Chapter 1

Regularly and singularly perturbed systems

The main purpose of this chapter is to briefly explain some preliminary mathematical results concerning the properties and analysis of perturbed differential equations. These are used throughout the book as background for a technique of approximate analysis and design of nonlinear control systems. In particular, the main notions of two-time analysis, as well as the conditions for the stability of regularly and singularly perturbed differential equations, are introduced. Quantitative criteria for degree of time-scale separation between fast and slow motions are considered.

1.1 Regularly perturbed systems

1.1.1 *Nonlinear nominal system*

Let us consider an autonomous (time-invariant) dynamical system given by

$$\dot{X} = f(X) + \mu g(X), \tag{1.1}$$

where

X is the state of the system (1.1), $X \in \mathbb{R}^n$, $X = \{x_1, x_2, \ldots, x_n\}^T$;

f and g are continuous functions of X on Ω_X;

Ω_X is an open bounded subset of $\mathbb{R}^n$;

μ is a positive small parameter.

Taking $\mu = 0$ in (1.1) we obtain the system

$$\dot{X} = f(X), \tag{1.2}$$

which is called the nominal system. The system (1.1) is called a perturbation or perturbed system of the nominal system (1.2).

First, let us make some assumptions regarding the properties of the nominal system.

Let $0 \in \Omega_X \subset \mathbb{R}^n$ and let $X = 0$ be an equilibrium point of (1.2), i.e., $f(X)|_{X=0} = 0$. Let us assume that a Lyapunov function $V(X)$ exists such that the inequalities

$$c_1\|X\|^2 \le V(X) \le c_2\|X\|^2, \tag{1.3}$$

$$\dot{V}(X) = \frac{\partial V}{\partial X} f(X) \le -c_3\|X\|^2, \tag{1.4}$$

$$\left\|\frac{\partial V}{\partial X}\right\| \le c_4\|X\| \tag{1.5}$$

are satisfied for all $X \in \Omega_X$, where c_i are some positive constants and

$$\frac{\partial V}{\partial X} = \left\{\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n}\right\}$$

is a row vector.

From (1.3) the inequalities

$$\frac{V(X)}{c_2} \le \|X\|^2 \le \frac{V(X)}{c_1} \tag{1.6}$$

result. Then from (1.4) and (1.6) we have

$$\dot{V}(X) \le -c_3\|X\|^2 \le -\frac{c_3}{c_2} V(X). \tag{1.7}$$

Consequently

$$\int_0^t \frac{dV}{V} = \ln \frac{V(X(t))}{V(X(0))} \le -\frac{c_3}{c_2} t$$

and

$$V(X(t)) \le V(X(0)) \exp\left(-\frac{c_3}{c_2} t\right), \tag{1.8}$$

where t denotes the time variable.

In accordance with (1.3), (1.6), and (1.8), we have

$$\begin{aligned} \|X(t)\| &\le \frac{V^{1/2}(X(t))}{c_1^{1/2}} \le \frac{V^{1/2}(X(0))}{c_1^{1/2}} \exp\left(-\frac{c_3}{2c_2} t\right) \\ &\le \left[\frac{c_2}{c_1}\right]^{1/2} \|X(0)\| \exp\left(-\frac{c_3}{2c_2} t\right). \end{aligned} \tag{1.9}$$

From (1.9) it follows that

$$\lim_{t \to \infty} X(t) = 0$$

and, moreover, that $X = 0$ is the exponentially stable equilibrium point of (1.2). The result may be formulated as a theorem.

Theorem 1.1 *Let $X = 0$ be an equilibrium point for system (1.2), and suppose the Lyapunov function $V(X)$ exists such that the conditions (1.3) and (1.4) are satisfied. Then the origin of the system (1.2) is exponentially stable.*

1.1.2 *Linear nominal system*

Let us consider a linear time-invariant (LTI) dynamical system of the form

$$\dot{X} = AX, \tag{1.10}$$

where

(i) A is an $n \times n$ real matrix;
(ii) $\det(A) \neq 0$ and so $X = 0$ is the isolated equilibrium point of (1.10);
(iii) $\operatorname{Re} \lambda_i(A) < 0$, $\forall\, i = 1, \ldots, n$ and so A is a stability matrix[1] (Hurwitz matrix.)

Let us consider a quadratic Lyapunov function

$$V(X) = X^T P X, \tag{1.11}$$

where P is a real symmetric positive definite matrix and P is the unique solution of the Lyapunov equation

$$PA + A^T P = -Q \tag{1.12}$$

for the given real symmetric positive definite matrix Q.

[1] $\operatorname{Re} \lambda_i(A)$ is the real part of the eigenvalue λ_i of A.

Then the inequalities (1.3), (1.4), (1.5) appropriate to (1.10), (1.11) may be rewritten in the following form:

$$\lambda_{\min}(P)\|X\|_2^2 \le V(X) \le \lambda_{\max}(P)\|X\|_2^2, \tag{1.13}$$

$$-\lambda_{\max}(Q)\|X\|_2^2 \le \dot{V}(X) = \frac{\partial V}{\partial X}AX$$
$$= -X^TQX \le -\lambda_{\min}(Q)\|X\|_2^2, \tag{1.14}$$

$$\left\|\frac{\partial V}{\partial X}\right\|_2 = \|2X^TP\|_2 \le 2\|P\|_2\|X\|_2 = 2\lambda_{\max}(P)\|X\|_2. \tag{1.15}$$

Consequently, the inequality (1.7) may be rewritten as

$$\dot{V}(X) \le -\lambda_{\min}(Q)\|X\|_2^2 \le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(X). \tag{1.16}$$

Then from (1.16) an upper bound for the Lyapunov function $V(X)$ follows:

$$V(X(t)) \le V(X(0))\exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t\right).$$

Therefore, instead of (1.9), from the above we have

$$\|X(t)\|_2 \le \left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]^{1/2}\|X(0)\|_2\exp\left(-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t\right) \tag{1.17}$$

as an upper bound for the norm of the function $X(t)$.

Note that the ratio $\lambda_{\min}(Q)/\lambda_{\max}(P)$ is maximized if $Q = I$ (see in [Patel and Toda (1980)], [Khalil (2002), p. 372]).

Similarly, from (1.13) and (1.14) it follows that

$$\dot{V}(X) \ge -\lambda_{\max}(Q)\|X\|_2^2 \ge -\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}V(X)$$

is a lower bound for the derivative $\dot{V}(X)$ of the Lyapunov function $V(X)$ with respect to t, and hence

$$V(X(t)) \ge V(X(0))\exp\left(-\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}t\right) \tag{1.18}$$

is a lower bound for the Lyapunov function $V(X)$. Finally, from (1.13) and (1.18), we find that

$$\|X(t)\|_2 \ge \left[\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}\right]^{1/2}\|X(0)\|_2\exp\left(-\frac{\lambda_{\max}(Q)}{2\lambda_{\min}(P)}t\right) \tag{1.19}$$

is a lower bound for the norm of the solution $X(t)$ of the linear system (1.10).

1.1.3 *Vanishing perturbation*

Let us consider the system (1.1), where it is assumed that the above assumptions regarding the function f are satisfied and, moreover, that g is an unknown continuous function of X on Ω_X and $c_5 > 0$ exists such that the condition

$$\|g(X)\| \leq c_5\|X\|, \quad \forall\, X \in \Omega_X \tag{1.20}$$

holds.

From (1.20) follows that $g(X)|_{X=0} = 0$, and so the perturbation vanishes completely at the equilibrium point.

Obviously, the time derivative of $V(X)$ along trajectories of (1.1) is given by

$$\dot{V}(X) = \frac{\partial V}{\partial X} f(X) + \mu \frac{\partial V}{\partial X} g(X). \tag{1.21}$$

In accordance with the above assumption, (1.4)–(1.5), and (1.20), it is easy to see that

$$\dot{V}(X) \leq -c_3\|X\|^2 + \mu \left\|\frac{\partial V}{\partial X}\right\| \|g(X)\| \leq -(c_3 - \mu c_4 c_5)\|X\|^2. \tag{1.22}$$

As a result, if the inequality

$$0 < \mu < \frac{c_3}{c_4 c_5} \tag{1.23}$$

is satisfied, then

$$c_3 - \mu c_4 c_5 > 0$$

and, accordingly, we have

$$\dot{V}(X) < 0, \quad \forall\, X \neq 0, \quad \forall X \in \Omega_X. \tag{1.24}$$

From (1.3), (1.22), and (1.24) it follows that the origin is an exponentially stable equilibrium point of the perturbed system (1.1) if the parameter μ is small enough. So the result may be formulated as a theorem.

Theorem 1.2 *Let the origin of the nominal system (1.2) be an exponentially stable equilibrium point, and suppose the requirement (1.20) for the* *continuous function g holds. Then there exists* $\mu^\star > 0$ *such that for all* $\mu \in (0, \mu^\star)$, *the origin of perturbed system (1.1) is exponentially stable.*

Instead of (1.1), let us consider the system given by

$$\dot{X} = AX + \mu g(X), \tag{1.25}$$

which is the perturbation of the stable linear nominal system (1.10). Then the inequality (1.22) appropriate to (1.25) may be rewritten as

$$\dot{V}(X) \leq -\lambda_{\min}(Q)\|X\|_2^2 + 2\mu\lambda_{\max}(P)c_5\|X\|_2^2$$

and, from (1.23), the inequality

$$0 < \mu < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)c_5}$$

follows, where P is the solution of the Lyapunov equation (1.12).

1.1.4 *Nonvanishing perturbation*

Instead of (1.1), let us consider the perturbed system given by

$$\dot{X} = f(X) + \mu g(X, w), \tag{1.26}$$

where the above assumptions regarding the function f are satisfied and

(i) g is an unknown continuous function of X on Ω_X and of w on Ω_w;
(ii) w serves to represent a vector of external disturbances and varying parameters;
(iii) $w \in \Omega_w$, where Ω_w is a bounded subset of $\mathbb{R}^l$.

When we refer to a nonvanishing perturbation of the nominal system (1.2), we have in mind that

$$\exists\ \bar{w} \in \Omega_w \mid g(X = 0, \bar{w}) \neq 0 \tag{1.27}$$

and that a positive constant c_6 exists such that

$$\|g(X, w)\| \leq c_6, \quad \forall\ X \in \Omega_X \text{ and } \forall\ w \in \Omega_w.$$

Then, in accordance with (1.26), (1.4), and (1.5), the time derivative of $V(X)$ along trajectories of (1.26) can be found using the chain rule. It is given by

$$\dot{V}(X) = \frac{\partial V}{\partial X} f(X) + \mu \frac{\partial V}{\partial X} g(X, w) \leq -c_3 \|X\|^2 + \mu c_4 c_6 \|X\|$$
$$= -(1-d)c_3 \|X\|^2 + (\mu c_4 c_6 - d c_3 \|X\|)\|X\|.$$

Let $0 < d < 1$. If the inequality

$$\|X\| \geq \frac{\mu c_4 c_6}{d c_3}$$

is satisfied, then

$$\dot{V}(X) \leq -(1-d) c_3 \|X\|^2. \tag{1.28}$$

Hence, some finite time t_1 exists such that the condition (1.28) holds for all $t \in [0, t_1)$. Therefore, similar to (1.9), we have an upper bound for $\|X(t)\|$ on this finite time interval given by

$$\|X(t)\| \leq \left[\frac{c_2}{c_1}\right]^{1/2} \|X(0)\| e^{-\frac{(1-d)c_3}{2c_2} t}, \quad \forall\, 0 \leq t < t_1$$

and an upper bound for $\|X(t)\|$ on infinity defined by the inequality

$$\|X(t)\| \leq \frac{\mu c_4 c_6}{d c_3}, \quad \forall\, t \geq t_1. \tag{1.29}$$

So in the presence of the nonvanishing bounded perturbation discussed above, the solutions of (1.26) are ultimately bounded with an ultimate bound (1.29) that approaches zero as $\mu \to 0$.

Let us reconsider the perturbed system of the form (1.25) with the linear nominal model (1.10) and in the presence of the nonvanishing bounded perturbation (1.27). Then from (1.15) and (1.16) it follows that the inequality (1.29) may be rewritten as

$$\|X(t)\|_2 \leq \frac{2\mu \lambda_{\max}(P) c_6}{d \lambda_{\min}(Q)}, \quad \forall\, t \geq t_1.$$

1.2 Singularly perturbed systems

1.2.1 *Singular perturbation*

Let us consider the following set of differential equations:

$$\dot{X} = f(X, Z), \quad X(0) = X^0, \tag{1.30}$$

$$\mu \dot{Z} = g(X, Z), \quad Z(0) = Z^0, \tag{1.31}$$

where μ is a small positive parameter, $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^m$, and f and g are continuously differentiable functions of X and Z.

The system (1.30)–(1.31) is called the standard singular perturbation model of a finite-dimensional dynamical system.

Let us release Z from the initial condition; then, with $\mu = 0$, the system (1.30)–(1.31) of dimension $n + m$ degenerates into

$$\dot{X} = f(X, Z), \quad X(0) = X^0, \tag{1.32}$$

$$0 = g(X, Z), \tag{1.33}$$

where the system (1.32)–(1.33) has dimension n.

In accordance with the implicit function theorem, assume that

$$\det \left\{ \frac{\partial g(X, Z)}{\partial Z} \right\} \neq 0, \quad \forall\, Z \in \Omega_Z; \tag{1.34}$$

then a function

$$\bar{Z} = h(X) \tag{1.35}$$

exists such that the function (1.35) is an unique solution of the equation $g(X, \bar{Z}) = 0$. Accordingly, the equality

$$g(X, h(X)) = 0, \quad \forall\, X \in \Omega_X$$

holds.

Then the set

$$M = \{(X, Z) \mid g(X, Z) = 0\} \tag{1.36}$$

is an n dimensional manifold in the original $n + m$ dimensional state space and, in accordance with (1.32) and (1.33), the behavior of $X(t)$ on this manifold is described by the reduced system

$$\dot{X} = f(X, h(X)), \quad X(0) = X^0. \tag{1.37}$$

1.2.2 *Two-time-scale motions*

If a pair of functions $X(t), Z(t)$ is such that

$$g(X(t), Z(t)) = 0, \ \forall\, t \geq 0$$

then the equality

$$dg(X(t), Z(t))/dt = 0, \ \forall\, t \geq 0$$

also holds. Accordingly, we have

$$\frac{\partial g}{\partial X}\dot{X} + \frac{\partial g}{\partial Z}\dot{Z} = 0 \tag{1.38}$$

and from (1.34) and (1.38) it follows that the behavior of $Z(t)$ on the manifold (1.36) is described by the equation

$$\dot{Z} = -\left\{\frac{\partial g}{\partial Z}\right\}^{-1}\frac{\partial g}{\partial X}\dot{X}.$$

It follows that on the manifold M the ratio

$$\frac{\|\dot{Z}\|}{\|\dot{X}\|} \leq \left\|\left\{\frac{\partial g}{\partial Z}\right\}^{-1}\frac{\partial g}{\partial X}\right\|$$

is some regular numerical value that depends only on the functions f, g.

At the same time, in accordance with the system of equations (1.30) and (1.31), we find that at an arbitrary point $(X, Z) \notin M$ of the $n + m$ dimensional state space this ratio is given by

$$\frac{\|\dot{Z}\|}{\|\dot{X}\|} = \frac{1}{\mu}\frac{\|g(X,Z)\|}{\|f(X,Z)\|}$$

and depends on the small parameter μ. So if $\mu \to 0$, then beyond the manifold M two-time-scale motions appear in the solutions of the equations (1.30)–(1.31), where Z is a fast changing variable and X is a slow changing variable as shown in Fig. 1.1.

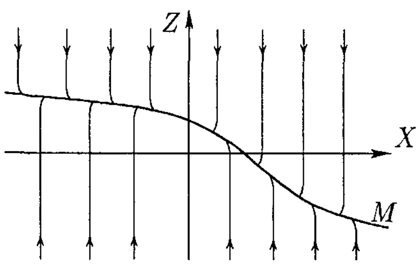

Fig. 1.1 Typical phase portrait in the case of a singularly perturbed system.

1.2.3 Boundary-layer system

Let us introduce a new variable $Y = Z - h(X)$, where Y is a deviation of Z from manifold (1.36). Then the equations (1.30)–(1.31) may be rewritten in the form

$$\frac{dX}{dt} = f(X, Y + h(X)), \quad X(0) = X^0, \tag{1.39}$$

$$\mu\frac{dY}{dt} = g(X, Y + h(X)) - \mu\frac{\partial h}{\partial X}f(X, Y + h(X)),\ Y(0) = Y^0, \tag{1.40}$$

where $Y(0) = Z(0) - h(X(0))$. After introducing a new time scale $t_0 = t/\mu$ into (1.39)–(1.40), we have

$$\frac{dX}{dt_0} = \mu f(X, Y + h(X)), \quad X(0) = X^0, \tag{1.41}$$

$$\frac{dY}{dt_0} = g(X, Y + h(X)) - \mu\frac{\partial h}{\partial X}f(X, Y + h(X)),\ Y(0) = Y^0. \tag{1.42}$$

From (1.41)–(1.42) it is easy to see that in the new time scale t_0 we have $dX/dt_0 \to 0$; that is, the rate of transients of $X(t)$ decreases as $\mu \to 0$. As a result, if μ tends to zero then from (1.41)–(1.42) the equation of a boundary-layer system

$$\frac{dY}{dt_0} = g(X, Y + h(X)), \quad Y(0) = Y^0 \tag{1.43}$$

follows as an asymptotic limit, where X is the frozen variable, i.e., $X \approx$ const.

1.2.4 Stability analysis

The investigation of conditions under which the trajectories of the full singularly perturbed system (1.30)–(1.31) approximate to the trajectories of the reduced model (1.37) is important both from a theoretical viewpoint and for practical applications in control system analysis and design. These conditions were considered in [Tikhonov (1948); Tikhonov (1952)] and [Vasileva (1963)] for a bounded time interval $t \in [0, t_1]$, and then in [Krasovskii (1963); Klimushchev and Krasovskii (1962)] and [Hoppensteadt (1966)] for an infinite time interval $t \in [0, \infty)$.

The simplified version of stability analysis of the singularly perturbed systems is provided below, while more detailed analysis may be found, for instance, in [Khalil (2002)].

Consider the singularly perturbed system (1.30)–(1.31)

$$\dot{X} = f(X, Z), \quad X(0) = X^0,$$
$$\mu \dot{Z} = g(X, Z), \quad Z(0) = Z^0,$$

where the following assumptions are satisfied:

- $f(0,0) = 0, \quad g(0,0) = 0.$
- The equation $g(X, Z) = 0$ has an unique isolated root $\bar{Z} = h(X)$ such that $h(0) = 0$ and

$$\|h(X)\| \leq m_1 \|X\|, \ \forall \ X \in B_{\rho_x} = \{X \in \mathbb{R}^n \mid \|X\| \leq \rho_x\}, \ m_1 > 0.$$

- The functions f, g, and h, along with their partial derivatives up to order 2, are bounded for all $Y = Z - h(X) \in B_{\rho_y}$, where

$$B_{\rho_y} = \{Y \in \mathbb{R}^m \mid \|Y\| \leq \rho_y\}.$$

In addition, we assume that a Lyapunov function $V(X)$ of the reduced system (1.37) exists such that

$$c_1 \|X\|^2 \leq V(X) \leq c_2 \|X\|^2, \tag{1.44}$$
$$\frac{d}{dt} V(X) = \frac{\partial V}{\partial X} f(X, h(X)) \leq -c_3 \|X\|^2, \tag{1.45}$$
$$\left\| \frac{\partial V}{\partial X} \right\| \leq c_4 \|X\|, \tag{1.46}$$

for all $X \in B_{\rho_x}$, where c_i are some positive constants. Therefore, in accordance with Theorem 1.1, the origin of the reduced system (1.37) is exponentially stable.

By introducing the new variable $Y = Z - h(X)$, let us rewrite equations (1.30)–(1.31) in the form (1.39)–(1.40) and consider the boundary-layer system (1.43). Assume that a Lyapunov function $W(Y)$ of (1.43) exists such that

$$b_1 \|Y\|^2 \leq W(Y) \leq b_2 \|Y\|^2, \tag{1.47}$$
$$\frac{d}{dt_0} W(Y) = \frac{\partial W}{\partial Y} g(X, Y + h(X)) \leq -b_3 \|Y\|^2, \tag{1.48}$$
$$\left\| \frac{\partial W}{\partial Y} \right\| \leq b_4 \|Y\|, \tag{1.49}$$

for all $Y \in B_{\rho_y}$, where the b_i are positive constants. Hence, by Theorem 1.1, the origin of the boundary-layer system (1.43) is exponentially stable.

Let us consider the function

$$\nu(X,Y) = (1-d)V(X) + dW(Y) \tag{1.50}$$

as a Lyapunov function candidate for the singularly perturbed system (1.39)–(1.40), where $0 < d < 1$. Then the derivative of (1.50) along the trajectories of (1.39)–(1.40) is given by

$$\frac{d\nu}{dt} = (1-d)\frac{\partial V}{\partial X}f(X, Y + h(X))$$
$$+\frac{d}{\mu}\frac{\partial W}{\partial Y}g(X, Y + h(X)) - d\frac{\partial W}{\partial Y}\frac{\partial h}{\partial X}f(X, Y + h(X)). \tag{1.51}$$

Because the function f and its partial derivatives up to order 2 are bounded for all $Y \in B_{\rho_y}$, and because $f(0,0) = 0$, the Taylor expansion of $f(X, Y + h(X))$ yields

$$f(X, Y + h(X)) = f(X, h(X)) + \frac{\partial f}{\partial Y}Y + O(\|Y\|^2),$$

where

$$\|f(X, Y + h(X))\| \le l_0\|X\|, \quad \left\|\frac{\partial f}{\partial Y}\right\| \le l_1, \quad \|O(\|Y\|^2)\| \le l_2\|Y\|^2,$$

and the l_i are some positive constants.

By taking into account the above assumptions, we obtain from (1.51) the inequality

$$\frac{d\nu}{dt} \le -(1-d)c_3\|X\|^2 + [(1-d)c_4l_1 + db_4m_1l_0]\|X\|\,\|Y\|$$
$$+\left[(1-d)c_4l_2\|X\| + db_4m_1l_1 + db_4m_1l_2\|Y\| - \frac{d}{\mu}b_3\right]\|Y\|^2. \tag{1.52}$$

Then (1.52) can be represented as

$$\frac{d\nu}{dt} \le -\eta^T\Gamma\eta,$$

where $\eta = \{\|X\|, \|Y\|\}^T$ and

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

in which

$$\gamma_{11} = (1-d)c_3,$$
$$\gamma_{12} = \gamma_{21} = -0.5[(1-d)c_4 l_1 + d b_4 m_1 l_0],$$
$$\gamma_{22} = \frac{d}{\mu} b_3 - (1-d)c_4 l_2 \|X\| - d b_4 m_1 l_1 - d b_4 m_1 l_2 \|Y\|.$$

Since $X \in B_{\rho_x}, Y \in B_{\rho_y}$, and $0 < d < 1$, there exists some small $\mu = \mu^\star > 0$ such that the matrix Γ is positive definite:

$$\Gamma > 0.$$

Then

$$\lambda_{\min}(\Gamma)\|\eta\|_2^2 \leq \eta^T \Gamma \eta \leq \lambda_{\max}(\Gamma)\|\eta\|_2^2.$$

From (1.44) and (1.47) it follows that some constants d_1, d_2 exist such that

$$d_1\|\eta\|^2 \leq \nu(\eta) \leq d_2\|\eta\|^2.$$

As a result we have

$$\frac{d\nu}{dt} \leq -\eta^T \Gamma \eta \leq -\lambda_{\min}(\Gamma)\|\eta\|_2^2 \leq -\frac{\lambda_{\min}(\Gamma)}{d_2}\nu.$$

Hence

$$\nu(X(t), Y(t)) \leq \nu(X(0), Y(0)) \exp\left(-\frac{\lambda_{\min}(\Gamma)}{d_2}t\right)$$

and, accordingly, we obtain

$$\|\eta(t)\| \leq \left[\frac{d_2}{d_1}\right]^{1/2} \|\eta(0)\| \exp\left(-\frac{\lambda_{\min}(\Gamma)}{2d_2}t\right).$$

Note that $\|h(X)\| \leq m_1\|X\|$ and $Y = Z - h(X)$; then there exists $m > 0$ such that

$$\|\hat{\eta}(t)\| \leq m\|\hat{\eta}(0)\| \exp\left(-\frac{\lambda_{\min}(\Gamma)}{2d_2}t\right),$$

where $\hat{\eta} = \{\|X\|, \|Z\|\}^T$ or, in other words, the origin of (1.30)–(1.31) is exponentially stable. The result may be formulated as the following theorem.

Theorem 1.3 *Consider the singularly perturbed system (1.30)–(1.31)*

$$\dot{X} = f(X, Z), \quad X(0) = X^0,$$
$$\mu\dot{Z} = g(X, Z), \quad Z(0) = Z^0,$$

under the following assumptions.

- $f(0,0) = 0, \quad g(0,0) = 0.$
- *The equation* $g(X, Z) = 0$ *has a unique isolated root* $\bar{Z} = h(X)$ *such that* $h(0) = 0$ *and* $\|h(X)\| \leq m_1\|X\|$, *where*

$$m_1 > 0, \ X \in B_{\rho_x}, \ B_{\rho_x} = \{X \in \mathbb{R}^n \mid \|X\| \leq \rho_x\}.$$

- *The functions* f, g, h *and their partial derivatives up to order 2 are bounded for all* $Y = Z - h(X) \in B_{\rho_y}$, *where*

$$Y \in B_{\rho_y}, \ B_{\rho_y} = \{Y \in \mathbb{R}^m \mid \|Y\| \leq \rho_y\}.$$

- *The Lyapunov function* $V(X)$ *of the reduced system (1.37) exists such that (1.44)–(1.46) are satisfied for all* $X \in B_{\rho_x}$.
- *The Lyapunov function* $W(Y)$ *of the boundary-layer system (1.43) exists such that (1.47)–(1.49) are satisfied for all* $Y \in B_{\rho_y}$.

Then there exists $\mu^\star > 0$ *such that for all* $\mu \in (0, \mu^\star)$, *the origin of (1.30)–(1.31) is exponentially stable.*

1.2.5 *Fast and slow-motion subsystems*

The above procedure for obtaining the boundary-layer system may be directly applied to (1.30)–(1.31) in order to obtain equations of fast-motion subsystem (FMS) and slow-motion subsystem (SMS). First, by introducing the new time scale $t_0 = t/\mu$ into (1.30)–(1.31) we have

$$\frac{dX}{dt_0} = \mu f(X, Z), \quad X(0) = X^0,$$
$$\frac{dZ}{dt_0} = g(X, Z), \quad Z(0) = Z^0,$$

where the FMS is given by

$$\frac{dZ}{dt_0} = g(X, Z), \quad Z(0) = Z^0 \tag{1.53}$$

in the new time scale t_0 and $X(t_0) \approx$ const during the transients in the subsystem (1.53).

By returning to the primary time scale t, from (1.53) the FMS equation

$$\mu\frac{dZ}{dt} = g(X,Z), \quad Z(0) = Z^0 \tag{1.54}$$

is obtained, where $X(t)$ is the frozen variable, i.e., $X(t) \approx$ const.

Second, let us assume that there is an unique equilibrium point (1.35) of (1.54) (more precisely, quasi-equilibrium point) that satisfies $g(X,Z) = 0$. Moreover, we assume that $Z = \bar{Z}$ is an exponentially stable equilibrium point of (1.54).

Finally, on the above assumption of exponential stability of the equilibrium point $Z = \bar{Z}$, we have that $Z(t) - \bar{Z} \to 0$, $\forall\ t > 0$ as $\mu \to 0$. So if the parameter μ is small enough, then after rapid decay of transients in the FMS (1.54) we find that the condition $Z = \bar{Z}$ is satisfied. Substitution of $Z = h(X)$ into (1.30) yields the SMS equation (1.37).

1.2.6 *Degree of time-scale separation*

Let us consider a linear standard singularly perturbed system

$$\dot{X} = A_{11}X + A_{12}Y, \tag{1.55}$$

$$\mu\dot{Y} = A_{21}X + A_{22}Y, \tag{1.56}$$

where μ is a small positive parameter, $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$, and the A_{ij} are matrices with appropriate dimensions.

In accordance with the above formal algorithm of time-scale separation, we have that

$$\mu\dot{Y} = A_{21}X + A_{22}Y \tag{1.57}$$

is the FMS equation, where $X =$ const.

Assume that $\det A_{22} \neq 0$ and, moreover, A_{22} is a Hurwitz matrix. Then it is easy to find that

$$\dot{X} = A_s X \tag{1.58}$$

is the SMS equation, where

$$A_s = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

and we assume that A_s is a Hurwitz matrix as well.

From a practical standpoint it is useful to have some quantitative criteria for the degree of time-scale separation between stable fast and slow motions.

The ratio

$$\eta = t_{s,SMS}/t_{s,FMS} \tag{1.59}$$

serves as a direct estimation of such a degree, where $t_{s,SMS}$ and $t_{s,FMS}$ are the settling times of the SMS and the FMS, respectively. We may also consider indirect estimates of the degree of time-scale separation between stable fast and slow motions, where the first estimate is based on solution of the Lyapunov equation and the second one is based on roots of the FMS and SMS characteristic polynomials.

Estimate based on solution of Lyapunov equation

The lower and upper bounds (1.17) and (1.19) of the linear differential equation solution may be used to introduce a quantitative criterion for degree of time-scale separation between stable fast and slow motions.

First, because the FMS (1.57) is stable, the Lyapunov function $V_F(Y) = Y^T P_F Y$ of the FMS (1.57) may be obtained by solving the Lyapunov equation

$$P_F A_{22} + A_{22}^T P_F = -Q_F, \tag{1.60}$$

where $Q_F = Q_F^T$, $Q_F > 0$, $P_F = P_F^T$, and $P_F > 0$. Then, in accordance with (1.17), the upper bound of the fast variable Y follows:

$$\|Y(t)\|_2 \leq \left[\frac{\lambda_{\max}(P_F)}{\lambda_{\min}(P_F)}\right]^{1/2} \|Y(0)\|_2 \exp\left(-\frac{\lambda_{\min}(Q_F)}{2\mu\lambda_{\max}(P_F)}t\right). \tag{1.61}$$

Next, since the SMS (1.58) is stable, the Lyapunov function $V_S(X) = X^T P_S X$ of the SMS (1.58) may be obtained by solving the Lyapunov equation

$$P_S A_S + A_S^T P_S = -Q_S,$$

where $Q_S = Q_S^T$, $Q_S > 0$, and $P_S = P_S^T, P_S > 0$. Then, in accordance with (1.19), the lower bound of the slow variable X follows:

$$\|X(t)\|_2 \geq \left[\frac{\lambda_{\min}(P_S)}{\lambda_{\max}(P_S)}\right]^{1/2} \|X(0)\|_2 \exp\left(-\frac{\lambda_{\max}(Q_S)}{2\lambda_{\min}(P_S)}t\right). \tag{1.62}$$

The ratio of the exponents in (1.61) and (1.62),

$$\eta_1 = \frac{\lambda_{\min}(P_S)\lambda_{\min}(Q_F)}{\mu\lambda_{\max}(P_F)\lambda_{\max}(Q_S)},$$

may be used as the degree of time-scale separation between fast and slow motions. In particular, by choosing $Q_F = Q_S = I$ we obtain[2]

$$\eta_1 = \frac{\lambda_{\min}(P_S)}{\mu\lambda_{\max}(P_F)}. \tag{1.63}$$

Estimate based on roots of FMS and SMS characteristic polynomials

Let us consider stable fast and slow subsystems of the linear standard singularly perturbed system (1.55)–(1.56). Denote by

$$A_{FMS}(s,\mu) = \det\left[sI - \frac{1}{\mu}A_{22}\right] = s^m + a_{m-1}^{FMS}\frac{1}{\mu}s^{m-1} + \ldots + \frac{1}{\mu^m}a_0^{FMS}$$

and

$$A_{SMS}(s) = \det[sI - A_S] = s^n + a_{n-1}^{SMS}s^{n-1} + \ldots + a_0^{SMS}$$

the FMS and SMS characteristic polynomials, respectively. Assume that $s_1^{FMS}, \ldots, s_m^{FMS}$ and $s_1^{SMS}, \ldots, s_n^{SMS}$ are the roots of the stable FMS and SMS characteristic polynomials, respectively. Denote

$$\omega_{FMS}^{\min} = \min_{i=1,\ldots,m}|\mathrm{Re}\, s_i^{FMS}|, \quad \omega_{SMS}^{\max} = \max_{i=1,\ldots,n}|\mathrm{Re}\, s_i^{SMS}|.$$

The ratio

$$\eta_2 = \frac{\omega_{FMS}^{\min}}{\omega_{SMS}^{\max}} \tag{1.64}$$

may be used as a criterion for the degree of time-scale separation between fast and slow motions.

We may also consider the ratio of FMS natural frequency to SMS natural frequency

$$\eta_3 = \frac{(a_0^{FMS})^{1/m}}{\mu(a_0^{SMS})^{1/n}} \tag{1.65}$$

as a quantitative criterion for the degree of time-scale separation between stable fast and slow motions instead of (1.63).

The estimate η_1 is more conservative than η_2 and η_3. The direct estimation (1.59) of the degree of time-scale separation between stable fast and

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[2]Here I is the identity matrix. A list of notation used throughout the book appears in Appendix B starting on p. 335.

slow motions can be based on the correlations (2.4) discussed in the next chapter. From (2.4) we get

$$t_{s,FMS} \approx \frac{4}{\omega_{FMS}^{\min}}, \quad t_{s,SMS} \geq \frac{4}{\omega_{SMS}^{\max}}.$$

Therefore, by (1.59), we obtain

$$\eta \geq \frac{\omega_{FMS}^{\min}}{\omega_{SMS}^{\max}}.$$

1.3 Discrete-time singularly perturbed systems

1.3.1 *Fast and slow-motion subsystems*

In this section the discrete-time counterpart of the singularly perturbed system (1.31) is discussed. We will deal with the system of state space difference equations given by

$$X_{k+1} = \{I_n + \mu A_{11}\}X_k + \mu A_{12}Y_k, \tag{1.66}$$

$$Y_{k+1} = A_{21}X_k + A_{22}Y_k, \tag{1.67}$$

where μ is a small parameter, $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$, and the A_{ij} are matrices with appropriate dimensions.

When $\mu = 0$, the system (1.66)–(1.67) of dimension $n + m$ degenerates into the system of dimension m given by

$$X_{k+1} = X_k,$$

$$Y_{k+1} = A_{21}X_k + A_{22}Y_k.$$

So if $\mu \to 0$, then the rate of transients of X_k decreases and, accordingly, the fast and slow modes are revealed in the system (1.66)–(1.67), where a time-scale separation between those modes is represented by the small parameter μ. If μ is sufficiently small, then from (1.66)–(1.67) the FMS equation

$$Y_{k+1} = A_{21}X_k + A_{22}Y_k \tag{1.68}$$

results, where $X_k \approx$ const during the transients in the system (1.68).

The characteristic polynomial of the FMS (1.68) is

$$A_{FMS}(z) = \det(zI_m - A_{22}).$$

Assume that all roots of $A_{FMS}(z)$ lie inside the unit circle so that the FMS (1.68) is stable. Then the steady-state of the FMS is given by

$$Y_k = \{I_m - A_{22}\}^{-1} A_{21} X_k. \tag{1.69}$$

Substitution of (1.69) into (1.66) yields the SMS

$$X_{k+1} = \{I_n + \mu[A_{11} + A_{12}(I_m - A_{22})^{-1} A_{21}]\} X_k,$$

where the characteristic polynomial of the SMS is

$$A_{SMS}(z) = \det(zI_m - A_{SMS}),$$

where

$$A_{SMS} = \{I_n + \mu[A_{11} + A_{12}(I_m - A_{22})^{-1} A_{21}]\}.$$

1.3.2 *Degree of time-scale separation*

Since the complex variables z and s are related by $z = e^{T_s s}$, the inverse mapping of the unit circle into the primary strip in the s-plane is given by

$$s = \frac{1}{T_s} \operatorname{Ln} z, \tag{1.70}$$

where T_s is the sampling period and $\operatorname{Ln} z$ is the principal value of $\ln z$. Here $z = 0$ is omitted and there is a cut along the negative real axis.

Assume that the following conditions are satisfied:

1. $z_1^{FMS}, \ldots, z_m^{FMS}$ and $z_1^{SMS}, \ldots, z_n^{SMS}$ are the roots of the FMS and SMS characteristic polynomials, respectively.
2. All roots lie inside the unit circle as shown in Fig. 1.2(a).
3. There are no roots on the cut or at the origin.

Then, by the mapping (1.70), we can obtain the sets of roots $s_1^{FMS}, \ldots, s_m^{FMS}$ and $s_1^{SMS}, \ldots, s_n^{SMS}$ as shown in Fig. 1.2(b), and construct two polynomials

$$A_{FMS}(s) = \prod_{i=1}^{m} (s - s_i^{FMS}), \qquad A_{SMS}(s) = \prod_{i=1}^{n} (s - s_i^{SMS}),$$

to which the previous criteria can be applied.

If we assume that the last mentioned condition is not satisfied, that is, there is at least one root on the cut or at the origin, then the following approach may be used.

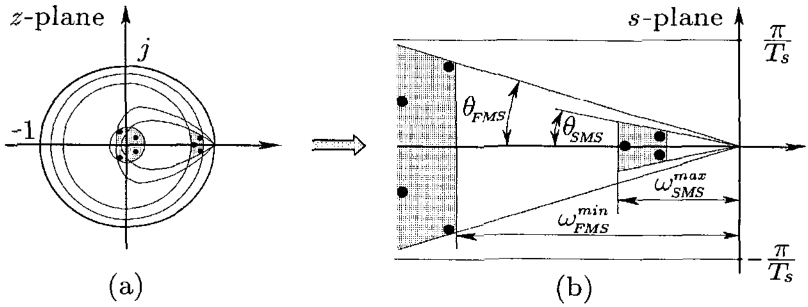

Fig. 1.2 Roots of the stable FMS and SMS characteristic polynomials in the discrete-time system (1.66)–(1.67) and their images in the primary strip on the s-plane.

Denote

$$r_{FMS} = \max_{i=1,\ldots,m} |z_i^{FMS}|, \quad \text{and} \quad r_{SMS} = \min_{i=1,\ldots,n} |z_i^{SMS}|,$$

where we assume that

$$0 < r_{FMS} < r_{SMS} < 1.$$

From (1.64) and (1.70) we obtain

$$\eta_2 = \frac{\ln r_{FMS}}{\ln r_{SMS}}. \tag{1.71}$$

The particular feature of the discrete-time FMS (1.68) is that a lower bound for the settling time exists, which is equal to the settling time of the deadbeat response.[3] If all roots of the FMS characteristic polynomial $A_{FMS}(z)$ are located at the origin, then the settling time of the discrete-time FMS (1.68) is equal to mT_s (the settling time of the deadbeat response for arbitrarily chosen initial conditions). However, from (2.4) and (1.70) we get

$$t_{s,FMS} \approx -\frac{4T_s}{\ln r_{FMS}}, \quad t_{s,SMS} \geq -\frac{4T_s}{\ln r_{SMS}}, \tag{1.72}$$

where $t_{s,FMS} \to 0$ as $r_{FMS} \to 0$. Therefore, from (1.72) the value $\bar{r}_{FMS}$ can be found such that the condition $t_{s,FMS}(\bar{r}_{FMS}) = mT_s$ is satisfied, where

$$\bar{r}_{FMS} = \exp(-4/m).$$

[3]The notion of the deadbeat response can be found, for instance, in [Lindorff (1965); Chen (1993); Ogata (1994)].

So the expressions (1.71) and (1.72) can be used only if the inequality $r_{FMS} > \bar{r}_{FMS}$ holds. If $r_{FMS} \leq \bar{r}_{FMS}$ then, by (1.59), we get

$$\eta \geq -\frac{4}{m \ \ln r_{SMS}}. \tag{1.73}$$

1.4 Notes

In this chapter we have discussed the basic principles for approximate analysis of the properties of the perturbed and singularly perturbed differential equations. The properties of the regularly and singularly perturbed differential equations that we have discussed are used throughout the book as the basis for an approximate analysis and design of nonlinear control systems.

Note that the numerical simulation of singularly perturbed differential equations has some particulars concerning the choice of step size. Usually, the higher order Runge-Kutta algorithms or Agams-Moulton methods allow us to obtain numerically stable solutions without special contrivance if the dimension of the equations is not too high.

There are many references devoted to consideration of particular details concerned with the analysis of regularly and singularly perturbed systems of differential equations. These may be found, for instance, in [Vasileva (1963); Gerashchenko (1975); Kokotović *et al.* (1986); Kokotović and Khalil (1986)] and [Sastry (1999); Khalil (2002)]. Various aspects of discrete-time singularly perturbed systems were considered in [Litkouhi and Khalil (1985); Naidu and Rao (1985); Naidu (1988)].

1.5 Exercises

1.1 The behavior of a dynamical system is described by the equation

$$x^{(2)} + 3x^{(1)} + 2x = 0, \quad x(0) = 1, \quad x^{(1)}(0) = 1.$$

Determine the lower and upper bounds for $\|X(t)\|$.

1.2 The behavior of a dynamical system is described by the equation

$$x^{(2)} + 1.5x^{(1)} + 0.5x + \mu\{2x^2 + [x^{(1)}]^2\}^{1/2} = 0.$$

Determine the region of μ such that $X = 0$ is an exponentially stable equilibrium point of the given system.

1.3 The behavior of a dynamical system is described by the equation

$$x^{(2)} + 1.5x^{(1)} + 0.5x + \mu|\sin(0.5t)| = 0.$$

Determine the parameter μ such that $\lim_{t\to\infty} \|X(t)\|_2 \le 0.4$.

1.4 The behavior of a dynamical system is described by the equations

$$\dot{x}_1 = x_1 - x_2, \quad \mu\dot{x}_2 = 2x_1 + x_2.$$

Obtain and analyze the stability of the SMS and FMS.

1.5 The behavior of a dynamical system is described by the equations

$$\dot{x}_1 = x_1 - x_2, \quad \mu\dot{x}_2 = 2x_1 - x_2. \tag{1.74}$$

Obtain and analyze the stability of the SMS and FMS. Plot the phase portraits of the system by computer simulation for $\mu = 0.1, 0.5, 1$ and compare the results.

1.6 Consider the system (1.74). Obtain and analyze the stability of the SMS and FMS. Determine the parameter μ such that $\eta_2 = 10$.

1.7 The behavior of a dynamical system is described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3 - 2x_2, \quad \mu\dot{x}_3 = x_4, \quad \mu\dot{x}_4 = -x_1 - x_3 - x_4.$$

Obtain and analyze the stability of the SMS and FMS. Determine the parameter μ, where: (a) $\eta_1 = 10$, (b) $\eta_2 = 10$, (c) $\eta_3 = 10$.

1.8 Consider the difference equations given by

$$x_1(k+1) = [1+\mu]x_1(k) + \mu x_2(k), \quad x_2(k+1) = ax_1(k) + bx_2(k).$$

Obtain and analyze the conditions for the SMS and FMS stability. Determine the parameter μ such that $\eta_2 = 10$, where $a = 0.35$, $b = 0.2$.

1.9 Consider the difference equations given by

$$\begin{aligned} x_1(k+1) &= [1-\mu]x_1(k) - \mu[x_2(k) + x_3(k)], \\ x_2(k+1) &= x_1(k) + 0.1x_2(k) + 0.2x_3(k), \\ x_2(k+1) &= 0.5x_1(k) + 0.2x_2(k) + 0.1x_3(k). \end{aligned}$$

Obtain and analyze the conditions for the SMS and FMS stability. Determine the parameter μ such that $\eta_2 = 10$.