Chapter 1

Introduction

1.1 What is Electrodynamics?

Electrodynamics as a theory deals with electric charges which move in space and with electromagnetic fields that are produced by the charges and again interact with charges. Electrodynamics is described by Maxwell's equations whose most important consequences are: (a) the electromagnetic nature of light (Maxwell), (b) the emission of electromagnetic waves by an oscillating dipole (H. Hertz), and (c) the unification of electric and magnetic forces. Unlike in Maxwell's days one does not refer to an aether anymore — it was Einstein who concluded that this does not exist — and instead uses the concept of fields in space. Einstein's Special Theory of Relativity unifies electrodynamics with classical mechanics.

1.2 Presentation of Macroscopic Electrodynamics

In general a first course in electrodynamics, or electricity and magnetism as it is also called, is preceded by a course in classical mechanics which even today is not always combined with the Special Theory of Relativity (as would be desirable). In addition the concept of a field in space and the distinction between directly observable field quantities and at best indirectly observable field potentials is at that stage still too vague to permit an immediate relativistic field theoretic approach to appear plausible. Moreover, in general the term electrodynamics is usually restricted to macroscopically observable phenomena, so that a quantised treatment with operator-valued fields is beyond its scope, the latter being dealt with in quantum electrodynamics. Thus the classical fields are c-numbers. As a consequence of this restriction, and also in order to establish classical electrodynamics as a theory in its own right, the fundamental equations of this theory, Maxwell's equations, are usually not derived from Hamilton's principle as a prior course in methods of classical mechanics might suggest. In addition the law of force between moving charges is considerably more complicated than that between pointlike masses so that an analogous procedure is not immediately advisable. Nonetheless Hamilton's variational principle is of such basic significance that it permits, of course, the derivation of Maxwell's equations as the Euler–Lagrange equations (and their consequences) of an appropriate variational principle. Thus here we do not adopt this procedure immediately (i.e. till Chapter 18). What are the other methods which suggest themselves? Perusing the literature, one observes two main procedures. The approach with emphasis on logic starts from an axiomatic presentation of Maxwell's equations, whereas the other more historical and phenomenological approach abstracts these from observations. A textbook which adopts the former procedure is that of Sommerfeld [1] who follows H. Hertz in this respect, but whereas H. Hertz starts from the differential form of Maxwell's equations, Sommerfeld chooses the vector integral form. Another text that follows this procedure is, for instance, that of Lim [2]. Books which choose the second procedure are those of Jackson [3] and Greiner [4].

In the axiomatic formulation of classical or macroscopic electrodynamics the *two principal axioms* are Faraday's law of induction and Ampère's flux theorem. In integral form and in units of the internationally agreed system of units (ISU), these are

$$\frac{d}{dt}\int_F \mathbf{B}\cdot d\mathbf{F} = -\oint_{C(F)} \mathbf{E}\cdot d\mathbf{l} \tag{1.1}$$

and

$$\frac{d}{dt}\int_F \mathbf{D}\cdot d\mathbf{F} + \int_F \mathbf{j}\cdot d\mathbf{F} = \oint_{C(F)} \mathbf{H}\cdot d\mathbf{l}, \tag{1.2}$$

where for a given fixed surface of integration F,

$$\frac{d}{dt}\int_F \mathbf{D}\cdot d\mathbf{F} = \int_F \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{F},$$

since (by assumption) F does not change with t.*

*Recall, for instance the analogy, that as shown in Whittaker and Watson [7], p. 67,

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}dx,$$

which shows the α-dependence of boundary values.

In the first or Faraday's law, $\mathbf{B}$ is the magnetic induction and $\mathbf{E}$ the electric field strength. The law says that every change of the magnetic flux[†] with time through an open, double-sided area F (a plane is single-sided) with boundary given by the closed curve $C(F)$ — this flux being the number of lines of force in Faraday's considerations — generates an equal but oppositely directed circuit potential, the *electromotive force* $\oint \mathbf{E} \cdot d\mathbf{l}$, along the boundary $C(F)$. An open surface can, for instance, be visualised as a container without a lid, and a closed surface as one closed with the lid. In the former case the rim of the opening corresponds to the curve $C(F)$. The word *double-sided* implies that the surface possesses direction normals directed towards inside or outside regions. A closed surface has no boundary, but we can imagine on it a closed curve $C(F)$ which divides the surface into two regions.

In the second or Ampère's law, $\mathbf{D}$ is the so-called dielectric displacement ($\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \epsilon_0$ the dielectric constant of vacuum, $\mathbf{P}$ the polarisation vector or dipole moment per unit volume of the medium), $\mathbf{j}$ is the density of the electron current and $\mathbf{H}$ the magnetic field strength. The law says that analogous to Faraday's law the time change of an electric flux through an area F is equal to a magnetic circuit potential in the boundary curve $C(F)$ in the same direction. The expression $\partial \mathbf{D}/\partial t$ obviously has the dimension of a current density. The expression

$$\frac{d}{dt}\int_F \mathbf{D} \cdot d\mathbf{F}$$

is called *Maxwell's displacement current.*

The two principal axioms are supplemented by two *subsidiary axioms* which are consequences of the principal axioms with implementation of some empirical findings. We arrive at these by considering a boundary curve $C(F)$ enclosing the area F, and by permitting $C(F)$ to shrink to zero, so that the line integrals vanish, i.e.

$$\lim_{C(F)\to 0} \frac{d}{dt}\int_F \mathbf{B} \cdot d\mathbf{F} = 0, \tag{1.3}$$

$$\lim_{C(F)\to 0} \left\{ \frac{d}{dt}\int_F \mathbf{D} \cdot d\mathbf{F} + \int_F \mathbf{j} \cdot d\mathbf{F} \right\} = 0. \tag{1.4}$$

The area F thus loses its boundary and becomes closed. In the case of magnetic dipoles (irrespectively of whether one considers magnets or current loops) every line of force leaving the area F is accompanied by a corresponding line of force toward it. The integral extended over the entire closed

[†]Flux is in general the scalar product of a vector field with an area.

surface thus yields the value zero, i.e.

$$\int_{F_{\text{closed}}} \mathbf{B} \cdot d\mathbf{F} = 0. \tag{1.5}$$

This is a general result in view of the empirical fact that no isolated magnetic poles exist and hence no region with only ingoing or only outgoing lines of force. With the definition of the divergence

$$div\mathbf{A}(\mathbf{r}) = \lim_{V \to 0} \frac{1}{V} \int_{F(V)} \mathbf{A} \cdot d\mathbf{F} \tag{1.6}$$

we obtain the differential form of Eq. (1.5), i.e.

$$div\mathbf{B}(\mathbf{r}) \equiv \boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r}) = 0. \tag{1.7}$$

One says: $\mathbf{B}(\mathbf{r})$ is *divergenceless.*

Equation (1.4) permits an analogous consideration but with different consequences. Equation (1.2), or rather (1.4), applies to cases in which the electron current

$$\int_{F(V)} \mathbf{j} \cdot d\mathbf{F}$$

leaves the surface F enclosing the volume $V(F)$. This implies that the volume with total charge q loses charge at the rate

$$\int_{F(V)} \mathbf{j} \cdot d\mathbf{F} = -\frac{dq}{dt}. \tag{1.8}$$

The minus sign indicates that $V(F)$ is losing charge. Put differently, as

$$\frac{dq}{dt} + \int_{F(V)} \mathbf{j} \cdot d\mathbf{F} = 0, \tag{1.9}$$

the equation says that the amount of charge in $V(F)$ remains constant (this is called charge conservation). Here one has to be careful, because if we write

$$\frac{dq}{dt} = + \int_F \mathbf{j} \cdot d\mathbf{F},$$

then q represents the amount of charge which passes through the area F per second *with no reference to* V. We can now write Eq. (1.4) as

$$\lim_{C(F) \to 0} \left\{ \frac{d}{dt} \int_F \mathbf{D} \cdot d\mathbf{F} - \frac{dq}{dt} \right\} = 0, \tag{1.10}$$

which implies

$$\int_{F_{\text{closed}}} \mathbf{D} \cdot d\mathbf{F} = q. \tag{1.11}$$

We could have added a constant of integration on the right side. But for charge $q = 0$ the field $\mathbf{D} = 0$, and thus this constant must be zero. The differential form of Eq. (1.11) again follows with (1.6); i.e.

$$\nabla \cdot \mathbf{D} = \rho, \quad \rho = \frac{q}{V}. \tag{1.12}$$

Here ρ is the charge density. Equations (1.11) and (1.12) are known as the Gauss law of electrodynamics. We can now write Eq. (1.8):

$$\frac{d}{dt}\int_V \rho dV = -\int_{F(V)} \mathbf{j} \cdot d\mathbf{F}. \tag{1.13}$$

Going to an infinitesimal volume and using the Gauss divergence theorem, this implies

$$\int_V \nabla \cdot \mathbf{j} dV = \int_{F(V)} \mathbf{j} \cdot d\mathbf{F} \tag{1.14}$$

and hence the *equation of continuity*

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot j = 0. \tag{1.15}$$

A method of deriving electrodynamics essentially from this equation and the Lorentz force (the force exerted on a charge in an electromagnetic field) — as an alternative approach to electrodynamics — can be found in a paper of Bopp.[‡]

Returning to comments at the beginning we mention that attempts have repeatedly been made to derive Maxwell's equations from the Coulomb law and the Special Theory of Relativity. Since it is known that the General Theory of Relativity contains the generalisation of Newton's law of gravitation, i.e. a law which has the same form as the Coulomb law for charges, such an attempt cannot succeed without additional assumptions. A discussion of this topic can be found in the book of Jackson [3]. An entirely different approach which sees the only logical foundation of Maxwell's theory in a relativistic treatment of radiation theory is the book of Page and Adams [5]. In the following we adopt the historical and phenomenological approach starting with electrostatics, since this seems to be more suitable for an understanding of the physics of electrodynamics, particularly in a first course on the subject.

[‡]F. Bopp, *Z. Physik* **169** (1962) 45.

1.3 On the Choice of Units

In textbooks on electrodynamics nowadays two systems of units are in use: The more common internationally used MKSA-system of units with meter (m), kilogramme (kg), second (s) and ampere (A), and the still frequently used system of Gaussian units which is based on the c.g.s. units, i.e. centimeter (c), gramme (g) and second (s) (the current is then given in statampere, 1 A $= 3\times 10^9$ statampere). Either system has its advantages — the former has been agreed upon internationally and is therefore used particularly in applications, and the latter, the Gaussian system, is somewhat theory-oriented and therefore frequently used in considerations of singly charged particles. Here we employ the first system but will refer occasionally also to the other system. We add the following comments.

In discussions on units the word "dimension" is frequently referred to. The dimension of a physical quantity is nothing absolute. It is possible to choose units (like the so-called *natural units*) in which Planck's action quantum and the velocity of light in vacuum have the value 1 and are dimensionless. Since velocity= distance/time, one can then express lengths in units of time, i.e. seconds, or more commonly time intervals in meters. This arbitrariness was already pointed out by Planck. Some authors even amuse themselves today about dimensional considerations.[§] The example shows that also the number of fundamental units is a matter of choice (in "natural units" every quantity can be expressed in a power of length). In earlier days the units used in the literature on electrodynamics were the so-called electrostatic units (e.s.u.) and electromagnetic units (e.m.u.) which today are indirectly contained in the Gaussian system. In essence, in the Gaussian system of units electric quantities are expressed in e.s.u. and magnetic quantities in e.m.u. while the change to Gaussian units requires a factor c or $1/c$ (c the velocity of light in vacuum), which can be determined experimentally (cf. Appendix B). For instance

$$\mathbf{B}_{\text{e.s.u.}} = \frac{1}{c}\mathbf{B}_{\text{e.m.u.}}.$$

In particular one has in these units for the dielectric constant of the vacuum ϵ_0 and for the magnetic permeability of the vacuum μ_0

$$\epsilon_0(\text{e.s.u.}) = 1, \quad \epsilon_0(\text{e.m.u.}) = \frac{1}{c^2}, \quad \mu_0(\text{e.s.u.}) = \frac{1}{c^2}, \quad \mu_0(\text{e.m.u.}) = 1.$$

[§] See e.g. A. O'Rahilly, Vol. I [6], p. 65, where the author says: "*Maxwell invented a second system... But this system is never employed, it merely occurs in those pages of textbooks which profess to deal with something called 'dimensions'.*" See also p. 68.

We do not enter into a deeper discussion of these units here (sometimes it is not clear whether a quantity is electric or magnetic).

The choice of a system of units in electrodynamics depends on the choice of the magnitude as well as the dimension of two arbitrary constants, as an exhaustive investigation of Maxwell's equations shows.[¶] The appearance of one constant k can immediately be seen by looking at the Coulomb law in electrostatics, which determines the force $\mathbf{F}_{12}$ between two point charges q_1, q_2, separated by a distance $\mathbf{r}$, i.e.

$$\mathbf{F}_{12} = kq_1 q_2 \frac{\mathbf{r}}{r^3}.$$

For length, mass and time, we choose here as agreed upon internationally the ISU (international system of units, internationally abbreviated SI, see Appendix B) and thus the units meter (m), kilogramme (kg), second (s) (correspondingly in the Gaussian system centimeter (cm), gramme (g), second (s)). Depending on the choice of dimension and magnitude of k we obtain different units for the charge q. The electric field strength $\mathbf{E}(\mathbf{r})$ at the distance $\mathbf{r}$ away from the charge q is

$$\mathbf{E}(\mathbf{r}) = k\frac{q}{r^3}\mathbf{r}.$$

Thus the field $\mathbf{E}(\mathbf{r})$ can be defined as force per unit charge.

In the consideration of magnetic phenomena we have to deal with currents. Currents I are defined with respect to charges q, $I = dq/dt$. The connection between electric and magnetic phenomena or between $\mathbf{E}$ and $\mathbf{B}$ introduces another constant for the choice of units of $\mathbf{B}$. This constant can be introduced in a number of ways. With the help of the theorem of Stokes (see later), Eq. (1.1) can be written as

$$\nabla \times \mathbf{E} + k^* \frac{\partial \mathbf{B}}{\partial t} = 0,$$

where k^* is chosen to be 1 and dimensionless in the MKSA-system (the constants k and k^* can be chosen arbitrarily in magnitude and dimension).

In accordance with the ISU, i.e. the MKSA-system of units, the unit of current is taken as the *ampere* (the spelling in English being this, ampere, which is not part of SI). One ampere is defined as that amount of current which runs through two parallel, straight, infinitely long, thin conductors which are one meter apart when the force between these is 2×10^{-7} newton/m (per meter in length) (the expression for this attractive force will be derived later). Since current $\times$ time is charge, we obtain the definition of the unit of charge:

[¶]See Jackson [3], p. 811.

$$1 \text{ coulomb (C)} = 1 \text{ ampere-second (A s)}.$$

It remains to determine the dimensions of $\mathbf{D}$ and $\mathbf{H}$. For a large number of different materials the following "*matter*" or "*material equations*" are valid:

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}.$$

ϵ is the dielectric constant of the material and μ its magnetic permeability. Both have magnitudes and dimensions depending on the chosen system of units. The vacuum values ϵ_0, μ_0 obey, as we shall see, the important relation

$$\epsilon_0 \mu_0 = \text{const.}$$

In the Gaussian system of units ϵ_0, μ_0 are taken as 1 and dimensionless. The constant is then 1. In the MKSA-system the constant is found to be $1/c^2$, where c is the velocity of light in vacuum, i.e.

$$c = 2.998 \times 10^8 \text{ m/s}.$$

In this case

$$\epsilon_0 = \frac{10^7}{4\pi c^2} \text{ farad/meter}$$

with the dimension of current^2 time^4 mass^{-1} length^{-3}, or $\epsilon_0 = 8.854 \times 10^{-12}$ A s/(V m) or C^2/(joule-meter), where 1 volt (V) = 1 A^{-1} m^2 kg s^{-2} and 1 joule = 1 newton-meter (N m). In addition

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{newton-ampere}^{-2} \quad (\text{N A}^{-2}),$$

or $\mu_0 = 1.257 \times 10^{-6}$ with dimension of mass length current^{-2} time^{-2}.

In the *Gaussian system* the constants k, k^* are chosen as follows:

$$k = 1 \quad \text{(dimensionless)}, \quad k^* = \frac{1}{c} \quad \text{(dimension time/length)}.$$

In the MKSA-system the vacuum constants k, k^* are chosen as

$$k = \frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$$

with dimension kg m^3 s^{-4} A^{-2} and

$$k^* = 1$$

(dimensionless). We see that in comparison with the Gaussian system the MKSA-system requires the use of ϵ_0, μ_0.