

Contents

1. Introduction	1
1.1 Mathematical Modeling	1
1.2 Mathematical versus Physical Equations	3
1.3 Dimensionless Variables and Characteristic Scales	4
1.4 Construction of Mathematical Equations	6
1.4.1 Decay Equation	6
1.4.2 Logistic Equation	7
1.4.3 The Fisher Equation	8
1.4.4 Duffings' Equation	9
1.4.5 Budworm Population Dynamics	11
1.5 Nonlinearity	12
<i>Comments and References</i>	15
<i>Bibliography</i>	16
2. Trigonometric Relations and Fourier Analysis	19
2.1 Introduction	19
2.2 Euler's Formula and DeMoivre's Theorem	20
2.3 Derivation of Trigonometric Relations	22
2.4 Periodic, Even and Odd Functions	25
2.4.1 Periodic Functions	25
2.4.2 Even and Odd Functions	25
2.5 Fourier Series	28

2.6	Worked Examples for Fourier Series	33
2.6.1	Parseval's Identity	33
2.6.2	$f(x) = x , -\pi \leq x \leq \pi$	34
2.6.3	$f(x) = x, -\pi < x < \pi$	35
2.6.4	The Square Wave Function	35
2.6.5	Comparison of Problems 2.6.2 and 2.6.4	36
2.6.6	$f(x) = \sin x, 0 < x < \pi$	37
2.6.7	Fourier Series of $x^2, -\pi < x < \pi$ by Integration	39
2.6.8	Riemann-Lebesgue Theorem	40
2.7	Fourier Series for $(\cos \theta)^\alpha$ and $(\sin \theta)^\alpha$	40
2.8	Fourier Transforms	44
2.8.1	Definition of Fourier Transforms	44
2.8.2	Basic Properties of Fourier Transforms	45
2.9	Application of Fourier Transforms	47
2.9.1	Fourier Transform of the Square Pulse	47
2.9.2	Fourier Transform of the Gaussian Function	47
2.9.3	The Convolution Theorem	48
2.9.4	The Diffusion Equation	49
2.9.5	The Wave Equation	51
2.10	The Laplace Transform	53
2.11	Worked Problems Using the Laplace Transform	56
2.11.1	Laplace Transform of $t^{-1/3}$	56
2.11.2	The Square Wave Function	56
2.11.3	The Dirac Delta Function	57
	<i>Comments and References</i>	63
	<i>Bibliography</i>	65
3.	Gamma, Beta, Zeta, and Other Named Functions	67
3.1	Scope of Chapter	67
3.2	Gamma Function	67
3.3	The Beta Function	71
3.4	The Riemann Zeta Function	72
3.5	The Dirac Delta Function	76
3.6	Dirichlet Integrals	81
3.7	Applications	84
3.7.1	Additional Properties of $\Gamma(z)$	84

3.7.2	A Definite Integral Containing Logarithms	86
3.7.3	A Class of Important Integrals	87
3.7.4	A Representation for x^{-p}	87
3.7.5	Additional Properties of the Beta Function	88
3.7.6	Fermi-Dirac Integrals	89
3.7.7	An Integral Involving an Exponential	90
3.7.8	Fermi-Dirac Integrals Containing Logarithms	90
3.7.9	Magnetic Moment of the Electron	91
3.7.10	Relationship Between the Theta and Delta Functions	91
3.7.11	Evaluation of Integrals by Use of the Beta Function	93
3.8	Other Named Functions	94
3.9	Elliptic Integrals and Functions	95
3.9.1	Elliptic Integrals of the First and Second Kind	96
3.9.2	Jacobi Elliptic Functions	99
3.10	Evaluation of Integrals	104

<i>Comments and References</i>	112
--------------------------------	-----

<i>Bibliography</i>	113
---------------------	-----

4.	Qualitative Methods for Ordinary Differential Equations	115
4.1	Introduction	115
4.2	One-Dimensional Systems	116
4.2.1	Definition	116
4.2.2	Fixed-Points	116
4.2.3	Sign of the Derivative	117
4.2.4	Linear Stability	122
4.3	Worked Examples	124
4.3.1	Examples A	124
4.3.2	Example B	125
4.3.3	Example C	126
4.3.4	Example D	128
4.3.5	Example E	129
4.4	Two-Dimensional Systems	132
4.4.1	Definition	132
4.4.2	Fixed-Points	132
4.4.3	Nullclines	133
4.4.4	First-Integrals and Symmetries	133

4.4.5	General Features of Two-Dimensional Phase Space	134
4.4.6	The Basic Procedure for Constructing Phase-Space Diagrams	138
4.4.7	Linear Stability	139
4.5	Worked Examples	144
4.5.1	Example A	145
4.5.2	Example B	149
4.5.3	Example C	153
4.5.4	Example D	157
4.5.5	Example E	160
4.5.6	Example F	169
4.5.7	Example G	173
4.5.8	Example H	177
4.6	Bifurcations	180
4.6.1	Hopf-Bifurcations	184
4.6.2	Two Examples	187
	<i>Comments and References</i>	193
	<i>Bibliography</i>	195
5.	Difference Equations	197
5.1	Genesis of Difference Equations	197
5.1.1	Square-Root Iteration	198
5.1.2	Integral Depending on Two Parameters	199
5.2	Existence and Uniqueness Theorem	200
5.3	The Fundamental Operators	202
5.3.1	The Δ Operator	202
5.3.2	The Shift Operator, E	204
5.3.3	Difference of Functions	205
5.3.4	The Operator Δ^{-1}	206
5.4	Worked Problems Based on Section 5.3	209
5.4.1	Examples A	209
5.4.2	Example B	210
5.4.3	Example C	211
5.4.4	Examples D	211
5.5	First-Order Linear Difference Equations	215
5.5.1	Example A	216

5.5.2	Example B	218
5.5.3	Examples C	218
5.6	General Linear Difference Equations	220
5.6.1	General Properties	220
5.6.2	Linearly Independent Functions	221
5.6.3	Fundamental Theorems for Homogeneous Equations	222
5.6.4	Inhomogeneous Equations	223
5.7	Worked Problems	225
5.7.1	Example A	225
5.7.2	Example B	226
5.7.3	Example C	227
5.7.4	Example D	228
5.8	Linear Difference Equations with Constant Coefficients	230
5.8.1	Homogeneous Equations	231
5.8.2	Inhomogeneous Equations	232
5.8.3	Worked Examples	234
5.9	Nonlinear Difference Equations	238
5.9.1	Homogeneous Equations	239
5.9.2	Riccati Equations	239
5.9.3	Clairaut's Equation	241
5.9.4	Miscellaneous Forms	242
5.10	Worked Examples of Nonlinear Equations	243
5.10.1	Example A	243
5.10.2	Example B	243
5.10.3	Example C	244
5.10.4	Example D	245
5.10.5	Example E	245
5.11	Two Applications	246
5.11.1	Chebyshev Polynomials	246
5.11.2	A Discrete Logistic Equation	249
	<i>Comments and References</i>	256
	<i>Bibliography</i>	257
6.	Sturm-Liouville Problems	259
6.1	Introduction	259
6.2	The Vibrating String	260

6.2.1	Fixed Ends	260
6.2.2	One Fixed and One Free Ends	262
6.2.3	Both Ends Free	263
6.2.4	Discussion	264
6.3	Sturm Separation and Comparison Theorem	265
6.3.1	Example A	267
6.3.2	Example B	269
6.3.3	Example C	269
6.4	Sturm-Liouville Problems	270
6.4.1	Fundamental Definition	270
6.4.2	Properties of Eigenvalues and Eigenfunctions	270
6.4.3	Orthogonality of Eigenfunctions	271
6.4.4	Expansion of Functions	272
6.4.5	The Completeness Relation	273
6.5	Applications	273
6.5.1	The Special Functions	273
6.5.2	Fourier Expansion of $f(x) = x(1 - x)$	275
6.5.3	Representation of $\delta(x)$ in Terms of Cosine Functions	276
6.5.4	Reality of the Eigenvalues	277
6.5.5	A Boundary Value Problem	278
6.6	Green's Functions	280
6.7	Worked Examples for Green Functions	284
6.7.1	$y''(x) = -f(x) : y(0) = y(L) = 0$	284
6.7.2	$y''(x) = -f(x) : y(0) = 0, y'(L) = 0$	285
6.7.3	$y''(x) + k^2y(x) = -f(x) : y(0) = y(L) = 0$	286
6.7.4	$y''(x) + y(x) = x : y(0) = y(1) = 0$	286
6.8	Asymptotic Behavior of Solutions to Differential Equations	287
6.8.1	Elimination of First-Derivative Terms	287
6.8.2	The Liouville-Green Transformation	290
6.9	Worked Examples	292
6.9.1	The Airy Equation	292
6.9.2	The Bessel Equation	293
6.9.3	A General Expansion Procedure	296
	<i>Comments and References</i>	303
	<i>Bibliography</i>	304
7.	Special Functions and Their Properties	307

7.1	Introduction	307
7.2	Classical Orthogonal Polynomials	309
7.2.1	Differential Equation and Interval of Definition . . .	309
7.2.2	Weight Functions and Rodrique's Formulas	310
7.2.3	Orthogonality Relations	310
7.2.4	Generating Function	310
7.2.5	Recurrence Relations	311
7.2.6	Differential Recurrences	311
7.2.7	Special Values	312
7.2.8	Zeros of COP	312
7.3	Legendre Polynomials: $P_n(x)$	312
7.4	Hermite Polynomials: $H_n(x)$	314
7.5	Chebyshev Polynomials: $T_n(x)$ and $U_n(x)$	315
7.6	Laguerre Polynomials: $L_n(x)$	317
7.7	Legendre Functions of the Second Kind and Associated Legendre Functions	319
7.8	Bessel Functions	322
7.9	Some Proofs and Worked Problems	328
7.9.1	Proof of Rodriques Formula for $P_n(x)$	328
7.9.2	Integrals of x^m with $P_n(x)$	329
7.9.3	Expansion of $\delta(x)$ in Terms of $P_n(x)$	330
7.9.4	Laplace's Equation in Spherical Coordinates	331
7.9.5	Sphere in a Uniform Flow	334
7.9.6	The Harmonic Oscillator	335
7.9.7	Matrix Elements of the Electric Dipole	338
7.9.8	General Solution Hermite's Differential Equation for $n = 0$	338
7.9.9	Three Dimensional Harmonic Oscillator	339
7.9.10	Proof that $J_{-n}(x) = (-1)^n J_n(x)$	343
7.9.11	Calculation of $J_n(x)$ from $J_0(x)$	343
7.9.12	An Integral Representation for $J_n(x)$	344
7.9.13	Expansion of Cosine in Terms of $J_n(x)$	346
7.9.14	Derivation of a Recurrence Relation from the Generating Function	347
7.9.15	An Alternative Representation for $Y_n(x)$	348
7.9.16	Equations Reducible to Bessel's Equation	348

<i>Bibliography</i>	355
8. Perturbation Methods for Oscillatory Systems	357
8.1 Introduction	357
8.2 The General Perturbation Procedure	358
8.3 Worked Examples Using the General Perturbation Procedure	362
8.3.1 Example A	362
8.3.2 Example B	364
8.4 First-Order Method of Averaging	367
8.4.1 The Method	367
8.4.2 Two Special Cases for $f(x, dx/dt)$	370
8.4.3 Stability of Limit-Cycles	372
8.5 Worked Examples for First-Order Averaging	373
8.5.1 Example A	373
8.5.2 Example B	373
8.5.3 Example C	374
8.6 The Lindstedt-Poincaré Method	376
8.6.1 Secular Terms	376
8.6.2 The Formal Procedure	379
8.7 Worked Examples Using the Lindstedt-Poincaré Method . .	382
8.7.1 Example A	382
8.7.2 Example B	383
8.8 Harmonic Balance	385
8.8.1 Direct Harmonic Balance	386
8.9 Worked Examples for Harmonic Balance	389
8.9.1 Example A	389
8.9.2 Example B	392
8.9.3 Example C	393
8.10 Averaging for Difference Equations	394
8.11 Worked Examples for Difference Equations	396
8.11.1 Example A	396
8.11.2 Example B	398
<i>Comments and References</i>	405
<i>Bibliography</i>	407
9. Approximations of Integrals and Sums	409

9.1	Resume of Asymptotics	410
9.2	Integration by Parts	413
9.3	Laplace Methods	417
	9.3.1 Watson's Lemma	417
	9.3.2 Laplace's Method for Integrals	418
9.4	Worked Examples	419
	9.4.1 Stirling's Formula	419
	9.4.2 Integral Containing a Logarithmic Function	419
	9.4.3 Integral Containing a Complex Exponential Structure	420
	9.4.4 Cosine and Sine Integrals	421
9.5	Euler-Maclaurin Sum Formula	423
	9.5.1 Bernoulli Functions and Numbers	423
	9.5.2 Euler-Maclaurin Sum Formula	426
9.6	Worked Examples for the Euler-Maclaurin Sum Formula	427
	9.6.1 Sums of Powers	427
	9.6.2 Evaluation of $\ln(n!)$	429
	9.6.3 $f(k) = x^{-1/2}$	430
	<i>Comments and References</i>	434
	<i>Bibliography</i>	435
10.	Some Important Nonlinear Partial Differential Equations	437
	10.1 Linear Wave Equations	438
	10.2 Traveling Wave and Soliton Solutions	441
	10.3 A Linear Advective, Nonlinear Reaction Equation	442
	10.4 Burgers' Equation	444
	10.5 The Fisher Equation	448
	10.6 The Korteweg-de Vries Equation	455
	10.7 The Nonlinear Schrödinger Equation	457
	10.8 Similarity Methods and Solutions	459
	10.8.1 Similarity Methods	459
	10.8.2 Examples	461
	10.9 The Boltzmann Problem	464
	10.10 The Nonlinear Diffusion Equation: $uu_t = u_{xx}$	467
	<i>Comments and References</i>	474
	<i>Bibliography</i>	476

Appendix A	Mathematical Relations	479
A.1	Trigonometric Relations	479
A.1.1	Exponential Definitions of Trigonometric Functions	479
A.1.2	Functions of Sums of Angles	479
A.1.3	Powers of Trigonometric Functions	479
A.1.4	Other Trigonometric Relations	480
A.1.5	Derivatives and Integrals of Trigonometric Functions	481
A.2	Factors and Expansions	483
A.3	Quadratic Equations	483
A.4	Cubic Equations	484
A.5	Differentiation of a Definite Integral with Respect to a Parameter	485
A.6	Eigenvalues of a 2×2 Matrix	485
A.7	Routh-Hurwitz Theorem	486
A.8	Integration by Parts	486
A.9	Leibnitz's Relation	487
A.10	L'Hopital's Rule	487
A.11	Special Determinants	487
A.11.1	Jacobian Determinants	487
A.11.2	Hessian Determinants	488
A.11.3	Wronskian Determinants	488
A.12	Special Series	488
A.12.1	Power Series	488
A.13	Trigonometric and Hyperbolic Series	489
A.14	Exponential and Logarithmic Series	489
A.15	Certain Standard Indefinite Integral Involving Exponential Functions	490
A.16	Partial Fractions	491
A.17	Special Constants	492
	<i>Bibliography</i>	494
Appendix B	Asymptotics Expansions	495
B.1	Gauge Functions and Order Symbols	495
B.1.1	The Symbol O	495
B.1.2	The Symbols o	496
B.1.3	Combination of Order Relations	497
B.2	Asymptotic Expansion	497

B.3 Uniform Expansions	499
B.4 Elementary Operations on Asymptotic Expansions	499
B.4.1 Addition and Subtraction	499
B.4.2 Integration	500
B.4.3 Multiplication	500
B.4.4 Differentiation	501
B.5 Examples	501
B.5.1 The Symbol O	501
B.5.2 The Symbol o	502
B.5.3 The Function $\sin(x + \epsilon)$	502
B.5.4 The Function $\exp(-\epsilon x)$	502
B.6 Generalized Asymptotic Power Series	503
 <i>Bibliography</i>	 504
 <i>Index</i>	 505