, \quad \bar{x} = \frac{x}{L_1} = \left(\frac{\lambda_1}{D}\right)^{1/2} x, \tag{1.4.17}$$

the physical Fisher equation is transformed into the following dimensionless mathematical equation,

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \bar{u}(1 - \bar{u}). \tag{1.4.18}$$

Thus, in contrast to the original Fisher equation, where three parameters appeared, our transformed equation contains no free parameters.

Since there exists a characteristic length and a characteristic time, it is now possible to calculate a characteristic velocity; it is given by

$$c^* \equiv \frac{L_1}{T_1} = (\lambda_1 D)^{1/2}. \tag{1.4.19}$$

This c^* is an estimate of how fast phenomena modeled by the Fisher equation propagate. In fact, it is known that the minimum speed of propagation for the Fisher equation is [5],

$$c = 2(\lambda_1 D)^{1/2}. \tag{1.4.20}$$

Thus, our estimate, c^*, is in excellent agreement with this value of c.

1.4.4 *Duffings' Equation*

A variety of mechanical oscillating systems [6; 7] can be approximated by the following second-order, nonlinear differential equation,

$$m\frac{d^2x}{dt^2} + kx + k_1 x^3 = 0, \tag{1.4.21}$$

where the initial conditions are selected to be

$$x(0) = A, \qquad \frac{dx(0)}{dt} = 0. \tag{1.4.22}$$

This equation is called Duffings' equation and relates the oscillatory motions of a mass, m, subject to both linear and nonlinear elastic forces. As presented above the equation depends on three parameters (m, k, k_1) along with one essential initial condition, $x(0) = A$.

Since each term in Eq. (1.4.21) corresponds to a force, they each have units of MLT^{-2}. Thus, the parameters and essential initial condition have the units

$$[m] = M, \quad [k] = MT^{-2}, \quad [k_1] = MT^{-2}L^{-2}, \quad [A] = L, \tag{1.4.23}$$

and from them the following scales can be constructed,

$$T_1 = \left(\frac{m}{k}\right)^{1/2}, \quad L_1 = \left(\frac{k}{k_1}\right)^{1/2}, \quad L_2 = A. \tag{1.4.24}$$

The time scale T_1 is related to the period of the free oscillations of the linear system, i.e., when $k_1 = 0$. The two length scales have interpretations first as a measure of the intrinsic size of the system, i.e., L_1, and second as the scale associated with the (external to the system) initial condition. Two sets of time and length scales can be formed; they are (T_1, L_1) and (T_2, L_2), and give, respectively, two difference forms of dimensionless Duffings' equation.
(T_1, L_1) :

$$\bar{x} = \frac{x}{L_1}, \quad \bar{t} = \frac{t}{T_1}, \tag{1.4.25}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \bar{x}^3 = 0, \quad \bar{x}(0) = \frac{A}{L_1}, \quad \frac{D\bar{x}(0)}{d\bar{t}} = 0; \tag{1.4.26}$$

(T_2, L_2) :

$$\bar{x} = \frac{x}{L_2}, \quad \bar{t} = \frac{t}{T_1}, \tag{1.4.27}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \epsilon\bar{x}^3 = 0, \quad \bar{x}(0) = 1, \quad \frac{d\bar{x}(0)}{d\bar{t}} = 0; \tag{1.4.28}$$

$$\epsilon = \left(\frac{L_2}{L_1}\right)^2. \tag{1.4.29}$$

The dimensionless parameter ϵ is the square of the ratio of the initial displacement to the intrinsic size of the system. Thus, the scales (T_1, L_1) corresponds to the preparation of the system in an initial state where the value of $x(0)$ is close to the intrinsic length scale of the system. On the other hand, Eq. (1.4.28) is the equation to use when the displacements in x are small compared to the intrinsic length scale. This is the form to be used when applying a perturbation method to calculate solutions for Duffings equation.

It is true, in general, that the intrinsic scales of a system are determined entirely by the parameters appearing in the physical equation. They can consequently be considered as the quantities relevant for a full understanding of the dynamics of the system. This characterization of the scales is also helpful in determining conditions for small changes in the dynamics of the system. For example, if a system has a time scale associated with it of magnitude $\bar{T}$, then the dynamics will not change much if the interval of time over which the system is observed is small compared to $\bar{T}$. Similarly, if initial conditions take values that are small compared to those of the corresponding intrinsic scales, then the system can be considered as a perturbed state of one for which either the initial conditions are zero or the values of some of the parameters are zero.

1.4.5 *Budworm Population Dynamics*

An interesting problem in mathematical ecology is to model the dynamics of budworm populations [7]. These pesky insects attack the leaves of a certain type of tree and when an outbreak takes place, they defoliate and eventually kill most of the trees that they feed on in several years. The following first-order ordinary differential equation has been used for the initial study of the budworm-tree interaction,

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}, \tag{1.4.30}$$

where N is the budworm population. Note that this equation only models the budworm population.

The models contain four parameters (A, B, R, K) and our task is to determine a dimensionless mathematical equation equivalent to Eq. (1.4.30). An examination of this equation indicates that the units of the various quantities appearing in it are

$$[N] = \#, \quad [R] = T^{-1}, \quad [K] = \#, \quad [A] = \#, \quad [B] = \#T^{-1}. \tag{1.4.31}$$

From the four parameters, the following population and time scales can be constructed,

$$N_1^* = A, \quad N_2^* = K, \quad N_3^* = \frac{B}{R}, \tag{1.4.32}$$

$$T_1 = \frac{1}{R}, \quad T_2 = \frac{K}{B}, \quad T_3 = \frac{A}{B}, \quad T_4 = \left(\frac{KA}{B^2}\right)^{1/2}. \tag{1.4.33}$$

Out of the possible twelve sets of (N^*, T) scales, we will use the one containing (N_1^*, T_3). Therefore, we have

$$\bar{N} = \frac{N}{A}, \qquad \bar{t} = \frac{t}{T_3} = \left(\frac{B}{A}\right) t. \tag{1.4.34}$$

Substitution of these into Eq. (1.4.30) and simplifying the resulting expression gives

$$\frac{d\bar{N}}{d\bar{t}} = q\bar{N}\left(1 - \frac{\bar{N}}{p}\right) - \left(\frac{\bar{N}^2}{1+\bar{N}^2}\right), \tag{1.4.35}$$

where

$$p = \frac{K}{A}, \qquad q = \frac{RA}{B}. \tag{1.4.36}$$

In summary, we have transformed the original physical equation for the budworm problem into the equivalent dimensionless mathematical equation with the number of free parameters reduced from four to two.

1.5 Nonlinearity

Much of science, in particular physics, up to about the mid-twentieth century was centered on the construction and analysis of linear or linearized models for the various phenomena studied. However, the creation of digital computers allowed the beginning of a better understanding of nonlinear systems and the associated mathematical structures. This included processes for which realistic theories already existed, but for which no set of analytical techniques were available to provide solutions and other systems of interest for which no fundamental theories existed, yet knowledge of their dynamic behavior and related properties were needed.

A critical feature of systems modeled by linear differential equations is the property of linear superposition [9]. This means that if two different solutions, $x_1(t)$ and $x_2(t)$, are found, then their sum

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \tag{1.5.1}$$

is a solution for arbitrary constants c_1 and c_2. In many cases, a knowledge of the actual explicit functional form of one solution could then allow the calculation of a second solution. Of interest is the fact that for a linear system, the output or reaction to a given input is directly proportional to the input to the system. Thus, small changes in the initial data generally give rise to some perturbations in the future evolution of the system.

However, such nice features do not hold true for nonlinear systems: small perturbations in the input data (initial conditions, parameters, etc.) can have a major impact on the future states of the systems. Often this aspect of nonlinear systems is characterized by the phrase "sensitive dependence on initial conditions" [10]. Related to this phenomena is the possible sensitive dependence on changes in parameter values; this area falls under the general topic of bifurcation theory [11; 12].

Nonlinear systems have many other properties that do not appear in linear systems. Examples include chaotic behavior, with the associated structure of "strange attractor" [11], and solitary waves which have many of the properties of classical particles [13]. Special nonlinear systems can even possess normal modes and have localized solutions [14]. It is expected that other types of solution behaviors will be found from future research on these systems.

While the central theme of this book is not concerned with the above indicated systems and the mathematical techniques needed to study them, the various topics included here are needed to begin such an effort. In particular, the genesis of most investigations on the nonlinear differential equations modeling nonlinear systems begins with the determination of the equilibrium solutions, i.e., fixed-points, and proceeds then to study the stability properties of the system equations linearized about these constant solutions. Thus, the techniques to be presented in this book form the basis of the broad background of resources needed to understand the dynamics of these complex systems.

Problems

Section 1.3

1.3.1 Explain why the $(\ell_j : j = 1, 2, \ldots, k)$ in Eq. (1.3.1) are unique for a particular L_i. The same reasoning applies to the results of Eqs. (1.3.2) and (1.3.3).

1.3.2 Determine $F(\bar{x}, \bar{t}, \bar{u}, \bar{\lambda})$ in Eq. (1.3.6) explicitly in terms of the $f(x, t, u, p)$ given by Eq. (1.2.1).

Section 1.4

1.4.1 Rescale the budworm population equation using (N_2^*, T_1).

1.4.2 Which scales should be used and why if the following conditions hold for the budworm population problem:

(i) $A \gg K$,
(ii) $A \ll K$.

1.4.3 Construct a dimensionless mathematical equation for the system of differential equations:

$$\frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{K}\right) - \frac{\beta x_1 x_2}{\alpha + x_2},$$

$$\frac{dx^2}{dt} = sx_2\left(1 - \frac{x_2}{\nu x_1}\right),$$

where all the parameters $(r, s, K, \beta, \alpha, \nu)$ are positive. These coupled, nonlinear, first-order differential equations model an animal predator (x_2)-prey (x_1) system [8].

1.4.4 The Rayleigh equation [6]

$$m\frac{d^2x}{dt^2} - \left[\alpha - \left(\frac{\beta}{3}\right)\left(\frac{dx}{dt}\right)^2\right]\frac{dx}{dt} + ky = 0$$

models nonlinear oscillations of a mass m acted upon by a linear elastic restoring force and a nonlinear energy source. Construct the required scales and transform it into a dimensionless equation. Can all of the original parameters (m, α, β, k) be transformed away in the dimensionless equation?

Section 1.5

1.5.1 Do there exist nonlinear differential equations having the property that if $x(t)$ is a solution, then $cx(t)$ is also a solution for any constant c?

1.5.2 Can a linear system of ordinary differential equations have chaotic solutions?

Comments and References

[1] Differential equations are not the only mathematical structures that can be used to represent such models. See the books of Aris, Brauer and Castillo-Chávez, Rosenblatt and Bell, and Shier and Wallenius.

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[5] See, for example,
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