1 Introduction

Many types of waves involving different physical factors exist in the ocean. As in the elementary problem of a spring-mass system, all waves must be associated with some kind of restoring force. It is therefore convenient to make a crude classification of ocean waves according to the restoring force, as shown in Table 1.1.

Wind waves and swell, generated by local and distant storms, are the most directly experienced by mankind. Occurring less frequently but with occasionally disastrous consequences are the tsunamis which usually refer to long-period [$\sim O(1\ \text{h})$] oscillations caused by large submarine earthquakes or landslides. Within the same broad range of time scales, waves can also exist as a result of human activities (ship motion, explosion, and so on). Since these waves are the most prominent on the water surface and their main restoring force is gravity, they are called the *surface gravity waves*. The shorter term, *surface waves*, is often used if the exclusion of surface capillary waves is understood.

Important in the science of oceanography are the internal gravity waves along the thermoclines which are horizontal layers of sharp density stratification beneath the sea surface. The associated wave motion is generally not pronounced on the surface except for some indirect signs of its presence. These waves contribute to the process of mixing and affect the eddy viscosity of ocean currents. Storm surges are the immediate consequence of local weather and can inflict severe damage to human life and properties by innundating the coast.

In nature, several restoring forces can be in effect at the same time, hence the distinction between various waves listed in Table 1.1 is not always very sharp.

Table 1.1: Wave Type, Physical Mechanism, Activity Region.

Wave Type	Physical Mechanism	Typical Period[a]	Region of Activity
Sound	Compressibility	10^{-2}–10^{-5} s	Ocean interior
Capillary ripples	Surface tension	$< 10^{-1}$ s	Air–water interface
Wind waves and swell	Gravity	1–25 s	Air–water interface
Tsunami	Gravity	10 min–2 h	Air–water interface
Internal waves	Gravity and density stratification	2 min–10 h	Layer of sharp density change
Storm surges	Gravity and earth rotation	1–10 h	Near coastline
Tides	Gravity and earth rotation	12–24 h	Entire ocean layer
Planetary waves	Gravity, earth rotation and variation of latitude or ocean depth	O(100 days)	Entire ocean layer

[a]In seconds (s), minutes (min), hours (h), and days.

This book will be limited to wave motions having time scales such that compressibility and surface tension at one extreme and earth rotation at the other are of little direct importance. Furthermore, the vertical stratification of sea water is assumed to be small enough within the depth of interest. Therefore, we shall only be concerned with the surface gravity waves, that is, wind waves, swell, and tsunamis. Discussions of all other waves listed in Table 1.1 can be found in the oceanographic treatises by Hill (1962) and LeBlond and Mysak (1978).

In this chapter we first review the basic equations of fluid motion and some general deductions for inviscid, irrotational flows. Linearized equations for infinitesimal waves are then derived. After introducing the general notions of propagating waves, we examine the properties of simple harmonic progressive waves on constant depth. An elementary discussion of group velocity will be given from both kinematic and dynamic points of view.

1.1 Review of Basic Formulation for an Incompressible Fluid of Constant Density

1.1.1 Governing Equations

In a wide variety of gravity-wave problems, the variation of water density is insignificant over the temporal and spatial scales of engineering interest.

The fundamental conservation laws are adequately described by the following Navier–Stokes equations:

$$\text{mass}: \nabla \cdot \mathbf{u} = 0\,, \tag{1.1.1}$$

$$\text{momentum}: \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u} = -\nabla\left(\frac{P}{\rho} + gz\right) + \nu\nabla^2\mathbf{u}\,, \tag{1.1.2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity vector (u, v, w), $P(\mathbf{x}, t)$ the pressure, ρ the density, g the gravitational acceleration, ν the constant kinematic viscosity, and $\mathbf{x} = (x, y, z)$ with the z axis pointing vertically upward.

One of the most important deductions from these equations is concerned with the vorticity vector $\boldsymbol{\Omega}(\mathbf{x}, t)$ defined by

$$\boldsymbol{\Omega} = \nabla \times \mathbf{u}\,, \tag{1.1.3}$$

which is twice the rate of local rotation. By taking the curl of Eq. (1.1.2) and using Eq. (1.1.1), we can show that

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\Omega}\,. \tag{1.1.4}$$

Physically, the preceding equation means that following the moving fluid, the rate of change of vorticity is due to stretching and twisting of vortex lines and to viscous diffusion (see, e.g., Batchelor, 1967). In water where ν is small ($\cong 10^{-2}$ cm^2/s) the last term in Eq. (1.1.4) is negligible except in regions of large velocity gradient and strong vorticity. A good approximation applicable in nearly all of the fluid is

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla\mathbf{u}\,. \tag{1.1.5}$$

An important class of problems is one where $\boldsymbol{\Omega} \equiv 0$ and is called the *irrotational flow*. Taking the scalar product of Eq. (1.1.5) and $\boldsymbol{\Omega}$, we have

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\frac{\boldsymbol{\Omega}^2}{2} = \Omega^2[\mathbf{e}_\Omega \cdot (\mathbf{e}_\Omega \cdot \nabla\mathbf{u})]\,,$$

where $\mathbf{e}_\Omega$ is the unit vector along $\boldsymbol{\Omega}$. Since the velocity gradient is finite in any physically realizable situation, the maximum of $\mathbf{e}_\Omega \cdot (\mathbf{e}_\Omega \cdot \nabla\mathbf{u})$ must be a finite value, $M/2$, say. The magnitude $\boldsymbol{\Omega}^2(\mathbf{x}, t)$ following a fluid particle cannot exceed $\boldsymbol{\Omega}^2(\mathbf{x}, 0)e^{Mt}$. Consequently, if there is no vorticity anywhere at $t = 0$, the flow will remain irrotational for all time.

For an inviscid irrotational flow, the velocity $\mathbf{u}$ can be expressed as the gradient of a scalar potential Φ,

$$\mathbf{u} = \nabla\Phi\,. \tag{1.1.6}$$

Conservation of mass requires that the potential satisfies Laplace's equation

$$\nabla^2\Phi = 0\,. \tag{1.1.7}$$

If the velocity potential is known, then the pressure field can be found from the momentum equation (1.1.2). By using the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\frac{\mathbf{u}^2}{2} - \mathbf{u}\times(\nabla\times\mathbf{u})$$

and irrotationality, we may rewrite Eq. (1.1.2), with $\nu = 0$, as

$$\nabla\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2\right] = -\nabla\left(\frac{P}{\rho} + gz\right).$$

Upon integration with respect to the space variables, we obtain

$$-\frac{P}{\rho} = gz + \frac{\partial\Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 + C(t)\,, \tag{1.1.8}$$

where $C(t)$ is an arbitrary function of t and can usually be omitted by redefining Φ without affecting the velocity field. Equation (1.1.8) is called the Bernoulli equation. The first term, gz, on the right-hand side of Eq. (1.1.8) is the hydrostatic contribution, whereas the rest is the hydrodynamic contribution to the total pressure P.

1.1.2 Boundary Conditions for an Inviscid Irrotational Flow

Two types of boundaries interest us: the air–water interface which will also be called the free surface, and the wetted surface of an impenetrable solid. Along these two boundaries the fluid is assumed to move only tangentially. Let the instantaneous equation of the boundary be

$$F(\mathbf{x},t) = z - \zeta(x,y,t) = 0\,, \tag{1.1.9}$$

where ζ is the height measured from $z = 0$, and let the velocity of a geometrical point $\mathbf{x}$ on the moving free surface be $\mathbf{q}$. After a short time dt, the free surface is described by

$$F(\mathbf{x}+\mathbf{q}\,dt, t+dt) = 0 = F(\mathbf{x},t) + \left(\frac{\partial F}{\partial t} + \mathbf{q}\cdot\nabla F\right) dt + O(dt)^2\,.$$

In view of Eq. (1.1.9), it follows that

$$\frac{\partial F}{\partial t} + \mathbf{q}\cdot\nabla F = 0$$

for small but arbitrary dt. The assumption of tangential motion requires $\mathbf{u}\cdot\nabla F = \mathbf{q}\cdot\nabla F$ which, in turn, implies that

$$\frac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F = 0 \quad \text{on} \quad z = \zeta\,, \tag{1.1.10}$$

or, equivalently,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \Phi}{\partial z} \quad \text{on} \quad z = \zeta\,. \tag{1.1.11}$$

Equation (1.1.10) or (1.1.11) is referred to as the *kinematic* boundary condition. In the special case where the boundary is the wetted surface of a stationary solid S_B, $\partial\zeta/\partial t = 0$ and Eq. (1.1.10) reduces to

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on} \quad S_B\,. \tag{1.1.12}$$

On the sea bottom B_0 at the depth $h(x,y)$, Eq. (1.1.9) becomes $z + h(x,y) = 0$ and Eq. (1.1.12) may be written

$$-\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} \quad \text{on} \quad B_0\,. \tag{1.1.13}$$

On the air–water interface, both ζ and Φ are unknown, and it is necessary to add a *dynamical* boundary condition concerning forces.

For most of the topics of interest in this book, the wavelength is so long that surface tension is unimportant; the pressure just beneath the free surface must equal the atmospheric pressure P_a above. Applying Eq. (1.1.8) on the free surface, we have

$$-\frac{P_a}{\rho} = g\zeta + \frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 \quad \text{on} \quad z = \zeta\,. \tag{1.1.14}$$

The two conditions, (1.1.11) and (1.1.14), may be combined into one in terms of Φ, by taking the total derivative of Eq. (1.1.14):

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\frac{P_a}{\rho} + \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\left(\frac{\partial \Phi}{\partial t} + \frac{\mathbf{u}^2}{2} + g\zeta\right) = 0\,, \quad z = \zeta\,. \tag{1.1.15}$$

Using Eq. (1.1.11) and

$$\mathbf{u} \cdot \nabla \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2} \mathbf{u}^2 \,,$$

we have from Eq. (1.1.15)

$$\frac{D}{Dt}\frac{P_a}{\rho} + \left[\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} + \frac{\partial \mathbf{u}^2}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \nabla \mathbf{u}^2 \right] = 0 \,, \quad z = \zeta \,. \tag{1.1.16}$$

Furthermore, if P_a = constant, the above condition becomes

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial t}(\mathbf{u})^2 + \frac{1}{2}\mathbf{u} \cdot \nabla \mathbf{u}^2 = 0 \,, \quad z = \zeta \,, \tag{1.1.17}$$

which is essentially a condition for Φ. Not only do nonlinear terms appear in these boundary conditions, but the position of the free surface is also an unknown quantity. An exact analytical theory for water-wave problems is therefore almost impossible.

When the motion of the air above is significant, the atmospheric pressure cannot always be prescribed *a priori*; the motions of air and water are, in general, coupled. Indeed, the exchange of momentum and energy between air and sea is at the heart of the theory of surface-wave generation by wind. However, we shall limit our attention to sufficiently localized regions in the absence of direct wind action. Air can then be ignored for most purposes because of its comparatively small density.

1.2 Linearized Approximation for Small-Amplitude Waves

Let us assume that certain physical scales of motion can be anticipated *a priori*. In particular, let

$$\begin{pmatrix} \lambda/2\pi \\ \omega^{-1} \\ A \\ A\omega\lambda/2\pi \end{pmatrix} \quad \text{characterize} \quad \begin{pmatrix} x, y, z, h \\ t \\ \zeta \\ \Phi \end{pmatrix} , \tag{1.2.1}$$

where λ, ω, and A are the typical values of wavelength, frequency, and free-surface amplitude respectively. We have assigned the scale for Φ to be $A\omega\lambda/2\pi$ so that the velocity has the scale $A\omega$ which is expected near the free surface. We now introduce dimensionless variables and denote them by primes as follows:

$$\begin{pmatrix} \Phi \\ x, y, z, h \\ t \\ \zeta \end{pmatrix} = \begin{pmatrix} A\omega\lambda\Phi'/2\pi \\ \lambda(x', y', z', h')/2\pi \\ t'/\omega \\ A\zeta' \end{pmatrix}. \tag{1.2.2}$$

When these variables are substituted into Eqs. (1.1.7), (1.1.11), (1.1.12) and (1.1.14), a set of dimensionless equations is obtained:

$$\nabla'^2\Phi' = \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)\Phi' = 0\,, \quad -h' < z' < \varepsilon\zeta'\,, \tag{1.2.3}$$

$$\frac{\partial \Phi'}{\partial n'} = 0\,, \quad z' = -h'\,, \tag{1.2.4}$$

$$\frac{\partial \zeta'}{\partial t'} + \varepsilon\left(\frac{\partial \Phi'}{\partial x'}\frac{\partial \zeta'}{\partial x'} + \frac{\partial \Phi'}{\partial y'}\frac{\partial z'}{\partial y'}\right) = \frac{\partial \Phi'}{\partial z'} \tag{1.2.5}$$

$$\text{on} \quad z' = \varepsilon\zeta'\,,$$

$$\frac{\partial \Phi'}{\partial t'} + \left(\frac{2\pi g}{\omega^2\lambda}\right)\zeta' + \frac{\varepsilon}{2}(\nabla'\Phi')^2 = -P'_a = -\frac{2\pi P_a}{\rho A\omega^2\lambda} \tag{1.2.6}$$

where $\varepsilon = 2\pi A/\lambda = 2\pi \times$ amplitude/wavelength = wave slope. Since the scales are supposed to reflect the physics properly, the dimensionless variables must all be of order unity; the importance of each term above is measured solely by the coefficients in front.[1]

Let us now consider small-amplitude waves in the sense that the wave slope is small: $\varepsilon \ll 1$. The free-surface boundary conditions can be simplified by noting that the unknown free surface differs by an amount of $O(\varepsilon)$ from the horizontal plane $z' = 0$. Thus, we can expand Φ' and its derivatives in a Taylor series:

[1] If the scales have been chosen properly, the normalized variables and their derivatives should indeed be of order unity. The relative importance of each term in an equation is entirely indicated by the dimensionless coefficient multiplying the term. If under certain conditions the solution of the approximate problem exhibits behavior which violates the original assumptions on the order of magnitude, then the scales initially chosen are no longer valid. New scales, hence new approximations, must be found to reflect the physics. It is not an exaggeration to say that estimating the scales is the first step toward the approximate solution of a physical problem.

As a procedural point, when the choices of the scales are limited so that only one dimensionless parameter appears, the formalism of nondimensionalization can often be omitted for brevity, although its essence must always be clearly understood.

$$f'(x', y', \varepsilon\zeta', t') = f'\Big|_0 + \varepsilon\zeta' \frac{\partial f'}{\partial z'}\Big|_0 + \frac{(\varepsilon\zeta')^2}{2!} \frac{\partial^2 f'}{\partial z'^2}\Big|_0 + O(\varepsilon^3)\,,$$

where $f|_0$ means $f(x, y, 0, t)$, etc. To the leading order of $O(1)$, the free-surface conditions become approximately

$$\left.\begin{aligned} \frac{\partial \zeta'}{\partial t'} &= \Phi'_{z'} \\ \frac{\partial \Phi'}{\partial t'} + \frac{2\pi g}{\omega^2 \lambda}\zeta' &= -P'_a \end{aligned}\right\} \quad z' = 0\,.$$

Only linear terms remain in these conditions which are now applied at a known plane $z' = 0$. Together with Eqs. (1.2.3) and (1.2.4) the approximate problem is completely linearized. Returning to physical variables, we have

$$\nabla^2 \Phi = 0\,, \qquad -h < z < 0\,, \tag{1.2.7}$$

$$\frac{\partial \Phi}{\partial n} = 0\,, \qquad z = -h\,, \tag{1.2.8}$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \tag{1.2.9}$$

$$z = 0\,.$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = -\frac{P_a}{\rho} \tag{1.2.10}$$

Furthermore, Eqs. (1.2.9) and (1.2.10) may be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = -\frac{1}{\rho}\frac{\partial P_a}{\partial t}\,, \quad z = 0\,, \tag{1.2.11}$$

which can also be obtained by linearizing Eq. (1.1.16).

The total pressure inside the fluid can be related to Φ by linearizing the Bernoulli equation:

$$P = -\rho g z + p\,, \quad \text{where} \quad p = -\rho\frac{\partial \Phi}{\partial t} = \text{dynamic pressure}\,. \tag{1.2.12}$$

These conditions must be supplemented by initial conditions and the boundary conditions on the body and at infinity, if appropriate.

It is worthwhile to remark further on the assumption of zero viscosity in the context of linear approximation. Near a solid boundary, the potential theory allows a finite slip in the tangential direction, but in reality all velocity components must vanish. There must be a thin boundary layer to smooth the transition from zero to a finite value. Thus,

$$\frac{\partial}{\partial x_N} \gg \frac{\partial}{\partial x'_T}, \ \frac{\partial}{\partial x''_T},$$

where x_N, x'_T, and x''_T form a locally orthogonal coordinate system with x_N normal to the solid surface and x'_T and x''_T tangential to it. It follows from the linearized momentum equation that the tangential velocity $\mathbf{u}_T$ satisfies

$$\frac{\partial \mathbf{u}_T}{\partial t} \cong \nu \frac{\partial^2 \mathbf{u}_T}{\partial x_N^2} - \frac{1}{\rho}\nabla_T p$$

inside the boundary layer. With the wave period as the time scale, the boundary-layer thickness δ must be of the order

$$\delta \sim \left(\frac{2\nu}{\omega}\right)^{1/2}.$$

For water, $\nu \cong 0.01\ \text{cm}^2/\text{s}$; in model experiments the typical period is 1 s so that $\delta \sim 0.056$ cm, which is far too small compared with the wavelength of usual interest. In the ocean, swells of 10-s periods are common; $\delta \sim 0.17$ cm. But the boundary layer near the natural sea bottom is usually turbulent for most of the period. As will be discussed later, a typical experimental value of eddy viscosity is about 100ν; the thickness of the turbulent boundary layer for a 10-s period is then about $\leq O(10)$ cm, which is still quite small. Thus, the boundary-layer region is but a tiny fraction of a fluid volume whose dimensions are comparable to a wavelength, and the global influence on the wave motion is small over distances of several wavelengths or time of several periods.

1.3 Elementary Notions of a Propagating Wave

Let us consider a special form of the free surface

$$\zeta(x, y, t) = \text{Re}\, A e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = A\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)\,, \tag{1.3.1}$$

where i is the imaginary unit $(-1)^{1/2}$ and

$$\mathbf{k} = (k_1, k_2)\,, \quad \mathbf{x} \equiv (x, y)\,. \tag{1.3.2}$$

For the convenience of mathematical manipulation, the exponential form is often preferred, and for brevity the sign Re (the real part of) is often omitted, that is,

$$\zeta(x, y, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{1.3.3}$$

is used to mean the same as Eq. (1.3.1). What sort of free surface does this expression describe?

To a stationary observer, ζ oscillates in time with the period $T = 2\pi/\omega$ between the two extremes A and $-A$. If we take a three-dimensional snapshot at a fixed t with ζ as the vertical ordinate and (x, y) as the horizontal coordinates, the variation of ζ in (x, y) describes a periodic topography. In a plane $y = \text{const}$, ζ is seen to vary periodically in the x direction between A and $-A$ with the spatial period $2\pi/k_1$. Similarly, in a plane $x = \text{const}$, ζ varies periodically in the y direction between A and $-A$ with the spatial period $2\pi/k_2$. Thus, along the x direction the number of crests per unit length is $k_1/2\pi$, and along the y direction the number of crests per unit length is $k_2/2\pi$.

Let us define the *phase function* S by

$$S(x, y, t) = k_1x + k_2y - \omega t = \mathbf{k}\cdot\mathbf{x} - \omega t\,. \tag{1.3.4}$$

For a fixed time, $S(x, y, t) = \text{const} = S_0$ describes a straight line with the normal vector

$$\mathbf{e}_k = \left(\frac{k_1}{k}, \frac{k_2}{k}\right), \quad \text{where} \quad k = (k_1^2 + k_2^2)^{1/2} = |\mathbf{k}|\,. \tag{1.3.5}$$

Along this straight line, the surface height is the same everywhere. In particular, the waves are the highest (crests) when $S_0 = 2n\pi$ and the lowest (troughs) when $S_0 = (2n + 1)\pi$. When S_0 increases by 2π, the surface height is repeated. Lines of different S_0 are parallel to each other if k_1 and k_2 are constant. We call these lines the *phase lines*. If we take a snapshot and cut a cross section along the direction $\mathbf{e}_k$, the profile of ζ is sinusoidal with the wavelength $\lambda = 2\pi/k$. Alternatively, we may say that the number of waves per unit length along the $\mathbf{k}$ direction is $k/2\pi$. Hence k is called the *wavenumber*, and $\mathbf{k}$ the *wavenumber vector* with k_1 and k_2 as its components. The maximum deviation A from the mean $z = 0$ is called the *amplitude*.

Let us follow a particular phase line $S = S_0$. As time t progresses, the position of the phase line changes. What is the velocity of the phase line? Evidently, if the observer travels with the same velocity $d\mathbf{x}/dt$, the phase line appears stationary, that is,

$$dS = \nabla S \cdot d\mathbf{x} + \frac{\partial S}{\partial t}\,dt = 0\,.$$

From Eq. (1.3.4) it follows that

$$\mathbf{k} = \nabla S = \mathbf{e}_k |\nabla S| \,, \tag{1.3.6a}$$

$$-\omega = \frac{\partial S}{\partial t} \,, \tag{1.3.6b}$$

and that

$$\mathbf{e}_k \cdot \frac{d\mathbf{x}}{dt} = \frac{-\partial S/\partial t}{|\nabla S|} = \frac{\omega}{k} \equiv C \,. \tag{1.3.7}$$

Thus, the speed at which the phase line advances normal to itself is ω/k, which is called the *phase speed* C. Equations (1.3.6a) and (1.3.6b) can be regarded as the definitions of ω and $\mathbf{k}$, that is, the frequency is the time rate, and the wavenumber is the spatial rate of phase change.

1.4 Progressive Water Waves on Constant Depth

For simple harmonic motion with frequency ω, linearity of the problem allows separation of the time factor $e^{-i\omega t}$ as follows:

$$\left.\begin{aligned} \zeta(x,y,t) &= \eta(x,y) \\ \Phi(x,y,z,t) &= \phi(x,y,z) \\ \mathbf{u}(x,y,z,t) &\rightarrow \mathbf{u}(x,y,z) \\ P(x,y,z,t) + \rho g z &= p(x,y,z) \end{aligned}\right\} e^{-i\omega t} \,. \tag{1.4.1}$$

Note that the same symbol $\mathbf{u}$ is used for both the fluid velocity and its spatial factor. The linearized governing equations (1.2.7) to (1.2.10) can be reduced to

$$\nabla^2 \phi = 0 \,, \quad -h < z < 0 \,, \tag{1.4.2}$$

$$\frac{\partial \phi}{\partial z} = 0 \,, \quad z = -h \,, \tag{1.4.3}$$

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} + i\omega\eta = 0 \,, \quad &(1.4.4) \\ g\eta - i\omega\phi = \frac{-p_a}{\rho} \,, \quad &(1.4.5) \end{aligned}\right. \quad z = 0 \,,$$

where Eqs. (1.4.4) and (1.4.5) may be combined as

$$g\frac{\partial \phi}{\partial z} - \omega^2 \phi = \frac{i\omega}{\rho} p_a \,, \quad z = 0 \,. \tag{1.4.6}$$

Let us seek a two-dimensional solution which represents a progressive wave without direct atmospheric forcing, that is, $p_a = 0$ and

$$\eta = Ae^{ikx} \,. \tag{1.4.7}$$

The potential that satisfies Eqs. (1.4.2) and (1.4.3) is easily seen to be

$$\phi = B \cosh k(z+h) e^{ikx} \,.$$

To satisfy the free-surface conditions with $p_a = 0$, we require

$$B = -\frac{igA}{\omega} \frac{1}{\cosh kh}$$

and

$$\omega^2 = gk \tanh kh \tag{1.4.8}$$

so that

$$\phi = -\frac{igA}{\omega} \frac{\cosh k(z+h)}{\cosh kh} e^{ikx} \,. \tag{1.4.9}$$

Thus, for a given frequency ω the progressive wave must have the proper wavenumber given by Eq. (1.4.8). In dimensionless form

$$\omega \left(\frac{h}{g}\right)^{1/2} = (kh \tanh kh)^{1/2} \,.$$

The dimensionless frequency $\omega(h/g)^{1/2}$ and the dimensionless wavenumber kh vary as shown in Fig. 1.1. In particular, the limiting approximations are

$$\begin{aligned} \omega &\simeq (gh)^{1/2} k \,, \quad kh \ll 1 \,, \\ \omega &\simeq (gk)^{1/2} \,, \quad kh \gg 1 \,. \end{aligned} \tag{1.4.10}$$

Since $kh = 2\pi h/\lambda$ is essentially the depth-to-wavelength ratio, the terms *long waves* and *shallow-water waves* refer to $kh \ll 1$, while *short waves* and *deep-water waves* refer to $kh \gg 1$. For fixed h, shorter waves have higher frequencies. In shallow water, waves of a fixed frequency have shorter length in smaller depth since $k \simeq \omega/(gh)^{1/2}$.

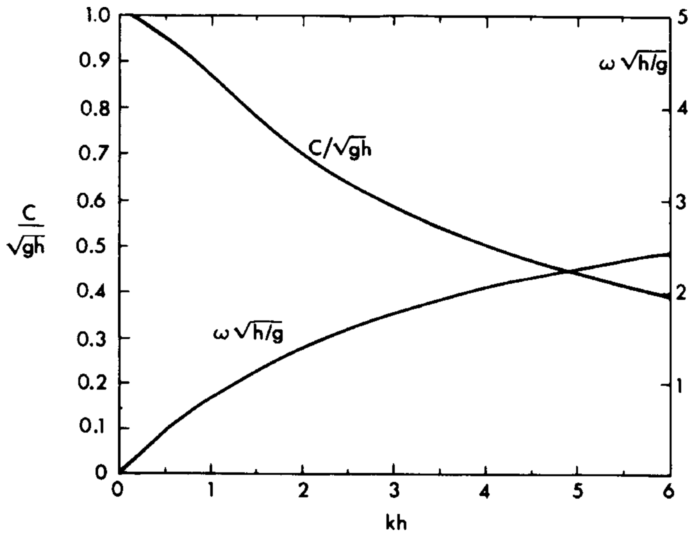

Figure 1.1: Dispersion curves for a progressive wave.

The phase speed C is given by

$$C = \frac{\omega}{k} = \left(\frac{g}{k}\tanh kh\right)^{1/2}, \tag{1.4.11}$$

which is plotted in dimensionless form in Fig. 1.1. For long and short waves the limiting relations are

$$\begin{aligned} C &= (gh)^{1/2}, \quad kh \ll 1, \\ C &= (g/k)^{1/2}, \quad kh \gg 1. \end{aligned} \tag{1.4.12}$$

In general, for the same depth, longer waves have faster speeds. It will be shown in Chapter Two that a localized initial disturbance can be thought of as a Fourier superposition of periodic disturbances with wavelengths ranging over a continuous spectrum. As time passes the longer waves lead the shorter waves. As the disturbances propagate outward the longest and shortest waves become further and further apart with intermediate waves marching in between. The phenomenon that waves of different frequencies travel at different speeds is called *dispersion*. Clearly, if the relation between ω and k for a sinusoidal wave is nonlinear, the medium is dispersive. Equation (1.4.8) or its equivalent, Eq. (1.4.11), is therefore called the *dispersion relation*.

From the linearized Bernoulli equation the dynamic pressure (without $-\rho g z$) is

$$\frac{p}{\rho} = i\omega\phi = gA\frac{\cosh k(z+h)}{\cosh kh}e^{ikx} = g\eta\frac{\cosh k(z+h)}{\cosh kh}\,. \tag{1.4.13}$$

The velocity field is

$$u = \frac{gkA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}e^{ikx}\,, \tag{1.4.14}$$

$$v = 0\,, \tag{1.4.15}$$

$$w = -\frac{igkA}{\omega}\frac{\sinh k(z+h)}{\cosh kh}e^{ikx}\,. \tag{1.4.16}$$

For very deep water, $kh \gg 1$,

$$(\phi, u, v, w, p) = \left(-\frac{ig}{\omega}, \frac{gk}{\omega}, 0, -\frac{igk}{\omega}, \rho g\right) A e^{kz} e^{ikx}\,, \tag{1.4.17}$$

and for very shallow water, $kh \ll 1$,

$$(\phi, u, v, w, p) = \left(-\frac{ig}{\omega}, \frac{gk}{\omega}, 0, 0, \rho g\right) A e^{ikx}\,. \tag{1.4.18}$$

Several distinctive features of the shallow-water results deserve mentioning: (i) The dependence on z disappears; (ii) the vertical velocity is negligible; and (iii) the dynamic pressure is $\rho g\eta$ and the total pressure $P = \rho g(\zeta - z)$, which is hydrostatic in terms of depth below the free surface.

Finally, we know from Section 1.2 that when the spatial scale is $1/k$, the condition for linearization is $kA \ll 1$. Let us check the linearizing assumption again by comparing a nonlinear term with a linear term, both evaluated at the free surface $z = 0$. For arbitrary kh, we have from Eqs. (1.4.11) and (1.4.14)

$$\left(\frac{u\partial u/\partial x}{\partial u/\partial t}\right)_{z=0} \sim \left(\frac{uk}{\omega}\right)_{z=0} \sim \left(\frac{u}{C}\right)_{z=0} = \frac{kA}{\tanh kh} \quad \text{for all } kh\,.$$

Note that for $kh \ll 1$, the above ratio becomes A/h. Therefore, in shallow water the linearized theory is indeed a very restricted approximation.

1.5 Group Velocity

One of the most important concepts in dispersive waves is the group velocity, for which two views may be examined to understand its significance.

1.5.1 A Kinematic View

Suppose that there is a group of sinusoidal waves with a continuous but narrow range of wavelengths centered around $k = k_0$. The free-surface displacement may be represented by

$$\zeta = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) e^{i[kx-\omega(k)t]}\, dk\,, \qquad \frac{\Delta k}{k_0} \ll 1\,, \tag{1.5.1}$$

where $A(k)$ is the wavenumber spectrum with ω and k satisfying the dispersion relation

$$\omega = \omega(k)\,. \tag{1.5.2}$$

By Taylor expansion we write

$$\omega = \omega[k_0 + (k-k_0)] = \omega(k_0) + (k-k_0)\left(\frac{d\omega}{dk}\right)_{k_0} + O(k-k_0)^2\,.$$

Denoting

$$\frac{k-k_0}{k_0} = \xi\,, \quad \omega_0 = \omega(k_0)\,, \quad \text{and} \quad \left(\frac{d\omega}{dk}\right)_0 = \left(\frac{d\omega}{dk}\right)_{k_0} \equiv C_g\,, \tag{1.5.3}$$

we have for sufficiently smooth $A(k)$ and to the crudest approximation

$$\begin{aligned}\zeta &\simeq A(k_0) e^{i(k_0x-\omega_0 t)} \int_{-\Delta k/k_0}^{\Delta k/k_0} \{\exp[ik_0\xi(x-C_g t)]\} k_0\, d\xi \\ &= 2A(k_0)\frac{\sin \Delta k(x-C_g t)}{(x-C_g t)} e^{i(k_0x-\omega_0 t)} = \tilde{A} e^{i(k_0x-\omega_0 t)}\,,\end{aligned} \tag{1.5.4}$$

where

$$\tilde{A} = 2A(k_0)\frac{\sin \Delta k(x-C_g t)}{(x-C_g t)}\,. \tag{1.5.5}$$

Because of the factor $\exp[i(k_0x-\omega_0 t)]$ in Eq. (1.5.4), ζ may be viewed as a locally sinusoidal wavetrain with a slowly modulated amplitude $\tilde{A}$. In particular, the envelope defined by $\tilde{A}$ is in the form of wave groups as shown in Fig. 1.2 and advances at the speed C_g. Therefore, C_g is called the *group*

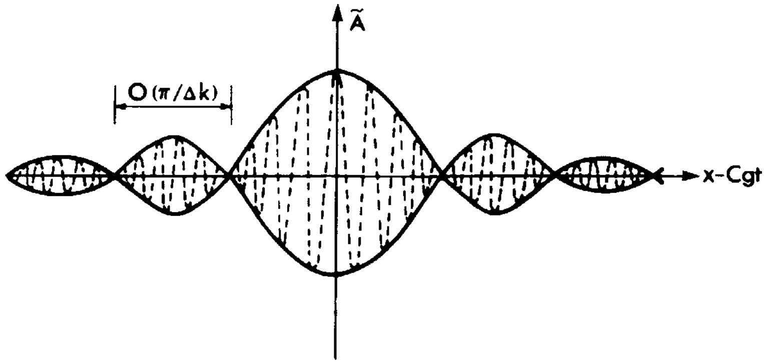

Figure 1.2: A group of waves within a narrow frequency band.

velocity. The distance between two adjacent nodes of an envelope, hence the modulation length scale of the amplitude, is roughly $\pi/\Delta k$ and is much greater than the length of the constituent waves $2\pi/k_0$.

For water waves on a constant depth, it follows by differentiating the dispersion relation (1.4.8) that

$$C_g = \frac{d\omega}{dk} = \frac{1}{2}\frac{\omega}{k}\left(1 + \frac{2kh}{\sinh 2kh}\right) = \frac{C}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{1.5.6}$$

For deep water $kh \gg 1$,

$$C_g \simeq \frac{1}{2}C \simeq \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}, \tag{1.5.7}$$

while for shallow water, $kh \ll 1$,

$$C_g \simeq C \simeq (gh)^{1/2}. \tag{1.5.8}$$

Since the phase velocity exceeds the group velocity for general depths, the individual wave crests travel from the tail toward the front of a group.

It will be shown more generally in Section 2.4 that C_g is the velocity of any slowly varying envelope, Eq. (1.5.5) being a special case.

1.5.2 A Dynamic View: Energy Flux

Let us first compute the average energy in a uniform progressive wavetrain beneath a unit square of the free surface. Denoting the time average over a period by an overhead bar, we have the kinetic energy in the whole fluid column

$$\text{K.E.} = \overline{\frac{\rho}{2}\int_{-h}^{\zeta} [\mathbf{u}(\mathbf{x},t)]^2\, dz}$$

$$\cong \frac{\rho}{2}\int_{-h}^{0} \overline{\{[\operatorname{Re} u(\mathbf{x})e^{-i\omega t}]^2 + [\operatorname{Re} w(\mathbf{x})e^{-i\omega t}]^2\}}\, dz\,, \qquad (1.5.9)$$

where for second-order $O(kA)^2$ accuracy the upper limit has been replaced by $z = 0$, and $\mathbf{u}$ may be approximated by the first-order result, Eqs. (1.4.14) and (1.4.16). Note that for any two sinusoidal functions,

$$a = \operatorname{Re} Ae^{-i\omega t} \quad \text{and} \quad b = \operatorname{Re} Be^{-i\omega t}\,,$$

the following formula is true:

$$\overline{ab} = \frac{1}{T}\int_0^T dt\, ab = \frac{1}{2}\operatorname{Re}(AB^*) = \frac{1}{2}\operatorname{Re}(A^*B)\,, \qquad (1.5.10)$$

where $(\ )^*$ denotes the complex conjugate. The proof is left as an exercise. With Eqs. (1.4.14), (1.4.16), and (1.5.10), Eq. (1.5.9) becomes

$$\text{K.E.} = \frac{\rho}{4}\left(\frac{gk|A|^2}{\omega}\right)^2 \frac{1}{\cosh^2 kh}\int_{-h}^{0}[\cosh^2 k(z+h) + \sinh^2 k(z+h)]\, dz$$

$$= \frac{\rho}{4}\left(\frac{gk|A|}{\omega}\right)^2 \frac{\sinh 2kh}{2k\cosh^2 kh} = \frac{1}{4}\rho g|A|^2\,, \qquad (1.5.11)$$

where use is made of the following formula:

$$\int_0^{kh} \cosh^2\xi\, d\xi = \frac{1}{4}(\sinh 2kh + 2kh)\,, \qquad (1.5.12)$$

and the dispersion relation. On the other hand, the potential energy in the fluid column due to wave motion is

$$\text{P.E.} = \overline{\int_0^{\zeta} \rho g z\, dz} = \frac{1}{2}\rho g\overline{\zeta^2} = \frac{1}{4}\rho g|A|^2 \qquad (1.5.13)$$

since $\rho g\, dz$ is the weight of a thin horizontal slice whose height above the mean-free surface is z. The total energy is

$$E = \text{K.E.} + \text{P.E.} = \frac{1}{2}\rho g|A|^2. \qquad (1.5.14)$$

Note that the kinetic and potential energies are equal; this property is called the *equipartition of energy*. Let us consider a vertical cross section of unit width along the crest. The rate of energy flux across this section is equal to the mean rate of work done by the dynamic pressure, that is,

$$\begin{matrix}\text{Rate of} \\ \text{energy flux}\end{matrix} = \begin{matrix}\text{Rate of} \\ \text{pressure working}\end{matrix} = \overline{\int_{-h}^{\zeta} p(\mathbf{x},t)u(\mathbf{x},t)\,dz} \cong -\rho \int_{-h}^{0} \overline{\Phi_t \Phi_x}\,dz\,, \tag{1.5.15}$$

which can be calculated to be

$$\text{Rate of energy flux} = -\frac{1}{2}\rho g A^2 \left[\frac{1}{2}\frac{\omega}{k}\left(1 + \frac{2kh}{\sinh 2kh}\right)\right] = EC_g\,. \tag{1.5.16}$$

Hence the group velocity has the dynamical meaning of the velocity of energy transport. In contrast, the phase speed is merely a kinematic quantity and is not always identifiable with the transport of any dynamical substance.

As an immediate application, consider a long wave tank of unit width with sinusoidal waves generated at one end. Many periods after the start of the wavemaker, the envelope is uniform almost everywhere except near the wave front which may look like Fig. 1.3. Since the rate of energy input by the wavemaker at the left (say at $x = 0$) is EC_g, the rate of lengthening of the wave region must be C_g. Thus the wave front must propagate at the group velocity. Details of the wave-front evolution will be discussed in Section 2.4.

Exercise 1.1

Consider a two-layered fluid system over a horizontal bottom. The lighter fluid above has the density of ρ, while the heavier fluid below has the density of ρ'. Let the free surface be at $z = 0$, the interface at $z = -h$, and the

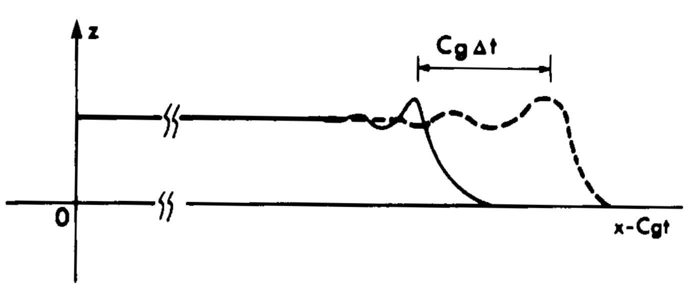

Figure 1.3: The envelope front of a sinusoidal wavetrain.

bottom at $z = -h'$. Show that a sinusoidal progressive wave must satisfy the dispersion relation:

$$\left(\frac{\omega^2}{gk}\right)^2 \{\rho' \coth kh \coth k(h'-h) + \rho\}$$

$$-\frac{\omega^2}{gk}\rho'\{\coth kh + \coth k(h'-h)\} + \rho' - \rho = 0\,.$$

Study the two possible modes corresponding to the two solutions ω_1^2 and ω_2^2 for the same k.

In particular, when $h' \sim \infty$ show that

$$\omega_1^2 = gk \quad \text{and} \quad \omega_2^2 = gk\frac{\rho'-\rho}{\rho' \coth kh + \rho} < \omega_1^2$$

and that the amplitude ratio of interface to free surface is

$$e^{-kh} \quad \text{and} \quad -\frac{\rho}{\rho'-\rho}e^{kh}$$

for the first and second mode, respectively. Plot the group velocity as a function of k for each mode.

Exercise 1.2: Capillary Waves

Surface tension on the free surface introduces a pressure difference between the atmospheric pressure P_a above and the water pressure P below. The difference is given by the Laplace formula (see, e.g., Landau and Lifshitz, 1959, p. 237 ff)

$$P - P_a \cong -T(\zeta_{xx} + \zeta_{yy})\,, \quad \text{on} \quad z \cong 0\,, \tag{1.5.17}$$

where the right-hand side is proportional to the surface curvature and T is the surface tension coefficient. For the water–air interface at 20°C, $T = 74$ dyn/cm in cgs units. Reformulate the boundary conditions on the free surface and study a plane progressive wave on deep water: $\Phi \propto e^{kz}e^{i(kx-\omega t)}$. Show that

$$\omega^2 = gk + \frac{Tk^3}{\rho}\,.$$

Show further that the phase velocity has a minimum value C_m which satisfies

$$\frac{C^2}{C_m^2} = \frac{1}{2}\left(\frac{\lambda}{\lambda_m} + \frac{\lambda_m}{\lambda}\right) = \frac{1}{2}\left(\frac{k_m}{k} + \frac{k}{km}\right),$$

where

$$\lambda_m = \frac{2\pi}{k_m} = 2\pi \left(\frac{T}{g\rho}\right)^{1/2} .$$

What are the numerical values of λ_m and C_m for water and air?

Discuss the variations of ω, C and C_g versus k or λ.