

# Chapter 1

## Overview

This book is a text on a subject – exactly solved quantum many-body problems – that is usually considered to be ‘difficult’. This subject belongs within the realm of mathematical physics – too mathematical to be ‘respectable’ physics, yet not rigorous enough to be ‘real’ mathematics. However, over a period of more than seventy years, there has been much success in understanding the detailed ‘how and why’ of such models. The results are quite interesting for their own sake, and so there are perennial attempts to translate this body of work into either respectable physics or real mathematics; this is not that sort of book. Instead, this book attempts to discuss the models and the solutions in their own ‘intrinsic’ language. But before we begin this program, in this chapter I would first like to try and sketch the motivation and strategy with a minimum of equations. So this first chapter is offered to encourage a general audience to read on; it can certainly be skipped by the experts.

### 1.1 Orientation

Let us begin with an informal description of the type of physical problems we will be trying to capture with our exactly solved models. The proper language to discuss these problems is the language of statistical mechanics; let us agree on terms. We start with a *closed*, insulated container filled with a fixed amount of chemically pure fluid – say water, for example. We might see something like the top picture of Fig. 1.1. We wait until nothing appears to be changing within the container – the fluid is in *equilibrium*. We know from experience that the fluid will then be uniform and have a unique temperature and pressure – it is in a *thermodynamic state*. (Actually, we show the fluid with a slight density and pressure gradient due to an external gravitational field. Thus, it is really temperature and chemical potential that are uniform,

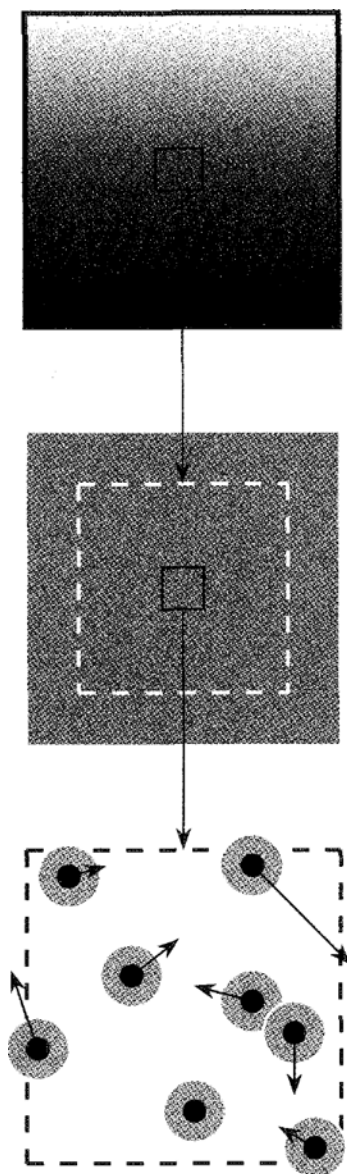


Fig. 1.1 Schematic pictures of an equilibrium fluid, seen on three different scales.

but chemical potential is less familiar than pressure.) Thus, since total mass  $M$  and energy  $E$  are *conserved*, and contained within the closed container of volume  $V$ , these quantities determine the state of the fluid,

and hence the temperature  $T$  and pressure  $P$  of the fluid. However, the set of quantities  $M, E, V$  are *extensive*: If we double them by placing two identical containers of fluid together, and remove an internal partition to give a container twice as big, we then have the same state, and thus the same  $T$  and  $P$  – which are *intensive*. Thus, the state of the system is really determined by two intensive densities – say  $E/V$  and  $M/V$  – rather than three extensive quantities.

There are several idealizations in this scenario: Wait ‘long enough’, stay away from the walls of a ‘big’ container, no ‘gravity’, *etc.* However, these are really experimental idealizations, as well as theoretical, in that they are systematic effects that must be minimized – or better, controlled – in order that ‘different experiments can be expected to agree. Most important among these idealizations is the so-called *thermodynamic limit* –  $M, E, V \rightarrow \infty$  with  $M/V \equiv \rho_M$  and  $E/V \equiv u$  fixed. We see that this is the ‘thermodynamic’ limit, not because temperature is directly involved, but rather because it is the limit we must approach in order for the system to have a ‘thermodynamics’ at all. (We shall even refer to *zero-temperature thermodynamics*, meaning ground state properties in the thermodynamic limit.)

However, by taking the thermodynamic limit  $M, E, V \rightarrow \infty$ , we are really dealing with an *open* system, since the walls of the container have been pushed to infinity. Within this infinite system, let us delineate a finite part of the whole fluid by an ‘imaginary’ volume  $V$ , as in the middle picture of Fig. 1.1. The rest of the system acts as a *reservoir*, holding the temperature and pressure at the fixed values  $T, P$ . Since the ‘walls’ of this open sub-system  $V$  are imaginary and hence porous, the mass and energy  $M, E$  within are no longer constant, but instead fluctuate. However, the averages are fixed by  $\langle M \rangle / V = \rho_M$  and  $\langle E \rangle / V = u$ . And for a thermodynamic system, the fluctuations such as

$$\delta E^2 \equiv \langle (E - \langle E \rangle)^2 \rangle \quad (1)$$

are typically extensive, so

$$\delta E / \langle E \rangle \rightarrow 0, \text{ as } 1/\sqrt{V} \quad (2)$$

in the thermodynamic limit. In this way we move from the *microcanonical* to the *grand canonical ensemble*. (We assume that  $V$  is so small that we can no longer detect the gradients due to gravity.)

We know that in reality a chemically pure fluid is not an infinitely divisible continuum, but instead is made up of many molecules – or ‘atoms’ – each of mass  $m$ . Further, these atoms are in motion, so when the fluid in an equilibrium state appears to be unchanging and hence ‘at rest’, in fact the atoms are really in a rapidly fluctuating *steady-state*. Let the number of such atoms within the volume  $V$  – either real or imaginary – be  $N$ , a large number typically of the order of  $N \approx 10^{24}$ . The number density  $\langle N \rangle / V = d$  is now more appropriate on this atomic scale, and since  $M = mN$ , then  $\rho_M = md$ . This fluid only appears to be a ‘smooth’ continuum – what is usually meant by a ‘fluid’ – because of the *coarse-graining* implied by the averaging with  $\langle N \rangle, \langle E \rangle, V \rightarrow \infty$ . This coarse-graining is simply the thermodynamic limit for an open system, and insures that the fluctuations of  $N$  and  $E$  are small, so that  $N/V \rightarrow d$  and  $E/V \rightarrow u$ .

Finally, the complete thermodynamics of the fluid is given when we know the *chemical potential*  $\mu(T, P) \equiv (E - TS + PV) / N$  as a function of temperature  $T$  and pressure  $P$ , through the standard relation  $Nd\mu = VdP - SdT$ . Here  $S$  is the entropy. We could equally well use  $P(T, \mu)$ . Derivatives give the *equations of state*.

Let us now move down to the atomic scale, as in the bottom picture of Fig. 1.1, where we will actually calculate the thermodynamic properties. We show the atoms schematically as small hard spheres of mass  $m$ , surrounded by their potential  $v(r)$ , and ‘jiggling about’ in a random thermal agitation. In fact, this is a classical picture, and particles of atomic mass really obey quantum mechanics. Thus, the ‘jiggling’ comes about by a combination of thermal agitation, and zero-point motion. This zero-point agitation is a purely quantum phenomenon which arises because of the uncertainty relation. A particle is localized, through crowding by the other particles, into a volume of size  $V/N = 1/d \equiv \lambda^3$ . However, the uncertainty relation says that we cannot simultaneously know both the position and the momentum of a particle. In fact, to know the location of the particle to within a volume  $\lambda^3$  implies that the momentum cannot be zero, but instead is of the order

$p = mv \approx \hbar / \lambda$ , with kinetic energy  $E_0 \approx \hbar^2 / m\lambda^2$ . Hence, we have zero-point motion, and in the case of a light atom such as helium, a 'permanent' fluid.

The particles within this volume  $V$  have a distribution of velocities or momenta, expressed by the *momentum distribution function*  $n(p)$  where  $Nn(p)4\pi p^2 dp$  gives the number of particles with momentum  $\vec{p}$  within the shell  $p < |\vec{p}| < p + dp$ . Similarly, we can define distribution functions for the location of the particles. The density of particles is uniform on the average. However the potential produces correlations between particles. Such correlations are expressed by the *pair correlation function*  $g(r)$ , defined so that  $g(r)4\pi r^2 dr$  is the average number of particles within a shell  $r < |\vec{r}| < r + dr$  centered on any given particle. Both of these distribution functions are intensive thermodynamic quantities, and so they fluctuate very little in a large system; we write the parametric dependence on the thermodynamic state as  $n(p|T, \mu)$  and  $g(r|T, \mu)$ .

A system of particles – the fluid – is defined when we give the *Hamiltonian*, which is simply the expression for the energy in terms of the positions and momenta of the particles. For our simple system of 'atoms', the energy is the sum of the kinetic energy of motion and the potential energy of interaction between pairs of particles, or as an equation,

$$H = \frac{1}{2m} \sum_{j=1}^N |\vec{p}_j|^2 + \sum_{j>i=1}^N v(|\vec{x}_j - \vec{x}_i|). \quad (3)$$

For a classical system,  $\vec{p} = m\vec{v} = m d\vec{x} / dt$ , while for a quantum system  $\vec{p} = -i\hbar \vec{\nabla}$  is an operator. (We have deliberately left it ambiguous whether this is to be a closed or an open system.) The 'value' of  $H$  is the energy  $E$ , and will be given by

$$E(T, \mu) / N = \frac{1}{2m} \int_0^\infty p^2 n(p|T, \mu) 4\pi p^2 dp + \frac{1}{2} \int_0^\infty v(r) g(r|T, \mu) 4\pi r^2 dr. \quad (4)$$

Statistical mechanics tells us how to calculate all thermodynamic quantities, given the Hamiltonian for the system. For instance, Gibbs says we find the pressure by calculating the grand partition function

$$\begin{aligned}
 Z(T, \mu) &= e^{VP(T, \mu)/T} = \sum_{N=0}^{\infty} \sum_{\text{quantum states}} e^{-E/T + \mu N/T} \\
 &- \sum_{N=0}^{\infty} \sum_{E \geq E_0} e^{S(E) - E/T + \mu N/T} \rightarrow \sum_{N=0}^{\infty} e^{\mu N/T} \left[ \iint \frac{dx dp}{\hbar} \right]^{3N} e^{-H/T}.
 \end{aligned} \tag{5}$$

The last expression is the *classical limit*, when  $\hbar \rightarrow 0$ . Using this expression, we can evaluate the fluctuations in an open system. For instance, we verify our assertion for the fluctuation of  $N$ , by calculating

$$\delta N^2 / N \equiv \langle (N - \langle N \rangle)^2 \rangle / \langle N \rangle = \left. \frac{T}{d} \frac{\partial^2 P}{\partial \mu^2} \right|_T = T d \chi, \tag{6}$$

where  $\chi$  is the *isothermal compressibility*. Thus,  $\delta N / N \rightarrow 0$  for a large system. (Throughout, we have taken energy as our unit of temperature; Boltzmann's constant  $k_B$  converts to degrees Kelvin.)

## 1.2 An experiment – ballistic expansion

We now want to describe an idealized experiment – *ballistic expansion* – which has proven to be quite useful in practice for determining the properties of cold, dilute atomic gases. We begin as before – with a fluid of  $N$  particles, having energy  $E$ , confined and isolated within a volume  $V$  – which has come to equilibrium. The system is large, and this equilibrium state is characterized by its temperature  $T$  and chemical potential  $\mu$ . Now, at a given instant in time, which we take to be  $t=0$ , we remove the walls of the container and allow the fluid to freely expand into a vacuum – a ballistic expansion. (We assume the potential to be repulsive, so the fluid does not bind with zero pressure.) This is not a *quasistatic process*, and the system does not come to equilibrium. However, energy is conserved. We show such a ballistic expansion experiment schematically in Fig. 1.2.

After a time, the particles are so far apart that they no longer interact by the pair potential, and in addition, the faster particles have overtaken the slower particles so they will never again interact. Each particle then moves with its constant *asymptotic momentum*  $k = mv = mr / t$ . We then see a constantly and uniformly expanding cloud with (number) density  $d(r, t) \rightarrow \rho(mr / t)m^3 / t^3$ , as  $t \rightarrow +\infty$ . Thus, by

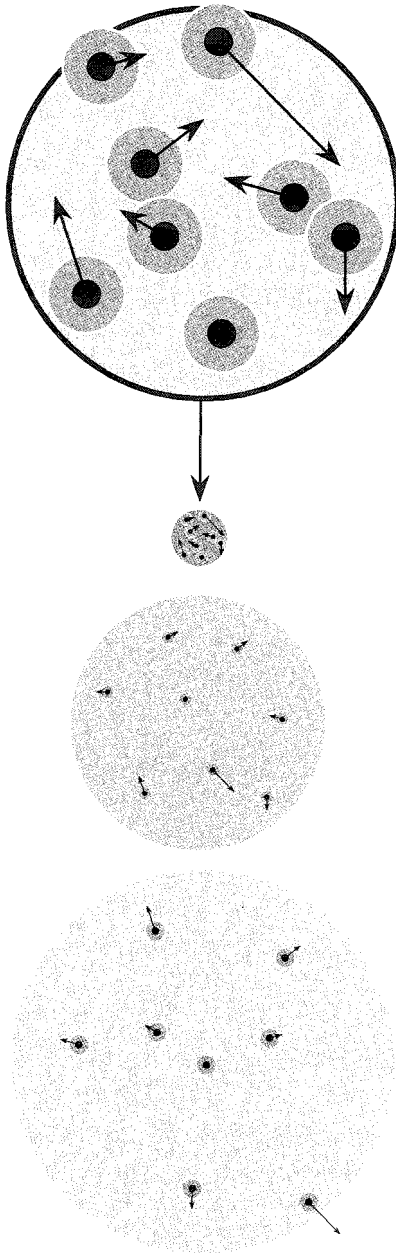


Fig. 1.2 Schematic picture of a ballistic expansion experiment. At top is the trapped system in equilibrium, enlarged. Below are successive pictures of the expansion.

*time-of-flight* measurement, we can extract the distribution of the asymptotic momenta  $\rho(k)$ , by

$$\rho(k) \equiv \frac{1}{m^3} \lim_{t \rightarrow \infty} t^3 d(kt/m, t). \quad (7)$$

When we repeat the experiment, we get the identical result, and so this distribution of the asymptotic momentum  $\rho(k|T, \mu)$  is itself a thermodynamic quantity depending only upon the thermodynamic state  $T, \mu$ . Since the energy is conserved, and the particles are so far apart that the potential energy is negligible, then the energy is now all kinetic energy, and we have an expression for the equilibrium energy

$$E(T, \mu)/N = \frac{1}{2m} \int_0^{\infty} k^2 \rho(k|T, \mu) 4\pi k^2 dk. \quad (8)$$

This is an alternate expression to Eq. (4).

Whether we actually do the experiment or not, this distribution of asymptotic momenta is still an important theoretical quantity. It allows us to calculate the thermodynamics of the system through Eq. (8). It is a quantity that is emphasized throughout this book, for reasons I hope to make clear in this overview.

### 1.3 One dimension *versus* two or three dimensions

In the previous two sections we have been slightly misleading, since we talk about three dimensions, draw two dimensions, yet this book is essentially about one dimension. Nothing we have said has been wrong. It is just that in two or three dimensions we cannot really calculate much. In this section we explain how and why one dimension is different.

It is useful to have a picture at hand. Consider Fig. 1.3 where we show a one-dimensional system – beads on a string, and a two-dimensional system – marbles on a table top. What is the difference?

First of all, in one dimension a particle has fewer neighbors than in two or more dimensions, since there is less space in which the neighbors can live. (Houses are spread along rural roads, arranged in blocks in small towns, but are stacked in apartments in crowded cities.) However, a particle in one dimension interacts much more strongly with its

two nearest-neighbors. In fact, a particle in one dimension cannot ‘go-around’ its neighbor, but instead must pass through, either by overcoming any potential barrier, or by quantum tunneling.

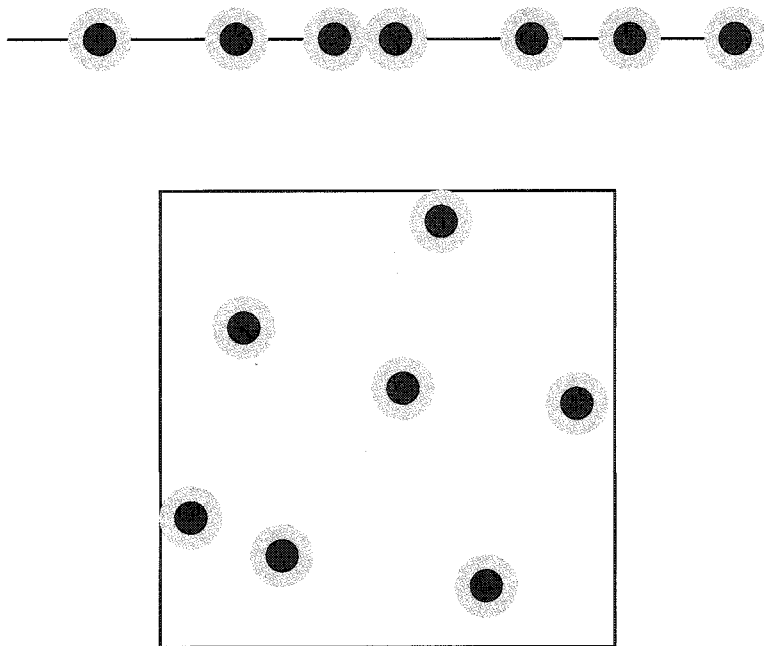


Fig. 1.3 Particles in one dimension (top), and in two dimensions (below).

Secondly, and for a similar reason, it is harder to maintain order in one dimension than in two or more. Consider a ferromagnet in one and two dimensions, as shown in Fig. 1.4.



Fig. 1.4 A ferromagnet with a single defect.

In one dimension, we destroy the order (all spins up) by a single mismatch, which only costs an energy  $\Delta E = +2J$ . However, this ‘defect’ can be at any one of  $N$  places, giving an entropy of  $\Delta S = \log N$ . Thus, the change of free energy when we create a defect is

$\Delta F = \Delta E - T\Delta S = 2J - T \log N < 0$  for a large system with  $T \neq 0$ . We conclude that in one dimension ordered phases only exist at zero temperature, and so if we want to study phase transitions in one dimension, we need look only in the ground state, where we can vary some parameter such as interaction strength. (Repeating the above reasoning in two dimensions gives Peierls' proof of ferromagnetism for the two-dimensional Ising model.)

Finally, let us examine *transport properties* in one dimension. Transport of conserved 'stuff' – mass, particle number, energy, momentum, charge – allows the system to relax to equilibrium. This equilibrium is established in stages. First, within a very short 'collision' time, local equilibrium is established with a local temperature. Second, this temperature gradient causes energy to flow from regions of higher to lower temperature. This transport of conserved energy is much slower.

Let us make a very elementary calculation. A temperature gradient  $T(x)$  leads to a gradient of the energy density  $E/V = u(x)$ , given as  $\partial u / \partial x = c \partial T / \partial x$ , where  $c$  is the specific heat. A small gradient leads to a small flux of heat  $J_E$ , and so we expect a transport relation of the form  $\bar{J}_E = -K_T \bar{\nabla} T$ , defining the thermal conductivity  $K_T$ . The kinetic picture for this is that an atom carries a 'memory' only of the last collision. Thus, looking at the net flux through a small area, we see particles crossing from each side with an average velocity  $v$ . But those crossing from the 'left' had their last collision at a different place – and so at different temperature – than those crossing from the right. Let  $\ell$  be the *mean-free-path* – the average distance between collisions. Thus, we argue that the net flux of energy should be given by

$$J_E \approx v[u(x) - u(x + \ell)] \approx -v\ell \frac{\partial u}{\partial x} = -v\ell c \frac{\partial T}{\partial x}, \quad (9)$$

and so  $K_T \approx v\ell c$ . The specific heat is familiar, the average velocity is related to the average kinetic energy, and we now estimate the mean-free-path  $\ell$ . Imagine the region of interaction about a particle as a sphere, which casts a shadow of area  $\sigma$ . Then moving a distance  $L$ , a particle sweeps out a volume  $\sigma L$ , colliding with the  $d\sigma L$  particles within this volume, so  $\ell \approx 1/\sigma d$ . (See Fig. 1.5.) In one dimension, how-

ever, two particles cannot miss, so in this case  $\ell \approx 1/d$ . The time to attain local equilibrium is the *collision time*  $\tau = \ell/v$ .

Everything here is pretty much as you would find it in an elementary thermal physics book. In two dimensions, one cannot determine the

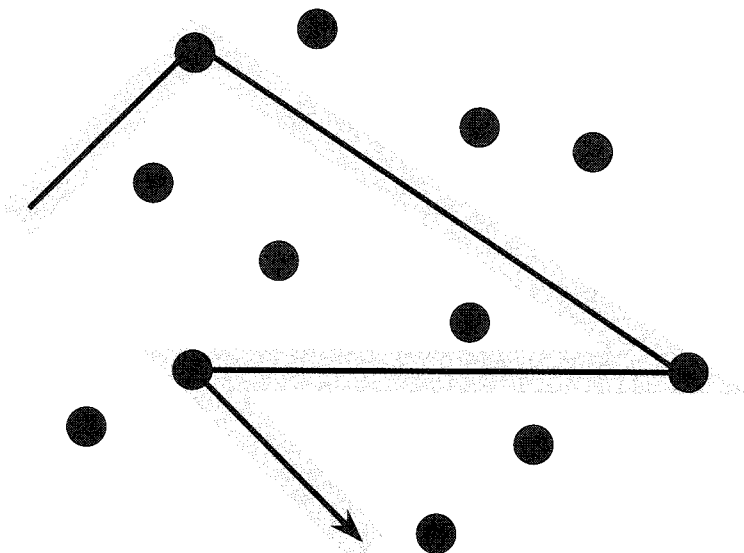


Fig. 1.5 Schematic picture for calculating the mean-free-path.

outcome of two-body scattering simply by knowing only the total momentum and energy of the two colliding particles. We also need to know the 'impact parameter', or distance 'off-center'; large impact parameter means a glancing collision. However, scattering in one dimension is rather different, because all collisions in one dimension are 'head-on'. Another way to say it: If we look at two identical particles that have just collided, as far as we can tell they might as well have simply passed through one another, like 'ghosts'. The only indication of a collision is a possible time delay (classical) or phase shift (quantum mechanics). Thus, in one dimension, two-body collisions – although inevitable – cannot thermalize the fluid of identical particles.

Instead, we must rely on the much less likely 3-body collisions. If  $\sigma$  now represents the 'size' of a particle – its region of interaction – then the density of interacting pairs is  $\sigma d^2$ , and so the mean-free-path for 3-body collisions is  $\ell_3 \approx 1/\sigma d^2$ . Of course, it need not really concern us all

that much if a system is slow to come to equilibrium; after all, the ideal gases – both classical and quantum – have been extremely useful. We are willing to depend on ‘dirt’, ‘walls’, or other ‘small corrections’ to bring the systems to equilibrium.

Still, what is remarkable about one-dimensional systems: They are strongly interacting, yet at best weakly dissipative. And one should stay near the ground state – i.e., zero temperature – if one wants to see interesting physics.

## 1.4 Quantum mechanics

Quantum mechanics – not classical mechanics – determines how particles ‘really’ behave. Quantum mechanics is also sometimes called ‘wave mechanics’ and indeed there is something in the theory that ‘waves’ – it is called the *wave function* (!) – and is almost always denoted by  $\psi(x,t)$ . (This is Greek ‘psi’ – small  $\psi$  for one particle (an *orbital*) or big  $\Psi$  for many particles.) This wave function tells us everything we can know about a particle.

However, this function is usually complex – it takes values that are complex numbers. For instance, a possible wave function for a free particle is a plane wave  $\psi(x,t) = \psi_0 e^{i(kx - \omega t)}$ ; the amplitude is  $\psi_0$  and the phase is  $\theta(x,t) = kx - \omega t$ . The frequency  $\omega(k)$  is a particular function  $\omega(k) = \hbar k^2 / 2m$  – the *dispersion relation* – of the wave number  $k$ . These parameters are related to the energy and momentum of the particle through the de Broglie relations,

$$\begin{aligned} E &= \hbar\omega, \\ P &= \hbar k. \end{aligned} \tag{10}$$

(The dispersion relation is simply  $E = P^2 / 2m$ .) The symbol  $\hbar$  is Planck’s constant – a fundamental constant of nature – divided by  $2\pi$ .

## 1.5 The essence of the Bethe ansatz

We will start with a very simple example; one which has essentially every feature of the more complicated problems. In exchange for a broad overview, you will have to simply accept certain remarkable properties on good faith for now, with the justification coming later. These properties of the models are precisely those which allow us to

solve them, and so they are bound to be rather remarkable; you will just have to suspend disbelief though, if you wish to glimpse the whole landscape before starting the detailed journey.

We begin with what I consider to be a 'fair' problem – nothing artificial – a system of  $N$  identical particles interacting in one dimension by a pair potential  $v(r)$ . The limitation to one-dimensional problems is essential for us to be able to solve them. Thus the dynamics is governed by a Hamiltonian

$$H = \frac{1}{2m} \sum_{j=1}^N p_j^2 + \sum_{j>k=1}^N v(x_j - x_k). \quad (11)$$

For a quantum system in the Schrödinger representation, the Hamiltonian operator is

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^N v(x_j - x_k). \quad (12)$$

Now I will tell you the first remarkable fact: There are reasonable two-body potentials  $v(r)$  for which the scattering of the particles of the  $N$ -body system is exactly equivalent to a sequence of two-body scatterings. Because of the conservation of momentum and energy, the scattering of two identical particles in one dimension simply multiplies the outgoing wave by a *phase factor*  $e^{-i\theta(p)}$ , where  $\theta(p)$  is the *two-body phase shift*, which is a function of the relative momentum  $p = p_1 - p_2$ . One such potential, which we will often use as an example, is the *inverse-square potential*  $v(r) = g/r^2$ . Since the particles are identical, we can simply say that the particle with momentum  $p_1$  passed through the particle with momentum  $p_2$ , and so when far from the scattering, the only effect is a phase factor  $e^{-i\theta(p)}$ . In terms of the wavefunction, with  $r = x_2 - x_1$ ,

$$\Psi(r) \rightarrow \begin{cases} \Psi_0 e^{i(kr - \omega t)}, & r \ll 0; \\ \Psi_0 e^{i(kr - \omega t - \theta)}, & r \gg 0; \end{cases} \quad (13)$$

If we now consider a particular particle ( $j$ ) with momentum  $p_j$ , and allow it to scatter from all the other  $N-1$  particles, then when far apart – the *asymptotic region* – the phase factor is

$$\prod_{\substack{k=1 \\ (k \neq j)}}^N e^{-i\theta(p_j - p_k)}. \quad (14)$$

We omit the factor in the product with  $k = j$ , since a particle does not scatter from itself. As physicists, what we usually want to study is a many-body problem, such as a fluid, at a given density. To this end, we place the system of  $N$  particles in a 'box' of size  $L$ . What kind of box, or equivalently, boundary conditions, is not important to us, because to eliminate the effect of these boundary conditions, we will eventually take the thermodynamic limit, discussed previously, where  $N, L \rightarrow \infty$ , with the (number) density  $N/L \equiv d$  fixed. So let us put our system on a very large ring of circumference  $L$ . After the particular particle with momentum  $p_j$  has scattered from all the other  $N-1$  particles, and entered the asymptotic region where the pair-potentials have decayed to essentially nothing, we allow the particle to continue around the ring, back to where it started, for an additional phase factor of  $e^{iLp_j/\hbar}$ .

Now the periodicity of the wave function requires that the total phase factor be unity, so

$$1 = e^{iLp_j/\hbar} \prod_{k=1}^N e^{-i\theta(p_j - p_k)}. \quad (15)$$

There is such an equation for each of the  $N$  particles. Taking the logarithm of these  $N$  coupled equations, we have the following *fundamental equation(s)*:

$$p_j/\hbar = \frac{2\pi I_j}{L} + \frac{1}{L} \sum_{k=1}^N \theta(p_j - p_k), \quad j=1, \dots, N. \quad (16)$$

The  $N$  quantities  $I_j$  are integers which come from the ambiguity in the phase – the 'modulo  $2\pi$ ' of the phase – and serve as quantum numbers for the state.

We can determine the energy  $E$  and momentum  $P$  of this state by looking in the asymptotic region where the wave function is simply a product of plane waves with (asymptotic) momenta  $p_j$ . Thus the energy and momenta are given by the free particle expressions,

$$P = \sum_{j=1}^N p_j, \quad (17)$$

and

$$E = \frac{1}{2m} \sum_{j=1}^N p_j^2 \quad (18)$$

Let us suppose that the particles are (spinless) fermions, and that the potential has a coupling constant  $g$  which we can turn off, so  $\theta \rightarrow 0$  as  $g \rightarrow 0$ . Then we see that  $p_j = 2\pi\hbar I_j / L$ , and we can identify the  $I_j$  with the free fermion momenta. In particular, they should be distinct integers, and we choose  $I_1 < \dots < I_N$ .

Now it would appear that our solution is only good for very low densities, since we have assumed a very large circumference  $L$ , in order for there to even be an asymptotic region. This brings us to the second remarkable fact: This solution is in fact exact for all densities. This is true only in the thermodynamic limit, but this is precisely the limit we want – when the exact choice of box is not important, nor is it important whether we have  $N = 10^8$  or  $N = 10^8 + 1$  particles.

## 1.6 A simple example – the $1/r^2$ potential

We thus have everything we need, in principle, to completely determine the spectrum of our many-body problem. We do need some modest ‘technology’ to extract the actual quantities of physical interest, but that can wait for now. Let us continue with this overview, and take as our example  $v(r) = g/r^2$ . We need for our Eq. (16) the two-body phase shift  $\theta(p)$  for this potential. The solution for  $\theta(p)$  is a nice problem from elementary quantum mechanics, which we give in Appendix A, along with some other important examples.

This solution gives us the (unremarkable) fact: The two-body phase shift for the potential  $v(r) = g/r^2$  is a constant, independent of the relative momentum  $p$ , so with the requirement that the phase shift be an odd function of  $p$ , we find

$$\theta(p) = \pi(\lambda - 1)p/|p| \equiv \pi(\lambda - 1)\text{sign}(p), \quad (19)$$

where  $gm/\hbar^2 = \lambda(\lambda - 1)$ . This result is unremarkable because when we observe that the combination  $gm/\hbar^2$  is dimensionless, we see that there is no momentum scale, and hence the phase shift  $\theta$  must be a constant. Similarly, by dimensional considerations, we can conclude that the ground state energy must be of the form  $E_0/L \equiv e_0(d|\lambda) = d^3 e_0(\lambda)$ , and the pressure of the form  $P(T, \mu|\lambda) = T^{3/2} P(\mu/T|\lambda)$ .

Thus, since the momenta are ordered  $p_1 < \dots < p_N$ , the coupled equations determining the momenta are

$$p_j/\hbar = 2\pi I_j/L + \pi(\lambda - 1)(2j - N - 1)/L, \quad j = 1, \dots, N. \quad (20)$$

This amounts to a uniform expansion of the spacing of the momenta from their free particle values when  $\lambda = 1$ , since

$$(p_{j+1} - p_j)/\hbar = 2\pi(I_{j+1} - I_j)/L + 2\pi(\lambda - 1)/L. \quad (21)$$

In Fig. 1.6., we show the effect of varying  $\lambda$  on a free fermion state consisting of the ground state with a particle and a hole.

For the non-degenerate ground state with  $N$  odd, which has quantum numbers  $I_j = j - (N + 1)/2$ , then,  $p_j = \pi\hbar\lambda(2j - N - 1)/L$ . This is just  $\lambda$  times the free fermion values, so the ground state energy  $E_0$  is simply  $\lambda^2$  times the free fermion value,

$$E_0/L = d^3 (2\pi\hbar\lambda)^2 / 3m. \quad (22)$$

For  $\lambda = 1, g = 0$ , this is the familiar free fermion result, and for  $\lambda \neq 1, g \neq 0$ , this result is exact for all densities  $d$ , in the thermodynamic limit. (Even for free fermions, there will be corrections for finite systems due to approximating the sum, even-odd effects, periodic versus hard-wall boundary conditions, impurities and defects, *etc.*) For the zero-temperature thermodynamics, the pressure  $P$  is given by

$$P = -\partial E_0 / \partial L = 2d^3 (2\pi\hbar\lambda)^2 / 3m = 2E_0 / L, \quad (23)$$

while the chemical potential is

$$\mu = \partial E_0 / \partial M = d^2 (2\pi\hbar\lambda)^2 / m = 3E_0 / M. \quad (24)$$

The second property of a many-body system we are often interested in, is the nature of the low-lying excitations. For our example, we will be guided by the free fermion case, where the excitations are holes and

particles, obtained by removing a particle from an orbital with quantum number  $I_h$  and placing it in an orbital with quantum number  $I_p$ , where  $|I_h| < N/2 < |I_p|$ . If  $I_p < 0$ , then preserving the ordering of the

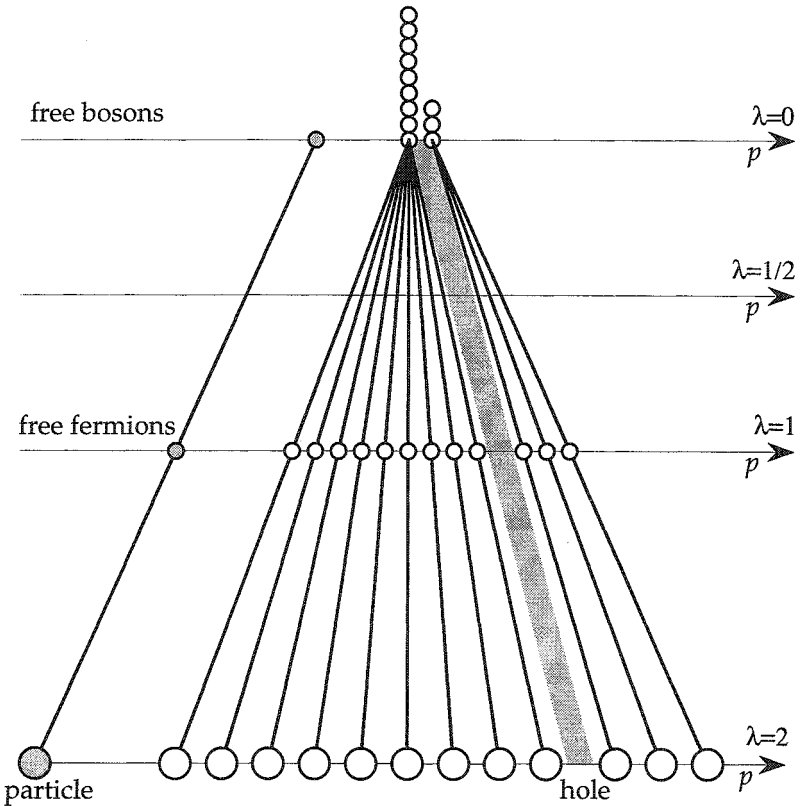


Fig. 1.6 Distribution of asymptotic momenta for the inverse-square potential at various interaction strengths.

quantum numbers, the ground state quantum numbers are changed to:  $I_1 \rightarrow I_p$ ,  $I_j \rightarrow I_j - 1$  for  $h \geq j > 1$ , all others unchanged. The total momentum of this state  $P$  is independent of  $\lambda$ , so  $P = 2\pi\hbar(I_p - I_h)/L$ , while the energy is

$$E - E_0 = \frac{(2\pi\hbar/L)^2}{2m} \left[ (I_p + \lambda - 1)^2 - \lambda^2 I_h^2 \right]. \quad (25)$$

Evaluation of the finite temperature thermodynamics, the third property of interest, although straightforward in principle, requires some technical tricks, and is thus deferred to Ch. 3.

Let us recall that the reader is owed a big debt – explanation and justification for the two remarkable facts on which our arguments depend:

Remarkable fact 1: There are reasonable two-body potentials  $v(r)$  for which the scattering of the particles of the  $N$ -body system is identical to a sequence of two-body scatterings;

and

Remarkable fact 2: The asymptotic solution given by the fundamental equation is in fact exact for all densities, in the thermodynamic limit.

## 1.7 References and history

Because this book is meant as a textbook which covers a very large subject, rather than as a technical monograph, we have kept references to a minimum. Those that we have included are reviews or monographs on selected subtopics, which will give the reader alternate derivations, additional related topics, and a more complete bibliography to provide access to the extensive research literature. In addition, we have included certain basic historical references – who first did what and when. In this section, we will now survey what is to come in the rest of the book, chapter by chapter, with a very brief collection of such references. For certain topics I have included a bit of personal anecdote, indicated by a shift to the first person, from the usual ‘conspiratorial’ we.

### 1.7.1 Overview

Since the first chapter is almost over, and is indeed an ‘overview’, let me give some general references. When we need basic results from quantum mechanics, Landau & Lifshitz [1958] will do; all the two-body potentials reviewed in Appendix A are taken from Landau. Also, in Landau there is a very nice discussion of the peculiarities of the  $1/r^2$  po-

tential, and a summary of all special functions except the elliptic functions. For the elliptic functions, integrals and various mathematical relations, we will send the reader to Gradshteyn & Ryzhik [1994]. For basic results from statistical mechanics, we will refer to Huang [1987].

Much of the content of this book comes under the label 'Bethe's ansatz'. Monographs on Bethe's ansatz include Gaudin [1983] and Korepin *et al.* [1993]. When we evaluate wavefunctions and correlations, we will need basic results from the study of random matrices; we will then refer to Mehta [1967] and [1991]. These are two editions of the same book, but we will need both, since certain results in the first do not appear in the second. For more recent results on time-dependent correlations, beyond the reach of this book, the monograph of Ha [1996] is recommended; it also has much on the Bethe ansatz. Except for a single example to illustrate the technique, we do not treat finite temperature thermodynamics in this book; however, the monograph of Takahashi [1999] is an excellent guide to this topic. For the connection between exactly solved one-dimensional quantum many-body problems and exactly solved two-dimensional statistical mechanics problems, as well as the XYZ model, the monograph of Baxter [1989] is essential.

All of these monographs are beautifully written expositions of subtle and interesting topics, and are to be highly recommended. One of the reasons for this book – the one you hold in your hands – is to provide access to these more advanced and necessarily more specialized works.

In addition, there is a collection by Mattis [1993] of reprints of journal papers on exactly solved one-dimensional many-body problems. When a reference is reprinted in this book, it is so indicated in the Bibliography. Beginning each chapter, Mattis provides a brief historical summary of selected topics that the reader might find useful. Also, C.N. Yang [1983] has reprinted many of his own related papers, with fascinating commentary; again such reprinted papers are so indicated in the Bibliography.

### 1.7.2 *Integrability and nondiffraction*

Chapter 2 is essentially a leisurely argument for what is usually called the 'asymptotic Bethe ansatz'. This is an extension to certain long-

ranged potentials of Bethe's [1931] original wavefunction – the 'Bethe ansatz' – which was valid for contact potentials only. The asymptotic Bethe ansatz was first introduced by Sutherland [1971a] for the  $1/r^2$  potential, shown to be integrable by Calogero [1971]. It was then presented as a general technique applicable to any integrable or nondiffractive system in Sutherland [1978] and [1985].

A proof of integrability for the hyperbolic potential was first presented – although not quite in this form – by Calogero *et al.* [1975a] and Calogero [1975b]. These papers also included the sinh-cosh potential that we study in Ch. 9. Calogero's method was the quantum version of a method due to Moser [1975] for the classical  $1/r^2$  potential, and it, in turn, was based on a famous paper of Lax [1968]. The whole technique is often called the 'quantum Lax method'. I learned the technique as here presented from B.S.Shastry [1993]. It was also found independently by Ujino *et al.* [1992].

The second example of an integrable system is bosons interacting by the  $\delta$ -function potential. This was first solved by Bethe's ansatz for scattering states by MacGuire [1964], and then for finite density by Lieb & Liniger [1963a] and Lieb [1963b].

### 1.7.3 Techniques

Chapter 3 introduces the basic mathematical techniques used for extracting the physical properties of a system from the Bethe ansatz wave function. Suitably generalized, they will be used throughout the remainder of the book. These techniques were first applied to the Heisenberg antiferromagnetic chain. This development began soon after Bethe's original paper, with Hulthén [1938] who introduced the distribution of asymptotic momenta and the fundamental integral equation to determine the ground state energy. Griffiths [1964] extended the integral equation to treat arbitrary magnetization, and des Cloizéaux & Pearson [1962] extracted the dispersion relation for low-lying excitations. At about the same time, Lieb & Liniger [1963a] and Lieb [1963b] were applying the same techniques to the  $\delta$ -function boson gas. Techniques to extract thermodynamic properties for the  $\delta$ -function boson gas were first introduced by Yang & Yang [1969]. These techniques were

first applied to a long-ranged potential through the asymptotic Bethe ansatz by Sutherland [1971b].

#### 1.7.4 *The classical limit*

The long-ranged potentials have a non-trivial classical limit, and this is explored in Chapter 4, to give the reader a more intuitive ‘feel’ for the systems studied in this book. This was immediately realized in Sutherland [1971b], where the partition function was evaluated for a classical one-dimensional gas interacting by a  $1/r^2$  potential. This was further explored in Sutherland [1978], and in particular, the classical results for the Toda [1989] lattice were recovered from the integrable quantum systems. This is the first time that classical solitons were identified as the classical limit of the low-lying excitations of the quantum system. Later, in Sutherland & Campbell [1994a] and Sutherland [1995c], the classical limit was further explored.

#### 1.7.5 *Ground state wavefunctions of product form*

Certain systems – including the  $1/r^2$  potential – have a ground state wavefunction which is of product form. In many cases, this allows an exact evaluation of correlations and excited states. We explore these results in Chapter 5.

The early history of the long-ranged potentials – and of the  $1/r^2$  system, in particular – shows a fertile interplay between Calogero and coworkers, and Sutherland. I will take a few paragraphs to tell how I remember this exchange – always and still, respectful and friendly – with complementary interests.

This history begins with two papers of Calogero [1969a,b], in which he first solves the 3-body problem of particles interacting in one dimension by an inverse-square plus harmonic pair potential. He then finds the exact ground state wavefunction (and a 1-parameter family of excited states) for the corresponding  $N$ -body problem, and makes a conjecture for the complete spectrum.

As a post-doc at Berkeley, I read these papers and wondered about the properties of a fluid of particles interacting by a pure inverse-square potential. Such a system would have a thermodynamic limit, in

contrast to Calogero's systems. This led to two papers, Sutherland [1971a,b].

In the first paper, I gave the exact ground state wavefunction for  $N$  particles interacting by an inverse-square pair potential, confined in a harmonic well, and used this result to extract the ground state properties of a fluid of particles interacting by a pure inverse-square potential at finite density. Even more importantly, a connection was made between this system, and the extensive body of exact results on random matrices by Dyson, Gaudin, Mehta, *et al.*; see Mehta [1967, 1991]. This allowed evaluation of certain correlations.

In the second paper, I introduced the asymptotic Bethe ansatz, and showed how it gives a complete solution to the problem by evaluating the ground state properties, low-lying excitations, thermodynamics and classical limit. Meanwhile, I received preprints from Calogero [1971] in which he proved his conjecture, and from Marchioro & Presutti [1970] who used this result to evaluate the grand partition function for Calogero's system; our results agreed.

I then went on to write two more papers, Sutherland [1971c,1972], the first at Berkeley and the second at Utah. These papers were both on the inverse-sine-square potential – a periodic version of the inverse-square potential. The first paper related the ground state to further results from random matrices and evaluated momentum distributions, as well as pair correlations; the second paper found the excited states as certain orthogonal polynomials. (Unknown to me, these same polynomials had been introduced a few years earlier in the mathematics literature; see Jack [1969] and Macdonald [1995].) An elliptic product wave function was soon introduced in Sutherland [1975a,b], and Calogero [1975c] showed this to be the most general case.

In recent years, these earlier results have been considerably generalized to yield time-dependent correlations; see Ha [1996].

### 1.7.6 *Heisenberg–Ising magnet*

Chapter 6 presents the solution to a one-dimensional model for a magnet, which interpolates between the isotropic Heisenberg magnets, a

free particle lattice gas, and the Ising magnets which are trivial in one-dimension.

We have already spoken of the contributions of Bethe, Hulthén, Griffiths and des Cloizéaux & Pearson to the solution of the Heisenberg antiferromagnet. These techniques were extended to the full Heisenberg-Ising model by Orbach [1958] and Walker [1959]. The collection of papers by Yang & Yang [1966a-e] were a summary of all work to date, as well as presenting new results. Taken together, they provide a beautiful monograph on the Heisenberg-Ising model. They are also where I began my graduate work.

Low-lying excitations were first studied in des Cloizéaux & Gaudin [1966], and the treatment of twisted boundary conditions and calculation of the spin stiffness were first presented in Shastry & Sutherland [1990] and Sutherland & Shastry [1990]. Baxter [1973] was the first to evaluate the spontaneous staggered magnetization. The thermodynamics of this system is treated at length in the monograph of Takahashi [1999].

### 1.7.7 Consistency

Chapter 7 is very important, and necessarily long. It begins with a review of various approaches to proving integrability. We then introduce the consistency equations for multicomponent scattering, and solve them within a general class of physical situations. This gives a necessary condition on the  $S$ -matrix for integrable systems. Assuming such integrable systems do exist, we proceed to determine the equations for the asymptotic momenta when we impose either periodic or twisted boundary conditions; *i.e.*, we quantize the systems and determine the eigenvalues.

The references for proving integrability were given in Ch. 2. The consistency conditions are also called Yang-Baxter equations; they were first emphasized by Yang [1967, 1968] and then made the foundation for a general method by Baxter [1989]. They are also the basis of the constructive approach for proving integrability, discussed in this chapter. Many authors have searched for solutions to the consistency conditions, in order to identify new exactly solvable models. The work

of Zamolodchikov & Zamolodchikov [1980] studies the problem in the context of a relativistic field theory.

Finally, the method of determining the eigenvectors of the transfer matrix in the long Sects. 7.6 and 7.7 – usually called the quantum inverse scattering method – I first learned from Faddeev, Takhtajan and Korepin. Good pedagogical treatments are in Takhtajan & Faddeev [1979] and Takhtajan [1985]. Although historically, these results were not first derived by these methods – Sec. 7.6 was shown by Baxter [1971a] and Sec. 7.7 was shown by Yang [1967] and Sutherland [1968b] – this is probably the most efficient derivation.

### 1.7.8 *Exchange models*

Since we view the scattering of particles as an exchange of quantum numbers among ordered particles, any potential that allows transmission deserves to be called an exchange model. If transmission is to not vanish, then the even and odd phase shifts must necessarily be different. This can easily be achieved if the potential is not too repulsive at the origin; the  $\delta$ -function potential is an example. However, one can also take an integrable potential such as the  $1/r^2$  potential, which is impenetrable, and modify it so that even and odd states feel different interaction strengths. Such a potential is  $v(r) = \lambda(\lambda - Q)/r^2$ , where  $Q$  is the permutation operator. For two components, we can write this ‘exchange’ operator as  $Q = (1 + \vec{\sigma} \cdot \vec{\sigma}')/2$ . We show that in some cases such potentials are integrable, hence solvable, and discuss the nature of the solutions.

The  $\delta$ -function potential was first solved for two-component fermions by Gaudin [1967a,b, 1968] in the attractive case, by Yang [1967] in the repulsive case, and by Sutherland [1968b] for the general multi-component problem. This paper of Sutherland first introduced the nested Bethe ansatz, or ‘Bethe-Yang ansatz’ as it was then called, and is an alternate derivation of the results in Sec. 7.7.

Although exchange potentials had long been used on a lattice – the Heisenberg magnets, for example – they were first introduced into the continuum case by Polychronakos [1992], who proved integrability. The complete solution was given by Sutherland & Shastry [1993a].

### 1.7.9 *Sinh-Cosh model*

Chapter 9 studies a two-component continuum system, where like particles interact with a potential  $v(r) = \lambda(\lambda - 1) / \sinh^2 r$ , and unlike particles interact by  $v(r) = -\lambda(\lambda - 1) / \cosh^2 r$ . We first show integrability, and then discuss the solution. This system is particularly rich in its physical properties, exhibiting bound pairs which can have an excitation spectrum, and in another case, paramagnetic spin-wave excitations within the quantum fluid.

Such a system was first introduced by Calogero *et al.* [1975a] and Calogero [1975b] who proved it to be integrable; we here modify the method of Shastry [1993] to show integrability. Soon after Calogero's papers, Sutherland [1978] introduced the asymptotic Bethe ansatz to show that such a system could be solved, and the solution was presented for a single component system. However, the equations for the two-component system were sufficiently complicated that an explicit solution was postponed until Sutherland & Römer [1993b]; this solution makes use of the results from Sec. 7.6. Motivation for taking up the problem once again, was that Mathematica could easily employ the logarithmic derivative of the gamma function in the complex plane!

### 1.7.10 *Exchange lattice systems by freezing*

In Chapter 10, we solve exchange lattice problems by 'freezing' the systems of Chapter 8. The technique and the resulting solutions are studied in detail.

Exchange lattice systems of course include the Heisenberg magnets of Chapter 6. The first extension was due to Lai [1974] and Sutherland [1975c], who introduced the nearest-neighbor exchange model. Let me give a few details of the history of this work. C.K.Lai, another of C.N.Yang's graduate students from Stony Brook, had come to Utah as my post-doc. During a visit of Yang to Utah as an Adjunct Professor, the three of us began working together to find a lattice version of the multi-component  $\delta$ -function potential. The obvious candidate was the Hubbard model – see Ch. 11 – which worked for two-component fermions, as shown by Lieb & Wu [1968], but this model was known to be non-integrable for other cases. After examining a class of models that in-

cludes what is now known as the  $t$ - $J$  model, we discovered that one case was integrable. The next year I visited Stony Brook, and while there, rewrote the solvable point as an exchange model to make it appear more natural.

Long-ranged exchange models first appeared – independently of one another – with Shastry [1988] and Haldane [1988]. They introduced and solved a Heisenberg magnet with  $1/r^2$  exchange interaction. The idea of ‘freezing’ the continuum exchange models of Ch. 8, to give solutions to corresponding exchange lattice models occurred to both Sutherland & Shastry [1993a], Sutherland *et al.* [1994b], and to Polychronakos [1993]. Explicit evaluation was given in Sutherland & Shastry [1993a] for the  $1/r^2$  exchange, agreeing with numerical results of Haldane [1991], and in Sutherland *et al.* [1994b] for the hyperbolic case.

### 1.7.11 Hubbard model

The Hubbard model – a lattice version of the  $\delta$ -function potential – is a system of great interest to condensed matter physicists, and the solution in one-dimension is outlined in Chapter 11.

This model was first solved by Lieb & Wu [1968], whose solution was based on Yang’s [1967] earlier solution of the two-component fermion  $\delta$ -function gas. This model is one of the most studied, and we refer the reader to a recent review of Deguchi *et al.* [2000] for a comprehensive history and presentation of known results.

I sincerely apologize to those authors whose work I have not referenced, but I am limited by space constraints and by my ignorance. I hope that the general references to which I have directed the reader will help set the record straight.