Numerical Analysis and Misspecifications in Finance: From Model Risk to Localization Error Estimates for Nonlinear PDEs

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In this paper we aim to illustrate the power of probabilistic techniques to analyze numerical errors arising in the study of financial issues and related to various misspecifications. We first describe the global convergence rate of approximation of quantiles of components of diffusion processes when one combines a time discretization of the model and a Monte Carlo simulation in view of computing VaR type quantities. We then present a worst case approach to the problem of controlling model risk; this approach leads to a stochastic game problem and a fully nonlinear PDE in an unbounded domain. To approximate its unique viscosity solution one needs to localize the PDE under consideration and to define artificial boundary conditions. We show that backward stochastic differential equations (BSDEs) are a useful tool to estimate the error due to misspecified Neumann boundary conditions on the artificial boundary.

1. Introduction

If the trader *knows* the model followed by the real market, then, in a complete market, she/he is able to perfectly hedge an option. When the unknown market model is close to the log–normal model and the trader has a rather precise information, then she/he is protected by the robustness of Black and Scholes model (see, e.g., El Karoui, Jeanblanc–Picqué and Shreve [10], Romagnoli and Vargiolu [27]). However, often information is missing, models are unstable, and calibration procedure converge slowly (or diverge!). Then can one measure the risk and find a strategy which guarantees tolerable performances whatever the unknown model is?

In view of proposing (partial) answers to these questions we consider the following system of asset prices (S_t^i) and value of the portfolio (P_t):

$$(1)\qquad \begin{cases} dS_t^i &= S_t^i[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j] \text{ for } 0 \le i \le n, \\ dP_t &= P_t \sum_{i=1}^n \pi_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j\right] + rP_t \left(1 - \sum_{i=1}^n \pi_t^i\right) dt. \end{cases}$$

Here $\{\pi^i\}$ is the set of prescribed strategies (model risk measurement) or of controls chosen by the trader (model risk management). We aim to

- Measure the risk, e.g., by computing quantiles of P_T by means of Monte Carlo methods,
- Control the risk due to misspecifications of the b_t^i's and the σ_t^{ij}'s.

Concerning the first issue, we will describe the global convergence rate of the algorithm in terms of the time step of the Euler discretization scheme of (1) and the number of simulations. We will pay a particular attention to the constants which are involved in the error estimates. This is done in Section 2.

Concerning the second issue, we will show that a worst case approach can be developed. This approach leads to a stochastic game problem and a fully nonlinear PDE in an unbounded domain whose we show existence and uniqueness of the viscosity solution. This is done in Section 3.

The numerical resolution of PDEs cannot be performed in unbounded domains. One thus needs to localize the PDE under consideration and to define artificial boundary conditions. The study of the global error is a huge problem. Here we simply show that backward stochastic differential equations (BSDEs) are a useful tool to estimate the error due to misspecified boundary conditions: in Section 4 we remind various results on BSDEs which allow us, in Section 5, to introduce the BSDEs corresponding to localized problems; finally, in Section 6, we study the stability of the exact solutions with respect to artficial Neumann boundary conditions used to solve the localized equations.

We emphasize that there is essentially no new result on BSDEs even if, to our knowledge, the results of our Section 5 are not explicitly written and proven in the literature because they concern a particular situation which is just between general cases solved in the papers summarized in Section 4. Our objective is simply to illustrate the power of probabilistic techniques to analyze numerical errors due to very various misspecifications arising in the study of financial issues.

2. Model Risk Measurement, Approximation of Quantiles of Diffusion Processes

Classical risk measurements such as VaR are based on quantiles of profit and losses of given portfolios. When one considers stochastic models such as (1), one needs to compute quantiles of one component (or of a system of components) of a multidimensional diffusion process. In this Section we aim to describe the global error of the algorithm consisting in a time discretization of the stochastic model and in a Monte Carlo estimation of the quantiles.

Consider the general SDE in $\mathbb{R}^d$ with smooth coefficients

$$X_t(x) = x + \int_0^t A_0(s, X_s(x))ds + \sum_{i=1}^r \int_0^t A_i(s, X_s(x))dW_s^i,$$

and the Euler discretization scheme of this SDE:

$$\bar{X}^n_{(p+1)T/n}(x) = \bar{X}^n_{pT/n}(x) + A_0(pT/n, \bar{X}^n_{pT/n}(x))\frac{T}{n} + \sum_{i=1}^r A_i(pT/n, \bar{X}^n_{pT/n}(x))(W^i_{(p+1)T/n} - W^i_{pT/n}).$$

For functions f with polynomial growth at infinity one has

$$\mathbb{E}f(X_T(x)) - \mathbb{E}f(\bar{X}^n_T(x)) = \frac{C_f(T,x)}{n} + \frac{Q_n(f,T,x)}{n^2},$$

with

$$|C_f(T,x)| + \sup_n |Q_n(f,T,x)| \;\leq\; C\,(1 + \|x\|^\alpha)\frac{1+K(T)}{T^q}$$

for some constants C, α, q and increasing function K depending on b and σ: see Talay and Tubaro [29] for smooth f's, Bally and Talay [2] for measurable f's when a uniform hypoellipticity condition holds, and Gobet and Munos [16], Kohatsu–Higa and Pettersson [20], etc., for extensions. Thus, Romberg extrapolation techniques are available:

$$\mathbb{E}\{2f(\bar{X}^{2n}_T(x)) - f(\bar{X}^n_T(x))\} - \mathbb{E}f(X_T(x)) = O\left(\frac{1}{n^2}\right).$$

We are interested in estimates on the approximation by the Euler scheme of the quantile $\rho(x,\delta)$ of the law of $X^d_T(x)$. As $\bar{X}^{n,d}_T$ may not have a density when the diffusion matrix is not strongly uniformly elliptic, one may need to deal with the perturbed Euler scheme:

$$\tilde{X}^{n,d}_T(x) = \bar{X}^{n,d}_T(x) + Z^n,$$

where $\{Z^n\}$ is, e.g., a family of independent Gaussian variables with variance $1/n$ which are independent of (W_t). The discretization error of this scheme satisfies the same properties as that of the Euler scheme. For $0 < \delta < 1$, set

$$\rho(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}[X_T^d(x) \le \rho] = \delta\},$$

and

$$\tilde{\rho}^n(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}[\tilde{X}_T^{n,d}(x) \le \rho] = \delta\}.$$

When $(X_t(x))$ models some asset prices and the value of a portfolio, i.e., when $X_t(x) \equiv (S_t, P_t)$, the Malliavin covariance matrix of $X_t(x)$ is not invertible. However one can often suppose that the inverse of the variance of $X_T^d(x)$ (or, more generally, of the covariance matrix of a system of components of $(X_t(x))$) has moments of all order. Consider partially hypoelliptic diffusions. Let $(X_s^t(x'),\ 0 \le s \le T-t)$ be a smooth version of the stochastic flow

$$X_s^t(x') = x' + \int_0^s A_0(t+\theta, X_\theta^t(x'))d\theta + \sum_{i=1}^r \int_0^s A_i(t+\theta, X_\theta^t(x'))dW_{t+\theta}^i.$$

Denote by $M(t,s,x')$ the Malliavin covariance matrix of $X_s^t(x')$, and suppose

(M-1) $\forall p \ge 1$, there exist a positive number r, a non decreasing function $K(\cdot)$ and a positive function $\Psi(\cdot)$ such that

$$\max_{0\le t\le T} \max_{0<s<T-t} \left\| \frac{1}{M_d^d(t,s,x')} \right\|_p \le \frac{K(T)}{s^r} \Psi(t,x'),$$

(M-2) $\forall \lambda \ge 1$, there exists a function $\Psi_\lambda(\cdot)$ such that

$$\sup_{t\in[0,T]} \mathbb{E}[\Psi(t,X_t(x))^\lambda] < \Psi_\lambda(x) \text{ and } \sup_{n>0} \sup_{t\in[0,T]} \mathbb{E}[\Psi(t,\bar{X}_t^n(x))^\lambda] < \Psi_\lambda(x).$$

Under Condition (M), the d-th marginal distribution of $X_T(x)$ has a smooth density $p_T^d(x,y)$ which is strictly positive at all point y in the interior of its support (cf. Nualart [24]).

Theorem 2.1. *Under Condition* (M) *we have*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \le \frac{K(T)}{T^q} \frac{1+\|x\|^Q}{p_T^{d\sharp}(\rho(x,\delta))} \Psi_\lambda(x) \frac{1}{n}.$$

This theorem has been obtained in Talay and Zheng [32] (see also [31]). The method recently used by Gobet and Munos [16]) and inspired by Kohatsu–Higa and Pettersson [20] allows one to get the same convergence rate under the hypothesis: the inverse of $M_d^d(0,T,x)$ belongs to $L^p(\Omega)$ for all integer $p \geq 1$, which is weaker than the condition (M).

In view of this theorem, for classical estimators of the quantile of a sample, the global error of the Monte Carlo method is of order

$$O\left(\frac{1}{p_T^{d\sharp}(\rho(x,\delta))n}\right)+O\left(\frac{1}{\bar{p}_T^{n,d}(x,\rho(x,\delta))\sqrt{N}}\right),$$

where $\bar{p}_T^{n,d}(x,\xi) :=$ density of $\bar{X}_T^{n,d}(x)$. In practice, as must choose the numerical parameters N and n in terms of the desired accuracy, one needs estimates on $\bar{p}_T^{n,d}(x,\xi)$. We know:

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\ p_T(x,y) - \bar{p}_T^n(x,y) = -\frac{1}{n}\pi_T(x,y) + \frac{1}{n^2}R_T^n(x,y),$$

with

$$|\pi_T(x,y)| + |R_T^n(x,y)| \leq \frac{K(T)}{T^q}\exp\left(-c\frac{\|x-y\|^2}{T}\right),$$

see Bally and Talay [3] and, for related results, Kohatsu–Higa [19] and Hu and Watanabe [18]. Consequently, one also needs accurate estimates from below of $p_T^d(x,\rho(x,\delta))$. Estimates are available, either when the generator of (X_t) is strictly uniform elliptic (see, e.g., Azencott [1]), or, when the generator is hypoelliptic, under restrictive assumptions (the generator needs to be in divergence form or almost in divergence form: see Kusuoka and Stroock [21]. Here we can take advantage of the particular structure of financial models. For example, in Talay and Zheng [31] one considers the problem of computing a risk measurement for the profit and loss of a misspecified strategy aiming to hedge a zero coupon bond European option. An easy calculation shows that an appropriate forward price of the zero coupon bond and the value of the trader's portfolio satisfy

$$(2)\qquad \begin{cases} X_t^1(x^1) &= x^1 + \int_0^t X_s^1(x^1)u_1(s)ds + \int_0^t X_s^1(x^1)u_2(s)dW_s, \\ X_t^2(x^1,x^2) &= x^2 + \int_0^t \varphi(s,X_s^1(x^1))X_s^1(x^1)u_1(s)ds \\ &\quad + \int_0^t \varphi(s,X_s^1(x^1))X_s^1(x^1)u_2(s)dW_s, \end{cases}$$

where $\varphi(s,z)$ is a prescribed function related to the payoff of the option under consideration. One has to approximate $\rho(x,\delta)$, the quantile of $X_T^2(x)$ at level δ. Supposing that the coefficients of the stochastic differential equation (2) are smooth and that

$$(3) \qquad |\varphi(t,x^1)u_2(t)| \geq a > 0 \text{ for all } t > 0 \text{ and } x^1 \in \mathbb{R}^+,$$

one can show that the law of $X^2_T(x)$ has a smooth density p^2_T (see [32]). In addition, for some K explicited below, for all $\rho(x,\delta) > K$, one has

$$p^2_T(\rho(x,\delta)) \geq \mathbb{E}\left[g_{\Lambda(T)}(\mathrm{H}^{-1}(\rho(x,\delta)))\mathcal{J}(\rho(x,\delta))\right].$$

Here, g_ϵ denotes the Gaussian density $N(0,\epsilon)$, and

$$\Lambda(t) := \int_0^t u_2^2(s)ds, \;\; \Upsilon(s,z) := \int_0^z \varphi(\Lambda^{-1}(s),\alpha)d\alpha, \;\; \bar{U}_t := \int_0^T \left(\tfrac{u_1(\Lambda^{-1}(s))}{u_2(\Lambda^{-1}(s))} - \tfrac{1}{2}\right)ds,$$

$$W_t^\Lambda := \sqrt{\Lambda^{-1}(t)}W_{\Lambda^{-1}(t)},$$

$$h(s,z) := \frac{\partial \Upsilon}{\partial s}(s,z) + \frac{1}{2}\frac{\partial \varphi}{\partial z}(\Lambda^{-1}(s),z),$$

$$\mathrm{H}(x,z,\omega) := x^2 - \Upsilon(0,x^1) + \Upsilon\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z\right)\right) - \int_0^{\Lambda(t)} h\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}_s^\Lambda - \frac{s}{\Lambda(t)}\tilde{W}^\Lambda_{\Lambda(t)} + \frac{zs}{\Lambda(t)}\right)\right)ds,$$

$$K := \sup_\omega \mathrm{H}\left(x, \log(\tfrac{C\Lambda(t)}{ax^1}) - \overline{U}_t, \omega\right),$$

and $\mathcal{J}$ is the Jacobian matrix of $\mathrm{H}^{-1}(x,\cdot,\omega)$ (see Talay and Zheng [32]). This expression results from rather simple calculations involving change of time and Brownian bridge but no Malliavin calculus technique. We have supposed that there exists a constant C such that

$$|\varphi(t,z)| \leq C$$

and

$$\left|\int_0^{\Lambda(t)} \frac{\partial \Upsilon}{\partial s}(s,z)ds\right| \leq C$$

for all t in $[0,T^O]$ and $z \in \mathbb{R}^+$, which ensures that K is finite a.s. The bound from below has a rather complex form but, however, one can easily deduce accurate estimates from it.

Open questions are: can one expand w.r.t. $1/n$ the error on quantiles, can one develop efficient variance reduction methods and find tractable lower bound estimates on (marginal) densities of degenerate diffusions more general than above?

3. A Worst Case Analysis for Model Risks

In this section we aim to present a worst case approach for model risk control.

Cvitanić and Karatzas [9] have proposed the dynamic measure of risks

$$\inf_{\pi(\cdot)\in\mathcal{A}(x)} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu(F(P_T^\pi)|P_0^\pi = x),$$

where P^π denotes the value of the portfolio corresponding to the strategy π, $\mathcal{A}(x)$ is the class of admissible portfolio strategies, $\mathcal{D}$ is a set of measures having the same risk-neutral equivalent martingale measure and $\mathbb{E}_\nu$ is the expectation under $\mathbb{P}_\nu$. Such a measure corresponds to the case where one is concerned by model risk on stock appreciation rates only. Here, we include model risk on volatilities, stock appreciation rates, yield curves, etc., and introduce a stochastic game problem (see Talay and Zheng [30]) as follows: the trader acts as a minimizer of the risk; on the other hand, suppose that the market systematically behaves against the interest of the trader, and acts as a maximizer of the risk. Thus the model risk control problem can be set up as a two players (trader versus market) zero-sum stochastic differential game problem. We study viscosity solutions of the related fully nonlinear Hamilton–Jacobi–Belmann–Isaacs PDE. The solution at time 0 is the minimal amount of money that the financial institution needs to contain the worst possible damage.

In our setting, an admissible control process $u(\cdot) := (b(\cdot), \sigma(\cdot))$ for the market on $[t,T]$ is a progressively measurable process taking value in a compact subset K_u of $\mathbb{R}^n \times \mathbb{R}^{nd}$. An admissible control process $\pi(\cdot)$ for the investor on $[t,T]$ is a progressively measurable process taking value in a compact subset K_π of $\mathbb{R}^n$. The set of all admissible controls for the market on $[t,T]$ is denoted by $Ad_u(t)$ and the set of all admissible controls for the investor on $[t,T]$ is denoted by $Ad_\pi(t)$. An admissible strategy Π for the investor on $[t,T]$ is a mapping $\Pi : Ad_u(t) \to Ad_\pi(t)$. The set of all admissible strategies for the investor on $[t,T]$ is denoted by $Ad_\Pi(t)$. For given $\Pi \in Ad_\Pi(t)$ and $u(\cdot) \in Ad_u(t)$, we set $\pi(\cdot) := \Pi(u(\cdot))(\cdot)$.

The controlled system of prices and value of the portfolio is (1). Given a suitable function F define a cost function as

$$J(\theta, x, p, \Pi, u(\cdot)) := \mathbb{E}_{\theta,x,p} F(S_T, P_T),\ 0 \le \theta \le T.$$

The corresponding value function is

$$V(\theta, x, p) := \inf_{\Pi\in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} J(\theta, x, p, \Pi, u(\cdot)).$$

For $x \in \mathbb{R}^d$, $p \in \mathbb{R}$, and σ in the set of $d\times n$ matrices, define the $(n+1)\times(n+1)$

symmetric matrix $a(x,p,\sigma,\pi)$ as

$$\begin{cases} a^i_j(x,p,\sigma,\pi) & := \sum_{k=1}^d (x^i\sigma^i_k x^j\sigma^j_k) \text{ for } 1 \le i,j \le n, \\ a^{n+1}_j(x,p,\sigma,\pi) & := \sum_{k=1}^d \sum_{\ell=1}^n (p\pi^\ell\sigma^k_\ell x^j\sigma^j_k) \text{ for } 1 \le j \le n, \\ a^{n+1}_{n+1}(x,p,\sigma,\pi) & := \sum_{k=1}^d \sum_{\ell=1}^n (p^2|\pi^\ell|^2|\sigma^k_\ell|^2), \end{cases}$$

and the $n+1$ dimensional vector $q(x,p,b,\pi)$ as

$$q(x,p,b,\pi) := \left(x^1b^1, \ldots, x^nb^n, p\left(r + \sum_{i=1}^d \pi^i(b^i - r) \right) \right).$$

For all admissible $u=(b,\sigma)$ and π and all $(n+1)\times(n+1)$ matrix A we set

$$\mathcal{H}_{u,\pi}(A,z,x,p) := \left\{ \frac{1}{2}\mathrm{Tr}\left(a(x,p,\sigma,\pi)A\right) + z\cdot q(x,p,b,\pi) \right\}.$$

We have (see Talay and Zheng [30]):

Theorem 3.1. *Under an appropriate locally Lipschitz condition on F, the value function $V(\theta,x,p)$ is the unique viscosity solution in the space*

$$S := \{\varphi(t,x,p) \text{ is continuous on } [0,T]\times\mathbb{R}^n\times\mathbb{R};\ \exists\overline{A}>0,$$
$$\lim_{|p|^2+x^2\to\infty} \varphi(t,x,p)\exp(-\overline{A}|\log(|p|^2+x^2)|^2) = 0 \text{ for all } t\in[0,T]\}$$

to the Hamilton-Jacobi-Bellman-Isaacs equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x,p) + \mathcal{H}^-(D^2v(t,x,p), Dv(t,x,p), x,p) = 0 \ \text{ in } [0,T)\times\mathbb{R}^{n+1}, \\ v(T,x,p) = F(x,p). \end{cases}$$

where

$$\mathcal{H}^-(A,z,x,p) := \max_{u\in K_u}\min_{\pi\in K_\pi} \left\{ \frac{1}{2}\mathrm{Tr}\left(a(x,p,\sigma,\pi)A\right) + z\cdot q(x,p,b,\pi) \right\}.$$

When the controlled system has bounded coefficients and F is a bounded Lipschitz function, the theorem results from Fleming and Souganidis [14]. Here, the existence of a solution is obtained owing to a localization technique consisting in approximating all the unbounded functions of the problem by sequences of bounded functions. To get uniqueness, one adapts a result and a proof designed by Barles, Buckdahn and Pardoux [5] for other families of PDEs.

Open questions are: the convergence of numerical discretization schemes of the HJBI equation, the cost of the optimal strategy and the analysis of its dependency on the model misspecification.

Concerning the numerical discretization of the HJBI equation, one needs to reduce the integration domain to a bounded domain, and it is difficult to define boundary conditions which ensure a good global accuracy. We now aim to provide a first step to the analysis of the effects of artificial boundary conditions for nonlinear PDEs in Finance (for linear PDEs, see [8], [22], [6]). For example, we consider the American option pricing problem. It is well known that the price at time t corresponding to the spot stock price x is the unique viscosity solution of the variational inequality

(4)

$$\begin{cases} \min\left\{v(t,x)-\phi(t,x); -\frac{\partial v}{\partial t}(t,x) - A_t v(t,x) - rv(t,x)\right\} = 0, \ (t,x)\in[0,T)\times\mathbb{R}, \\ v(T,x)=\phi(T,x), \ x\in\mathbb{R}, \end{cases}$$

where ϕ is the payoff function, A is the infinitesimal generator of the stock price, and r is the instantaneous interest rate. To solve this equation numerically one needs to localize it into a bounded domain O with smooth boundary, and to define Dirichlet or Neumann artifial boundary conditions on ∂O. One then discretizes the localized equation. As the solution $v(t,x)$ is unknown on the boundary, the artificial boundary conditions cannot be chosen in such a way that the solution $u(t,x)$ of the localized problem coincides with $v(t,x)$ inside O: this induces a localization error. A first step in the global error analysis consists in finding estimates on the localization error. Here we address a small part of the problem. We consider an equation which is slightly more general than (4):

(5)

$$\begin{cases} \min\left(\tilde{u}(t,x)-L(t,x); -\frac{\partial \tilde{u}}{\partial t}(t,x) - A_t\tilde{u}(t,x) - f(t,x,\tilde{u}(t,x),(\nabla\tilde{u}\sigma)(t,x))\right) = 0, \\ \qquad\qquad (t,x)\in[0,T)\times\mathbb{R}^d, \\ \tilde{u}(T,x)=\phi(x) \text{ for all } x\in\mathbb{R}^d. \end{cases}$$

We then localize it and choose unhomogeneous Neumann boundary conditions:

(6)

$$\begin{cases} \min\left\{u(t,x)-L(t,x); -\frac{\partial u}{\partial t}(t,x) - A_t u(t,x) - f(t,x,u(t,x),(\nabla u\sigma)(t,x))\right\} = 0, \\ \qquad\qquad (t,x)\in[0,T)\times O, \\ u(T,x)=\phi(x), \ x\in\overline{O}, \\ (\nabla u(t,x); n(x)) + g(t,x) = 0, \ (t,x)\in[0,T)\times\partial O, \end{cases}$$

where, for all x in $\partial\mathcal{O}$, $n(x)$ denotes the inward unit normal vector at point x. In order to compare $u(t,x)$ and $\tilde{v}(t,x)$ we construct a reflected backward stochastic differential equation (RBSDE) coupled with a reflected forward stochastic differential equation, and we show that the solution of the RBSDE provides the viscosity solution $u(t,x)$ of (6). We then apply our result to estimate localization errors.

4. Various RBSDEs

In this section we remind results due to El Karoui et al. [11], Pardoux and Zhang [26], Ma and Cvitanić [23] which contain the essential of our stochastic representation of the solution of (6). We recommend the reader to consult these papers to see the results summarized below in their full generality and to get an extensive bibliography on BSDEs. See also, e.g., Hamdene and Lepeltier [17], Pardoux [25].

In all the sequel the time origin t is arbitrary in $[0,T]$. The various constants do not depend on t (but depend on T). We consider a d-dimensional Brownian motion $(W_s, s \geq 0)$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with the augmented natural filtration $(\mathcal{F}_s, s \geq 0)$ of (W_s). We fix a bounded domain $\mathcal{O}$ in $\mathbb{R}^d$ with a smooth boundary. We will use the following spaces of random variables and processes:

$$\mathcal{L}^2 := \{\xi \text{ is } \mathcal{F}_T - \text{measurable and } \mathbb{E}|\xi|^2 < \infty\},$$

$$\mathcal{S}^2 := \{(\psi_s, 0 \leq s \leq T) \text{ is a progressively measurable process s.t. } \mathbb{E} \sup_{0 \leq s \leq T} |\psi_s|^2 < \infty\},$$

$$\mathcal{H}^2 = \{(\psi_s, 0 \leq s \leq T) \text{ is a progressively measurable process s.t. } \mathbb{E} \int_0^T |\psi_s|^2 ds < \infty\}.$$

4.1 Reflected BSDEs with an unreflected forward SDE and a Lipschitz hypothesis

Consider the forward SDE

$$X_s^{t,x} = x + \int_t^s b(\theta, X_\theta^{t,x}) d\theta + \int_t^s \sigma(\theta, X_\theta^{t,x}) dW_\theta, \; 0 \leq t \leq s \leq T, \tag{7}$$

where b is a continuous function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^d$ and σ is a continuous function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^{d \times d}$. Both b and σ are Lipschitz w.r.t. the x coordinates. In [11] one shows the existence and uniqueness of the triple $(Y_s^{t,x}, Z_s^{t,x}, R_s^{t,x})$ of progressively measurable processes solving the following

BSDE with reflection on the obstacle $(L(s, X_s^{t,x}))$:

(8)

$$\begin{cases} Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + R_T^{t,x} - R_s^{t,x} - \int_s^T Z_r^{t,x} dW_r, \ t \le s \le T, \\ Y_s^{t,x} \ge L(s, X_s^{t,x}), \ 0 \le t \le s \le T, \\ (R_s^{t,x}, 0 \le t \le s \le T) \text{ is a continuous increasing process such that} \\ \int_t^T (Y_s^{t,x} - L(s, X_s^{t,x}))dR_s^{t,x} = 0. \end{cases}$$

Among the assumptions made by the authors, the coefficient $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is assumed to satisfy the Lipschitz condition

$$(9) \quad \begin{cases} |f(t,x,0,0)| \le K(1 + |x|^p), \ t \in [0,T], \ x \in \mathbb{R}^d, \\ |f(t,x,y,z) - f(t,x,y',z')| \le K(|y - y'| + |z - z'|), \\ \qquad \qquad t \in [0,T], \ x, z, z' \in \mathbb{R}^d, \ y, y' \in \mathbb{R}, \end{cases}$$

and the function L is assumed to satisfy

$$L(t,x) \le K(1 + |x|^p), \ t \in [0,T], \ x \in \mathbb{R}^d,$$
$$L(T,x) \le \phi(x), \ x \in \mathbb{R}^d.$$

El Karoui et al. [11] also show that $\tilde{u}(t,x) := Y_t^{t,x}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ is the unique viscosity solution of (5), where A is the infinitesimal generator of the solution of (7). El Karoui et al. [12] have applied this result to represent American option prices.

4.2 Reflected BSDEs with an unreflected forward SDE and a monotonicity hypothesis

We aim to replace the condition (9) in subsection 4.1 by a. monotonicity condition. We therefore suppose

$$(10) \quad \begin{cases} |f(t,x,0,0)| \le K(1 + |x|^p), \ t \in [0,T], \ x \in \mathbb{R}^d, \\ \exists \gamma \in \mathbb{R}, \ (y_1 - y_2)(f(t,x,y_1,z) - f(t,x,y_2,z)) \le \gamma |y_1 - y_2|^2 \\ \qquad \qquad \text{for all } t, x, y_1, y_2, z. \end{cases}$$

To this end we first look at a BSDE without a forward SDE. Consider a map $q : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that, for some $\gamma \in \mathbb{R}$, $K \ge 0$, $(\psi_t) \in \mathcal{H}^2$ and all (ω, t, x, y),

1. $q(\cdot, y, z)$ is progressively measurable and $y \to q(t,y,z)$ is continuous,
2. $(y_1 - y_2)(q(t,y_1,z) - q(t,y_2,z)) \le \gamma |y_1 - y_2|^2$,
3. $|q(t,y,z_1) - q(t,y,z_2)| \le K|z_1 - z_2|$,
4. $q(t,y,z) \le \psi_t + K(|y| + |z|)$.

We have

Proposition 4.1. *Let $(\tilde{V}_s)$ in $\mathcal{H}^2$, $(\tilde{L}_s)$ in $\mathcal{S}^2$ and ξ in $\mathcal{L}^2$ satisfying $\xi \geq \tilde{L}_T$ a.s. There exists a unique triple $(\tilde{Y}, \tilde{Z}, \tilde{R})$ such that*

$$(11) \quad \begin{cases} \tilde{Y}_s = \xi + \int_s^T q(\theta, \tilde{Y}_\theta, \tilde{V}_\theta) d\theta + \tilde{R}_T - \tilde{R}_s - \int_s^T \tilde{Z}_\theta dW_\theta, \ 0 \leq s \leq T, \\ \tilde{Y}_s \geq \tilde{L}_s, \ 0 \leq s \leq T, \\ \int_0^T (\tilde{Y}_s - \tilde{L}_s) d\tilde{R}_s = 0. \end{cases}$$

Proof. The proof essentially is the same as the proof of the proposition 1.8 in Pardoux and Zhang [26] . We set $Q(s,y) := q(s,y,\tilde{V}_s)$ and $Q_n(s,y) := (\rho_n * Q(s,\cdot))(y)$, where $\rho_n : \mathbb{R} \to \mathbb{R}_+$ is a sequence of smooth functions such that

$$\int \rho_n(\theta) d\theta = 1, \ \sup_n \int |\theta| \rho_n(\theta) d\theta < \infty.$$

In view of the theorem 5.2 in [11]) there exists a unique solution to the BSDE with smooth coefficients

$$(12) \quad \begin{cases} \tilde{Y}_s^n = \xi + \int_s^T Q_n(\theta, \tilde{Y}_\theta^n) d\theta + \tilde{R}_T^n - \tilde{R}_s^n - \int_s^T \tilde{Z}_\theta^n dW_\theta, \ 0 \leq s \leq T, \\ \tilde{Y}_s^n \geq \tilde{L}_s, \ 0 \leq s \leq T, \\ \int_0^T (\tilde{Y}_s^n - \tilde{L}_s) d\tilde{R}_s^n = 0. \end{cases}$$

One then proceeds as in the proof of the proposition 1.8 in Pardoux and Zhang [26] . □

Equipped with the preceding proposition one can then use a fixed point argument as in the proof of the theorem 1.7 in Pardoux and Zhang [26] (see also the proof of the theorem 5.2 in [11] to check the condition $Y \geq L$) and get

Theorem 4.1. *Under the above assumptions there exists a unique triple $(\tilde{Y}, \tilde{Z}, \tilde{R})$ solution to*

(13)

$$\begin{cases} \tilde{Y}_s = \xi + \int_s^T q(\theta, \tilde{Y}_\theta, \tilde{Z}_\theta) d\theta - \int_s^T (\tilde{Z}_\theta; dW_\theta) + \tilde{R}_T - \tilde{R}_s, \ 0 \leq s \leq T, \\ \tilde{Y}_s \geq L_s, \ 0 \leq s \leq T, \\ (\tilde{R}_s, 0 \leq s \leq T) \text{ is a continuous increasing process s.t. } \int_t^T (\tilde{Y}_\theta - L_\theta) d\tilde{R}_\theta = 0. \end{cases}$$

Choosing $q(\omega, t, \cdot, \cdot) := f(X_t(\omega), t, \cdot, \cdot)$ where (X_t) is the solution to (7), one can then easily prove that, as in the preceding subsection but under the condition (10), $\tilde{u}(t,x) := \tilde{Y}_t^{t,x}$ is the unique viscosity solution of (5). In addition, standard calculations (see Ma and Cvitanić [23]) show that

$$(14) \quad |\tilde{Y}_{t_1}^{t_1,x_1} - \tilde{Y}_{t_2}^{t_2,x_2}|^2 \leq C(|x_1 - x_2|^2 + t_2 - t_1)$$

for all x_1, x_2 in $\mathbb{R}^d$ and $t \leq t_1 \leq t_2 \leq T$.

We now state a slight extension. We are given a coefficient f satisfying the condition (10), an obstacle $(L_s, 0 \leq s \leq T)$ in $\mathcal{S}^2$, a random variable ξ in $\mathcal{L}^2$ s.t. $\xi \geq L_T$ a.s., and a process $(H_s, 0 \leq s \leq T)$ in $\mathcal{S}^2$.

Consider the BSDE

(15)

$$\begin{cases} Y_s = \xi + \int_s^T q(\theta, Y_\theta, Z_\theta)d\theta - \int_s^T (Z_\theta; dW_\theta) + R_T - R_s + H_T - H_s, \ 0 \leq s \leq T, \\ Y_s \geq L_s, \ 0 \leq s \leq T, \\ (R_s, 0 \leq s \leq T) \text{ is a continuous increasing process s.t. } \int_t^T (Y_\theta - L_\theta)dR_\theta = 0. \end{cases}$$

Proposition 4.2. *Under the above assumptions there exists a unique triple (Y, Z, R) with Y in $\mathcal{S}^2$ and Z in $\mathcal{H}^2$ which solves (15).*

Proof. Set $\tilde{Y}_s := Y_s + H_s$, $\tilde{\xi} := \xi + H_T$, $\tilde{q}(\theta, y, z) := q(\theta, y - H_\theta, z)$ and $\tilde{L}_s = L_s + H_s$. The BSDE can be equivalently rewritten as

(16)

$$\begin{cases} \tilde{Y}_s = \tilde{\xi} + \int_s^T \tilde{q}(\theta, \tilde{Y}_\theta, \tilde{Z}_\theta)d\theta - \int_s^T (\tilde{Z}_\theta; dW_\theta) + \tilde{R}_T - \tilde{R}_s, \ 0 \leq s \leq T, \\ \tilde{Y}_s \geq \tilde{L}_s, \ 0 \leq s \leq T, \\ (\tilde{R}_s, 0 \leq s \leq T) \text{ is a continuous increasing process s.t. } \int_0^T (\tilde{Y}_\theta - \tilde{L}_\theta)d\tilde{R}_\theta = 0. \end{cases}$$

One easily checks that the theorem 4.1 applies. □

The solution (Y, Z, R) satisfies the following estimate:

Proposition 4.3. *There exists a positive number C such that*

$$(17) \quad \mathbb{E}\left\{ \sup_{0 \leq \theta \leq T} |Y_s|^2 + \int_0^T |Z_s|^2 d\theta + |R_T|^2 \right\} \leq C\mathbb{E}\left\{ \xi^2 + \int_t^T q(\theta, -H_\theta, 0)^2 d\theta + \sup_{0 \leq \theta \leq T} |L_\theta|^2 + \sup_{0 \leq \theta \leq T} |H_\theta|^2 \right\}.$$

Proof. Consider (16) and observe that the monotonicity condition (10) implies that

$$\begin{aligned} \tilde{Y}_\theta \tilde{q}(\theta, \tilde{Y}_\theta, \tilde{Z}_\theta) &\leq (\tilde{Y}_\theta - 0; \tilde{q}(\theta, \tilde{Y}_\theta, \tilde{Z}_\theta) - \tilde{q}(\theta, 0, \tilde{Z}_\theta)) + \tilde{Y}_\theta \tilde{q}(\theta, 0, \tilde{Z}_\theta) \\ &\leq \gamma |\tilde{Y}_\theta|^2 + K|\tilde{Y}_\theta| \, |\tilde{Z}_\theta| + |\tilde{Y}_\theta| \, |\tilde{q}(\theta, 0, 0)|. \end{aligned}$$

One can then proceed as in the proof of the proposition 3.5 in [11] to get

$$\mathbb{E}\left\{ \sup_{0 \leq \theta \leq T} |\tilde{Y}_\theta|^2 + \int_0^T |\tilde{Z}_\theta|^2 d\theta + |\tilde{R}_T|^2 \right\} \leq C\mathbb{E}\left\{ \tilde{\xi} + \int_0^T \tilde{q}(\theta, 0, 0)d\theta + \sup_{0 \leq \theta \leq T} |\tilde{L}_\theta|^2 \right\}.$$

The conclusion follows easily. □

4.3 BSDEs with a reflected forward SDE and a monotonicity hypothesis

In [26], Pardoux and Zhang prove the existence and uniqueness of the solution of a BSDE coupled with the **forward reflected SDE**

$$(18)\quad \begin{cases} X_s^{t,x} = x + \int_t^s b(\theta, X_\theta^{t,x})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,x})dW_\theta + \eta_s^{t,x},\ 0 \le t \le s \le T, \\ \eta_s = \int_t^s n(X_\theta^{t,x})d|\eta|_\theta^{t,x} \text{ with } |\eta|_s^{t,x} = \int_t^s \mathbb{I}_{\{X_\theta^{t,x} \in \partial O\}} d|\eta|_\theta^{t,x}. \end{cases}$$

Consider the multidimensional BSDE

$$(19)\qquad Y_s^{t,x} = \underline{\phi}(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r$$
$$+ \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d\eta_r^{t,x},\ 0 \le t \le s \le T,$$

where $\underline{\phi}$ is a mapping from O to $\mathbb{R}^p$.

We do not rewrite here the hypotheses made in [26]. We simply emphasize that, instead of assuming the Lipschitz condition (9) on f, the authors assume the monotonicity condition (10). The authors prove the existence and uniqueness of a solution to the forward-backward SDE (18-19), and that

$$u(t,x) := Y_t^{t,x} \text{ for all } (t,x) \in [0,T] \times O,$$

is the unique viscosity solution of the system of quasi-linear PDEs

$$(20)\quad \begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) + Au_i(t,x) + f_i(t,x,u(t,x), \nabla u_i \sigma(t,x)) = 0,\ (t,x) \in [0,T) \times O, \\ u_i(T,x) = \underline{\phi}_i(x),\ x \in \bar{O}, \\ \dfrac{\partial u_i}{\partial n}(t,x) + g_i(t,x,u(t,x)) = 0,\ (t,x) \in [0,T) \times \partial O, \end{cases}$$

where A is the infinitesimal generator of the solution of (18) and $1 \le i \le p$.

4.4 Reflected BSDEs with a reflected forward SDE and a monotonicity hypothesis

In [23] Ma and Cvitanić consider a BSDE with two obstacles L and U s.t. $L < U$, coupled with the forward reflected SDE (18). For all (ω, t, x) in $\Omega \times [0,T] \times \mathbb{R}^d$, set

$$O_2(\omega, t, x) := [L(\omega, t, x), U(\omega, t, x)].$$

Consider the BSDE

$$(21)\quad \begin{cases} Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r + R_T^{t,x} - R_s^{t,x}, \\ (Y_s^{t,x}) \text{ a.s. belongs to } O_2(\cdot, s, X_s^{t,x}), \\ (Y_s^{t,x} - \rho_s, dR_s^{t,x}) \le 0 \text{ for any continuous and progressively measurable} \\ \text{process } \rho \text{ s.t. } \rho_s \in O_2(\cdot, s, X_s^{t,x}) \text{ for all } s \in [t,T] \text{ a.s..} \end{cases}$$

The authors show the existence and uniqueness of the solution, and prove that

$$u(t,x) := Y_t^{t,x} \text{ for all } (t,x) \in [0,T] \times \mathcal{O},$$

is the unique viscosity solution of the following variational inequality with homogeneous Neumann boundary condition

$$(22) \quad \begin{cases} \max\Big\{u(s,x) - U(s,x); \\ \qquad \min\Big(u - L; -\dfrac{\partial u}{\partial t} - A_t u - f\big(t,x,u(t,x),(\nabla u\sigma)(t,x)\big)\Big)\Big\} = 0, \\ \qquad\qquad\qquad\qquad\qquad (t,x) \in [0,T) \times \mathcal{O}, \\ u(T,x) = \phi(x),\ x \in \bar{\mathcal{O}}, \\ \dfrac{\partial u}{\partial n}(s,x) = 0,\ (s,x) \in [0,T) \times \partial\mathcal{O}. \end{cases}$$

5. An Unhomogeneous Reflected BSDE with a Reflected Forward SDE and a Monotonicity Hypothesis

From now on, the results essentially come from Berthelot's Ph.D. thesis [6].

We again consider the forward reflected SDE (18). We assume that the functions b and σ satisfy the same conditions as in the subsection 4.1. Then there exists a unique continuous and progressively measurable solution $(X_s^{t,x}, \eta_s^{t,x}, t \le s \le T)$ to (18) which satisfies

$$(23) \qquad \mathbb{E} \sup_{t\le\theta\le T} |X_\theta^{t,x}|^4 < \infty \text{ and } \mathbb{E} \sup_{t\le\theta\le T} e^{\mu|\eta|_\theta^{t,x}} < \infty \text{ for all } \mu > 0.$$

In addition, we have the following estimates for all $t \le t_1 \le t_2 \le T$:

$$(24) \qquad \mathbb{E} \sup_{t_2\le\theta\le T} |X_\theta^{t_1,x_1} - X_\theta^{t_2,x_2}|^4 \le C(|x_1 - x_2|^4 + (t_2 - t_1)^2),$$

$$(25) \qquad \mathbb{E} \sup_{t_2\le\theta\le T} ||\eta|_\theta^{t_1,x_1} - |\eta|_\theta^{t_2,x_2}|^4 \le C(|x_1 - x_2|^4 + (t_2 - t_1)^2).$$

The above four inequalities are easy to prove: the three first ones readily follows from the corollary 2.3 in [28] and the argument used in [26]; the last one follows from this latter argument and the Itô formula applied to $p(X_s^{t,x})$, where p is a smooth function such that $\partial\mathcal{O} := \{p(x) = 0\}$ and $\mathcal{O} := \{p(x) > 0\}$.

Now consider the **unhomogeneous reflected BSDE with reflected for-**

ward SDE

(26)

$$\begin{cases} Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(\theta, X_\theta^{t,x}, Y_\theta^{t,x}, Z_\theta^{t,x})d\theta - \int_s^T (Z_\theta^{t,x}; dW_\theta) + R_T^{t,x} - R_s^{t,x} \\ \qquad\qquad + \int_s^T g(\theta, X_\theta^{t,x})d|\eta|_\theta^{t,x},\ t \le s \le T, \\ Y_s^{t,x} \ge L(s, X_s^{t,x}),\ t \le s \le T, \\ (R_s^{t,x}, t \le s \le T) \text{ is an increasing continuous process s.t.} \\ \int_t^T (Y_\theta^{t,x} - L(\theta, X_\theta^{t,x}))dR_\theta^{t,x} = 0. \end{cases}$$

We assume:

Hypotheses on the backward SDE.

(27)

$$\begin{array}{l} \text{The function } \phi : \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is Lipschtiz continuous,} \\ \text{The function } f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is continuous,} \\ \exists K \ge 0,\ \gamma \in \mathbb{R},\ |f(t,x_1,y,z_1) - f(t,x_2,y,z_2)| \le K(|x_1 - x_2| + |z_1 - z_2|), \\ \qquad\qquad\qquad\qquad\qquad 0 \le t \le T, \\ \exists \gamma > 0,\ (y_1 - y_2, f(t,x,y_1,z) - f(t,x,y_2,z) \le \gamma |y_1 - y_2|^2,\ 0 \le t \le T, \\ \exists K_f > 0,\ p \in \mathbb{N},\ |f(t,x,y,z)| \le K_f(1 + |x|^p + |y|),\ 0 \le t \le T, \\ \text{The function } g : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is continuous, } 0 \le t \le T. \end{array}$$

Hypothesis on the obstacle of the BSDE.

(28) The function $L(s,x)$ is of class $C^{1,2}([0,T] \times \mathbb{R}^d)$ and its derivatives have a polynomial growth at infinity.

Proposition 5.1. *For all $0 \le t \le T$ there exists a unique triple $(Y_s^{t,x}, Z_s^{t,x}, R_s^{t,x}, t \le s \le T)$ of progressively measurable processes which solves (26).*

Proof. Set

$$H_s^{t,x} := \int_t^s g(\theta, X_\theta^{t,x})d|\eta|_\theta^{t,x},$$

and then apply the proposition 4.2. □

We now prove that $u(t,x) := Y_t^{t,x}$ is the unique viscosity solution of the variational inequality (6).

By definition, a function $u(t,x)$ in $C([0,T) \times \overline{O})$ is a viscosity subsolution of (6) if $u(T,x) \le \phi(x)$ for all x in $\bar{O}$ and, for all function φ in $C^{1,2}([0,T] \times \overline{O})$ such that (t,x) is a global maximum of $u - \varphi$ one has

$$\min\left\{u(t,x) - L(t,x); -\frac{\partial \varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x))\right\} \le 0,$$
$$(t,x) \in [0,T] \times O,$$

and

$$\min\Big\{ -(\nabla\varphi(t,x);n(x)) - g(t,x);$$

$$\min\Big(u(t,x) - L(t,x); -\frac{\partial\varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x))\Big)\Big\} \leq 0,$$
$$(t,x) \in [0,T] \times \partial O.$$

Similarly, a function $u(t,x)$ in $C([0,T) \times \overline{O})$ is a viscosity supersolution of (6) if $u(T,x) \geq \phi(x)$ for all x in $\bar{O}$ and, for all function φ in $C^{1,2}([0,T] \times \overline{O})$ such that (t,x) is a global minimum of $u - \varphi$ one has

$$\min\Big\{u(t,x) - L(t,x); -\frac{\partial\varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x))\Big\} \geq 0,$$
$$(t,x) \in [0,T] \times O,$$

and

$$\max\Big\{ -(\nabla\varphi(t,x);n(x)) - g(t,x);$$

$$\min\Big(u(t,x) - L(t,x); -\frac{\partial\varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x))\Big)\Big\} \geq 0,$$
$$(t,x) \in [0,T] \times \partial O.$$

A viscosity solution is a function which both is a viscosity subsolution and supersolution.

Theorem 5.1. *The function $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (6).*

Proof. The proof essentially follows arguments developed in Cvitanić and Ma [23]. We sketch it for the reader's convenience. We only prove that u is a subsolution. We first observe that it is a continuous map: indeed, from (25) we know that the random field $(\eta_\theta^{t,x})$ almost surely depends continuously from (t,x) from which it is easy to deduce that $(\int_t^T g(\theta, X_\theta^{t,x})d|\eta|_\theta^{t,x})$ satisfies the same property; it then suffices to set $\tilde{Y}_t := Y_t + H_t$ and use the estimate (14). Now, admit for a while the following inequality where $\tilde{\mathbb{E}}$ denotes the expectation under a probability measure to be defined later on and τ is an arbitrary stopping time taking values in $[t,T]$:

(29)

$$0 \geq \tilde{\mathbb{E}} \int_t^\tau \left\{ -\frac{\partial\varphi}{\partial t}(\theta, X_\theta^{t,x}) - A_\theta\varphi(\theta, X_\theta^{t,x}) - f(\theta, X_\theta^{t,x}, u(\theta, X_\theta^{t,x}), (\nabla\varphi\sigma)(t, X_\theta^{t,x})) \right\} d\theta$$
$$- \tilde{\mathbb{E}} \int_t^\tau \left\{ g(\theta, X_\theta^{t,x}) + \left(\nabla\varphi(\theta, X_\theta^{t,x}); n(X_\theta^{t,x})\right)\right\} d|\eta|_\theta^{t,x} - \tilde{\mathbb{E}}(R_\tau^{t,x} - R_t^{t,x}).$$

Case 1: $\underline{x \in \mathcal{O}}$. Let $(t,x) \in [0;T] \times \mathcal{O}$ and φ a smooth function such that (t,x) is a local maximum of $u - \varphi$. We may suppose that $\varphi(t,x) = u(t,x)$. If $u(t,x) = L(t,x)$ then we are done. We thus consider the case where $L(t,x) < u(t,x)$. Set

$$G(t,x,u,\varphi) := -\frac{\partial\varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)).$$

We aim to prove that $G(t,x,u,\varphi) \leq 0$. Suppose that

$$\exists \varepsilon_0 > 0 \text{ s.t. } \Gamma_1(t,x) := \min(u(t,x) - L(t,x); G(t,x,u,\varphi); d(x,\partial\mathcal{O})) \geq \varepsilon_0.$$

Consider the stopping time $\tau_1 := \inf\{s \in [t,T];\ \Gamma_1(s, X_s^{t,x}) \leq \frac{\varepsilon_0}{2}\}$. From (29) we have

$$0 \geq \frac{\varepsilon_0}{2}\tilde{\mathbb{E}}(\tau_1 - t) - \tilde{\mathbb{E}}(R_\tau^{t,x} - R_t^{t,x}).$$

Observe that $(Y_s - \rho_s; dR_s) \leq 0$ for all process ρ satisfying $\rho_s \geq L_s$, and that $Y_\theta^{t,x} = u(\theta, X_\theta^{t,x})$ in view of the pathwise uniqueness of the solution to the RBSDE under consideration. Therefore

$$0 \geq \int_t^{\tau_1} (Y_\theta^{t,x} - (u(\theta, X_\theta^{t,x}) - \frac{\varepsilon_0}{2}))dR_\theta^{t,x} = \frac{\varepsilon_0}{2}(R_{\tau_1}^{t,x} - R_t^{t,x}).$$

Thus $\tilde{\mathbb{E}}(R_{\tau_1}^{t,x} - R_t^{t,x}) = 0$ since the process R is increasing. Consequently

$$0 \geq \frac{\varepsilon_0}{2}\tilde{\mathbb{E}}(\tau_1 - t),$$

which exhibits a contradiction since $\tau_1 > t$.

Case 2: $\underline{x \in \partial\mathcal{O}}$. We again consider the case $u(t,x) > L(t,x)$ only. We now aim to prove that, either $-g(t,x) - (\nabla\varphi(t,x); \nabla n(x)) \leq 0$, or $G(t,x,u,\varphi) \leq 0$. Suppose that it is not true and set

$$\Gamma_2(t,x) := \min(u(t,x) - L(t,x); G(t,x,u,\varphi); -g(t,x) - (\nabla\varphi(t,x); \nabla n(x))).$$

There exists $\varepsilon_0 > 0$ s.t. $\Gamma_2(t,x) \geq \varepsilon_0$. We consider the stopping time $\tau_2 := \inf\{s \in [t,T];\ \Gamma_2(s, X_s^{t,x}) \leq \frac{\varepsilon_0}{2}\}$. We then proceed as in case 1 and get

$$0 \geq \frac{\varepsilon_0}{2}\tilde{\mathbb{E}}(\tau_2 - t) + \tilde{\mathbb{E}}(|\eta|_{\tau_2}^{t,x} - |\eta|_t^{t,x}),$$

which again exhibits a contradiction since $\tau_2 > t$ and η is an increasing process.

It thus remains to prove the inequality (29). An easy calculation shows

$$u(\tau, X_\tau^{t,x}) = u(t,x) - \int_t^\tau f(\theta, X_\theta^{t,x}, Y_\theta^{t,x}, Z_\theta^{t,x})d\theta + \int_t^\tau (Z_\theta^{t,x}; dW_\theta) - (R_\tau^{t,x}) - R_t^{t,x})$$
$$- \int_t^\tau g(\theta, X_\theta^{t,x})d|\eta|_\theta^{t,x}.$$

Observe

$$f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) = f(s, X_s^{t,x}, u(s, X_s^{t,x}), (\nabla\varphi)\sigma(s, X_s^{t,x})) + (\alpha_s; Z_s^{t,x} - (\nabla\varphi)\sigma(s, X_s^{t,x}))),$$

with

$$\alpha_s := \int_0^1 \nabla f(s, X_s^{t,x}, u(s, X_s^{t,x}), \mu Z_s^{t,x} + (1-\mu)(\nabla\varphi)\sigma(s, X_s^{t,x}))d\mu.$$

Now, apply Itô's formula to $\varphi(\tau, X_\tau^{t,x}) - \varphi(t, X_t^{t,x})$ and remember that (t,x) is a global maximum of $u - \varphi$ and $u(t,x) = \varphi(t,x)$. It comes:

$$0 \geq \int_t^\tau \left\{ -\frac{\partial\varphi}{\partial t}(\theta, X_\theta^{t,x}) - A_\theta\varphi(\theta, X_\theta^{t,x}) - f(\theta, X_\theta^{t,x}, u(\theta, X_\theta^{t,x}), (\nabla\varphi\sigma)(t, X_\theta^{t,x})) \right\} d\theta$$
$$- \int_t^\tau \left\{ g(\theta, X_\theta^{t,x}) + \left(\nabla\varphi(\theta, X_\theta^{t,x}); n(X_\theta^{t,x})\right)\right\} d|\eta|_\theta^{t,x} - (R_\tau^{t,x} - R_t^{t,x})$$
$$- \int_t^\tau (\alpha_\theta; Z_\theta^{t,x} - (\nabla\varphi\sigma)(\theta, X_\theta^{t,x}))d\theta + \int_t^\tau (Z_\theta^{t,x} - (\nabla\varphi\sigma)(\theta, X_\theta^{t,x}); dW_\theta).$$

We now define a new probability $\tilde{\mathbb{P}}$ by the Girsanov transformation such that

$$(-\int_t^s (\alpha_\theta; Z_\theta^{t,x} - (\nabla\varphi\sigma)(\theta, X_\theta^{t,x}))d\theta + \int_t^s (Z_\theta^{t,x} - (\nabla\varphi\sigma)(\theta, X_\theta^{t,x}); dW_\theta),\ t \leq s \leq T)$$

is a $\tilde{\mathbb{P}}$-Brownian motion. The conclusion follows. □

We now get the uniqueness of the viscosity solution to (6) by using elementary arguments (for general equations and more elaborated techniques, see Barles [4]). Classical arguments show that it suffices to prove the

Theorem 5.2. *Let $u(t,x)$ be a viscosity subsolution such that $u(t,x) \geq L(t,x)$, and let $v(t,x)$ be a viscosity supersolution to (6). Then $u(t,x) \leq v(t,x)$ for all (t,x) in $[0,T] \times \mathbb{R}^d$.*

Proof. Let $\bar{O}^\alpha := \{x \in \bar{O};\ d(x, \partial O) \geq \alpha\}$. In view of the lemma 14.16 in [15] there exists $\alpha_0 > 0$ such that $\bar{O}^\alpha \neq \emptyset$ for all $0 < \alpha \leq \alpha_0$. Since subsolutions and supersolutions are continuous and since u and v coincide at time T, we only need to prove: $u(t,x) \leq v(t,x)$ for all $(t,x) \in [0;T) \times \bar{O}^\alpha$ and $\alpha \in (0;\alpha_0]$. Set

$$w(t,x) := u(t,x) - v(t,x).$$

Suppose that $M := \max_{(t,x)\in[0,T)\times\bar{O}^\alpha}(w(t,x)\exp(2\gamma t)) > 0$. Let (t_0, x_0) be such that $w(t_0,x_0)e^{2\gamma t_0} = M$. On $[0;T) \times \bar{O}^\alpha \times \bar{O}^\alpha$ define the function

$$\psi_\epsilon(t,x,y) = u(t,x) - v(t,y) - \frac{|x-y|^2}{\epsilon^2} - w(t_0,x_0)e^{-2\gamma(t-t_0)}.$$

Choose a sequence $(t_\epsilon, x_\epsilon, y_\epsilon)$ such that

$$\psi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) = \max_{[0;T]\times\bar{O}^\alpha} \psi_\epsilon(t,x,y).$$

In view of the proposition 3.7 in [7] we have

1. $(t_\epsilon, x_\epsilon, y_\epsilon) \longrightarrow (\bar{t}, \bar{x}, \bar{x})$ when ϵ tends to 0, where

$$(\bar{t}, \bar{x}) \in \operatorname{argmax}(w(t,x) - w(t_0,x_0)e^{-2\gamma(t-t_0)};\ (t,x) \in [0;T] \times \bar{O}^\alpha).$$

2. $\dfrac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2}$ is bounded and tends to 0 when ϵ tends to 0,

3. $\max_{[0;T]\times\bar{O}^\alpha} \psi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \longrightarrow \max_{[0;T]\times\bar{O}^\alpha}(w(t,x) - w(t_0,x_0)e^{-2\gamma(t-t_0)})$.

From the theorem 8.3 in [7] we deduce that there exist (a,x,p) in the jet space $\bar{\mathcal{P}}^{2,+}(u(t_\epsilon,x_\epsilon) - w(t_0,x_0)e^{-2\gamma(t_\epsilon-t_0)})$ and (a,p,Y) in the jet space $\bar{\mathcal{P}}^{2,-}v(t_\epsilon,y_\epsilon)$ where $p := \dfrac{2(x_\epsilon - y_\epsilon)}{\epsilon^2}$ and

$$-\frac{6}{\epsilon^2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{6}{\epsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{30}$$

By definition of the jet spaces, we have

$$(a - 2\gamma w(t_0,x_0)e^{-2\gamma(t_\epsilon-t_0)}, x, p) \in \bar{\mathcal{P}}^{2,+}u(t_\epsilon, x_\epsilon).$$

In addition, as $u(t,x) \geq L(t,x)$ and $u(t_0,x_0) \neq L(t_0,x_0)$ we also have $u(t_\epsilon,x_\epsilon) > L(t_\epsilon,x_\epsilon)$. Classical properties of the jet spaces and standard calculations lead to

(31)

$$\begin{aligned}2\gamma w(t_0,x_0)e^{-2\gamma(t_\epsilon-t_0)} \leq & \frac{2}{\epsilon^2}(b(t_\epsilon,x_\epsilon)-b(t_\epsilon,y_\epsilon);x_\epsilon-y_\epsilon) \\ & + \frac{1}{2}\mathrm{Tr}\{\sigma\sigma^*X\} - \frac{1}{2}\mathrm{Tr}\{\sigma\sigma^*Y\} \\ & + f(t_\epsilon,x_\epsilon,u(t_\epsilon,x_\epsilon),\sigma(t_\epsilon,x_\epsilon)p) - f(t_\epsilon,y_\epsilon,v(t_\epsilon,y_\epsilon),\sigma(t_\epsilon,y_\epsilon)p).\end{aligned}$$

Notice that, from the definition of $(\bar{t},\bar{x})$ we have $w(\bar{t},\bar{x})-w(t_0,x_0)e^{-2\gamma(\bar{t}-t_0)} \geq 0$, from which

$$w(\bar{t},\bar{x})e^{2\gamma\bar{t}} \geq w(t_0,x_0)e^{2\gamma t_0},$$

and thus, from the definition of (t_0,x_0),

$$w(\bar{t},\bar{x})e^{2\gamma\bar{t}} = w(t_0,x_0)e^{2\gamma t_0} > 0. \tag{32}$$

Therefore $u(t_\epsilon,x_\epsilon) - v(t_\epsilon,y_\epsilon)$ is positive for all ϵ small enough. From our monotonicity condition (10) on f we deduce

$$f(t_\epsilon,y_\epsilon,u(t_\epsilon,x_\epsilon),\sigma(t_\epsilon,x_\epsilon)p)-f(t_\epsilon,y_\epsilon,v(t_\epsilon,y_\epsilon),\sigma(t_\epsilon,y_\epsilon)p) \leq \gamma(u(t_\epsilon,x_\epsilon-v(t_\epsilon,x_\epsilon)).$$

Next, multiply each term of the right inequality in (30) by the symmetric non negative definite matrix

$$\begin{pmatrix}\sigma(t_\epsilon,x_\epsilon)\sigma^*(t_\epsilon,x_\epsilon) & \sigma(t_\epsilon,x_\epsilon)\sigma^*(t_\epsilon,y_\epsilon)\\ \sigma(t_\epsilon,y_\epsilon)\sigma^*(t_\epsilon,x_\epsilon) & \sigma(t_\epsilon,y_\epsilon)\sigma^*(t_\epsilon,y_\epsilon)\end{pmatrix},$$

and compute the trace of the product matrix; we deduce

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t_\epsilon,x_\epsilon)X) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t_\epsilon,y_\epsilon)Y) \\ &\qquad = \frac{3}{\epsilon^2}(\sigma(t_\epsilon,x_\epsilon)-\sigma(t_\epsilon,y_\epsilon))(\sigma^*(t_\epsilon,x_\epsilon)-\sigma^*(t_\epsilon,y_\epsilon)).\end{aligned}$$

It then remains to use the Lipschitz property of b and σ to get

$$2\gamma w(t_0,x_0)e^{-2\gamma(t_\epsilon-t_0)} - \gamma(u(t_\epsilon,x_\epsilon)-v(t_\epsilon,y_\epsilon)) \leq 2\gamma\left(|x_\epsilon-y_\epsilon| + \frac{|x_\epsilon-y_\epsilon|^2}{\epsilon^2}\right).$$

Making ϵ tend to 0 we get

$$2\gamma w(t_0,x_0)e^{-2\gamma(\bar{t}-t_0)} - \gamma w(\bar{t},\bar{x}) \leq 0. \tag{33}$$

Thus, in view of (32), we have found a contradiction with our hypothesis on (t_0,x_0). □

6. Localization Error

In this section we apply the results of the preceding sections to estimate $\tilde{u}(t,x) - u(t,x)$, where $\tilde{u}$ and u are the respective solutions to (5) and (6). From (14) and Rademacher's theorem we know that $\tilde{u}$ is differentiable almost everywhere. We suppose that one can find a smooth boundary ∂O such that $\frac{\partial \tilde{u}}{\partial x}(t,\cdot)$ is a continuous function. Here we do not discuss the choice of such a boundary; however this is an important and difficult issue in practice.

We then have $v(t,x) := \tilde{u}_{|O}(t,x)$ is the unique viscosity solution of

(34)

$$\begin{cases} \min\left\{v(t,x) - L(t,x); -\frac{\partial v}{\partial t}(t,x) - A_t v(t,x) - f(t,x,v(t,x),(\nabla v\sigma)(t,x))\right\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [0,T) \times O, \\ v(T,x) = \phi(x),\ x \in \overline{O}, \\ (\nabla v(t,x); n(x)) = (\nabla \tilde{u}(t,x); n(x)),\ (t,x) \in [0,T) \times \partial O. \end{cases}$$

Proposition 6.1. *Under the conditions of the preceding section and under the above assumption on ∂O, there exists $C > 0$ such that*

(35)

$$|u(t,x) - \tilde{u}(t,x)| \le Ce^{CT}\left\{\mathbb{E}\max_{t\le s\le T}\left|g(s,X_s^{t,x}) - (\nabla\tilde{u}(s,X_s^{t,x}); n(X_s^{t,x}))\right|^4 \mathbb{I}_{[X_s^{t,x}\in\partial O]}\right\}^{1/4},$$
$$0 \le t \le T,\ x \in O.$$

Proof. In view of the results of our section 5 we have $u(t,x) = Y_t^{t,x}$, where Y is the solution to (26). As well, in view of (34), for all x in O we have $\tilde{u}(t,x) = \check{Y}_t^{t,x}$ where $\check{Y}$ is the solution to the same BSDE as (26) except that $-(\nabla\tilde{u}(t,x); n(x))$ is substituted to g. Set

$$\begin{aligned} \Delta Y_s^{t,x} &:= \tilde{Y}_s - \check{Y}_s^{t,x}, \\ \Delta Z_s^{t,x} &:= \tilde{Z}_s - \check{Z}_s^{t,x}, \\ \Delta R_s^{t,x} &:= \tilde{R}_s - \check{R}_s^{t,x}, \\ \Delta G_s &:= \{g(s,X_s^{t,x}) - (\nabla\tilde{u}(s,X_s^{t,x}); n(X_s^{t,x}))\}\mathbb{I}_{[X_s^{t,x}\in O]}. \end{aligned}$$

Choose $\mu > 0$ and define $F(s,y,\beta) := e^{-2\gamma s}e^{\mu\beta}|y|^2$. Apply Itô's formula to $F(s,\Delta Y_s^{t,x},|\eta|_s^{t,x})$ from $s = t$ to $s = T$. Observe that, in view of the proposition 4.3 and the inequality (23), all the resulting integrals have a finite second moment. Then use (10) and the following obvious inequalities:

$$2\Delta Y_s^{t,x}(g(s,X_s^{t,x}) - (\nabla\tilde{u}(s,X_s^{t,x}); n(X_s^{t,x}))) - \mu(\Delta Y_s^{t,x})^2 \le \frac{1}{\mu}|(\nabla\tilde{u}(s,X_s^{t,x}); n(s,X_s^{t,x}))|^2,$$

$$\Delta Y_s^{t,x}\Delta R_s^{t,x} = (L(s,X_s^{t,x}) - \check{Y}_s^{t,x})d\tilde{R}_s - (\tilde{Y}_s - L(s,X_s^{t,x}))d\check{R}_s^{t,x} \leq 0.$$

It finally comes

$$|\Delta Y_t^{t,x}|^2 \leq \frac{2e^{2\max(0,-\gamma)T}}{\mu}\mathbb{E}\int_t^T (\Delta G_s)^2 e^{\mu|\eta|_s^{t,x}} d|\eta|_s^{t,x}.$$

As $(|\eta_s^{t,x}|)$ is an increasing process, it remains to use the Cauchy–Schwarz inequality and (23). □

In a forthcoming work we will study means of choosing g such as the right hand side of (35) is so small as possible. We will also handle the case of misspecified Dirichlet conditions.

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