

CHAPTER 1

Classical Electrodynamics

There are two quite disparate approaches to electromagnetic field theory. One is a deductive approach that begins with a single relativistic source potential and deduces from it the full slate of classical equations of electromagnetism. The other is an inductive approach that begins with the experimentally determined force laws and induces from them, incorporating new facts as needed, until the Maxwell equations are obtained. Although the theory was developed using the inductive approach, it is the deductive method that shows the majestic simplicity of electromagnetism.

The inductive approach is commonly used in textbooks at all levels. Coulomb's law is the usual starting point, with other effects included as needed until the full slate of measurable quantities are obtained. From this viewpoint, the potentials are but mathematical artifices that simplify force field calculations. They simplify the calculation necessary to solve for the force fields but are without intrinsic significance. The deductive approach begins with a limited axiomatic base and develops a potential theory from which, in turn, follow the force fields. In 1959 Aharonov and Bohm, using the premise that potential has a special significance, predicted an effect that was confirmed in 1960, the Aharonov–Bohm effect: Magnetic field quantization is affected by a static magnetic potential even in a region void of force fields. We conclude that the magnetic potential has a physical significance in its own right and has meaning in a way that extends beyond the calculation of force fields. There is physical significance contained in the deductive approach that is not present in the inductive one.

1.1. Introductory Comments

To begin the deductive approach, consider that the universe is totally empty of condensed matter but does contain light. What is the speed of the light? Since there is no reference frame by which to measure it, the question is moot. Therefore, introduce an asteroid large enough to support an observer

and his equipment, which determines the speed of light passing him to be v_A . Since there is nothing else in the universe, a question about the speed of the asteroid is moot. Next, introduce a second asteroid, identical to the first but separated far enough to be independent by any means of which we are currently aware. An observer on the second asteroid determines the speed of light passing him to be v_B . Will the measured values be the same? By the cosmological principle, an experiment run in one local four-space yields the same results as an identical experiment run in a different local four-space. Therefore we expect that $v_A = v_B = c$.

Next, bring the asteroids into the same local region. Either the speeds depend upon the magnitude of the local masses or they do not, and if they do not, there is no change in speed. However, in the local region, a relative speed between identical asteroids A and B may be determined. Since there is no way one asteroid can be preferred over the other in an otherwise empty universe, the two observers continue to measure the same speed. This condition requires that the speed of light be independent of the relative speed of the system on which it is measured. Next, bring in other material, bit by bit, until the universe is in its present form, and the conclusion remains the same. The speed of light is independent of the speed of the object on which it is measured, independently of the speed of other objects.

1.2. Space and Time Dependence upon Speed

Let a pulse of light be emitted from an origin in reference frame F and observed in reference frame F'. If the speed of light is the same in all reference frames, if the two frames are in relative motion, and if the origins coincide at the time the light is emitted, the light positions as measured in the two frames are:

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c'^2t'^2 \quad (1.2.1)$$

If the relative speed is such that F' is moving at speed ν in the z -direction with respect to F, then at low speeds:

$$x' = x; \quad y' = y; \quad z' = (z - \nu t); \quad t' = t \quad (1.2.2)$$

Since Eq. (1.2.1) is not satisfied by Eq. (1.2.2), it follows that Eq. (1.2.2) does not extend to speeds that are a significantly large fraction of c . To obtain a transition that is linear in the independent variables, and that goes

to Eq. (1.2.2) in the low speed limit, consider the linear transformation of the form:

$$x' = x; \quad y' = y; \quad z' = \gamma(z - vt); \quad t' = At + Bz \quad (1.2.3)$$

Parameters γ , A and B are undetermined but independent of both position and time. Since Eq. (1.2.3) approaches Eq. (1.2.2) in the limit of velocity v much less than c , in that limit:

$$\gamma = 1; \quad A = 1; \quad B = 0 \quad (1.2.4)$$

Since the coordinates are independent variables, combining Eqs. (1.2.1) and (1.2.3) and solving shows that:

$$\begin{aligned} z^2(\gamma^2 - 1 - c^2B^2) &= 0; & t^2(c^2 + \gamma^2v^2 - c^2A^2) &= 0; \\ zt(v\gamma^2 + ABc^2) &= 0 \end{aligned} \quad (1.2.5)$$

Solving Eq. (1.2.5) yields:

$$A = \gamma = (1 - v^2/c^2)^{-1/2}; \quad B = -(\gamma v/c^2) \quad (1.2.6)$$

Combining yields the Lorentz transformation equations:

$$x' = x; \quad y' = y; \quad z' = \gamma(z - vt); \quad t' = \gamma(t - (vz/c^2)) \quad (1.2.7)$$

This transformation preserves the speed of light in inertial frames.

Equation (1.2.7) forms a sufficient basis upon which to determine results if events in one frame of reference are observed in another one. Let the observer be in the unprimed frame. A stick of length L_0 as determined in the moving frame, in which it is stationary, lies along the z -axis. It moves at speed v past the observer in the z -direction. A flash of light illuminates the region, during which time the observer determines the positions of the ends of the moving stick, z_1 and z_2 . It follows from Eq. (1.2.7) that the measured positions are:

$$z'_1 = \gamma(z_1 - vt_0) \quad \text{and} \quad z'_2 = \gamma(z_2 - vt_0) \quad (1.2.8)$$

The length as measured in the stationary frame is:

$$L = (z_2 - z_1) = (z'_2 - z'_1)/\gamma = L_0/\gamma \quad (1.2.9)$$

It follows that:

$$L = L_0(1 - v^2/c^2)^{1/2} \leq L_0 \quad (1.2.10)$$

The observed length of the stick is less than that measured in the rest frame; this fractional contraction is the Lorentz contraction.

Next, pulses of light are issued at times t'_2 and t'_1 , again in the moving frame. When does a stationary observer see them, and what is the time interval between them? Using Eq. (1.2.7) gives:

$$t'_2 = \gamma(t_2 - vz_2/c^2) \quad \text{and} \quad t'_1 = \gamma(t_1 - vz_1/c^2) \quad (1.2.11)$$

From Eq. (1.2.11) the time difference in the frame at which the two sources are stationary is:

$$T_0 = t'_2 - t'_1 = \gamma[(t_2 - t_1) - v(z_2 - z_1)/c^2] = \gamma T(1 - v^2/c^2) \quad (1.2.12)$$

T is the time measured in the stationary frame. Solving for T gives:

$$T = \gamma T_0 = \frac{T_0}{(1 - v^2/c^2)^{1/2}} \geq T_0 \quad (1.2.13)$$

The observer measures the time duration between pulses to be more than that measured in the rest frame; this time expansion is time dilatation.

1.3. Four-Dimensional Space Time

The equality of the speed of light in all inertial frames is the basis for a system of 4-vectors. Let x_1, x_2, x_3 represent the three spatial axes x, y, z of three dimensions and $x_4 = ict$ where $i = \sqrt{-1}$. The four space-time dimensions are:

$$(x_1, x_2, x_3, x_4) \quad (1.3.1)$$

Since three of the axes determine lengths and one determines time, a three-dimensional rotation represents a change in spatial orientation and a four-dimensional rotation includes a change in time. Such four-dimensional rotations are Lorentz transformations. These transformations are usually simple and contain a high degree of symmetry. Such transformations are covariant with respect to changes in coordinate systems; that is, an equation that represents reality in one reference frame has the same form in all other inertial frames.

The imaginary property of the fourth dimension represents an essential difference from spatial ones: the squares of the space coefficients and time coefficients have different signs. For notational purposes we use Roman or Greek subscripts to indicate, respectively, three- or four-dimensional tensors. For example, the rotation matrix element in four dimensions is $c_{\mu\nu}$

where, for velocities v directed along the x_1 -axis:

$$c_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\gamma v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma v/c & 0 & 0 & \gamma \end{pmatrix} \quad (1.3.2)$$

Four-dimensional and three-dimensional direction cosines follow similar laws:

$$c_{\mu\nu} = c_{\nu\mu}; \quad c_{\mu\nu}c_{\mu\rho} = \delta_{\nu\rho}; \quad \det |c_{\mu\nu}| = 1 \quad (1.3.3)$$

The Lorentz direction cosines $c_{\mu\nu}$ are:

$$x'_\mu = c_{\mu\nu}x_\nu \quad (1.3.4)$$

The proper time interval, $\Delta\tau$, between two events with space-time coordinates spaced Δx_α apart is defined to be:

$$(\Delta\tau)^2 = -\frac{1}{c^2}\Delta x_\alpha\Delta x_\alpha \quad (1.3.5)$$

Using three-dimensional notation, the proper time difference is

$$(\Delta\tau)^2 = (\Delta t)^2 - \frac{(\Delta\mathbf{r})^2}{c^2} \quad (1.3.6)$$

Since $(\Delta\tau)^2$ can be zero, positive, or negative, $\Delta\tau$ may be zero, real, or imaginary. Since the speed of light is the same in all reference frames, by Eq. (1.2.1) the proper time is also the same in all reference frames. If it is real, it is “time-like” and if imaginary, it is “space-like”. If time-like, the proper time is the time separation of the two events in the same frame. If space-like, there is a frame in which c times the proper time is the spatial separation of the two events that are simultaneous in that frame.

With τ as proper time, consider the 4-vector defined by the expression:

$$U_\mu = \frac{dx_\mu}{d\tau} \quad (1.3.7)$$

Since both x_μ and τ are independent of details of the particular inertial frame in which it is measured, so is U_μ ; U_μ is therefore a 4-vector with the four components:

$$\begin{aligned} U_1 &= \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = \gamma v_x; & U_2 &= \frac{dy}{d\tau} = \frac{dy}{dt} \frac{dt}{d\tau} = \gamma v_y \\ U_3 &= \frac{dz}{d\tau} = \frac{dz}{dt} \frac{dt}{d\tau} = \gamma v_z; & U_4 &= \frac{d(ict)}{d\tau} = \gamma ic \end{aligned} \quad (1.3.8)$$

The three-dimensional velocity components are v_i and the 4-velocity components are U_μ .

A particle of mass m_0 with 4-velocity U_μ has 4-momentum given by:

$$P_\mu = m_0 U_\mu \quad (1.3.9)$$

Combining shows the momentum components to be:

$$\mathbf{p} = \gamma m_0 \mathbf{v}; \quad p_4 = \gamma m_0 i c = iW/c; \quad W = \gamma m_0 c^2 \quad (1.3.10)$$

The quantity W , defined by Eq. (1.3.10), is the energy associated with the moving mass.

The binomial expansion is:

$$(1 \pm a)^n = 1 \pm na + \frac{n}{2!}(n-1)a^2 \pm \dots \quad (1.3.11)$$

This equation combines with the definition of γ , see Eq. (1.2.6), to show that:

$$\gamma = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \dots \quad (1.3.12)$$

Combining Eqs. (1.3.10) and (1.3.12) shows the total energy of the particle:

$$W = m_0 c^2 \left[1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \dots \right] \quad (1.3.13)$$

In the rest frame m_0 is the rest mass. The particle energy is:

$$W_0 = m_0 c^2 \quad (1.3.14)$$

By Eq. (1.3.14), the first term of Eq. (1.3.13) is the self-energy of the mass. The second term is the kinetic energy at low speeds and the higher order terms complete the evaluation of the kinetic energy of the mass at any speed.

1.4. Newton's Laws

The Minkowski force is defined to be:

$$F_\mu = \frac{d}{d\tau} P_\mu \quad (1.4.1)$$

This force is a 4-vector with the x -directed component:

$$F_1 = \frac{d}{d\tau} (m_0 U_1) = \gamma \frac{\partial}{\partial t} (\gamma m_0 v_x) \quad (1.4.2)$$

The corresponding three-dimensional force component is:

$$F_x = \frac{\partial}{\partial t} (\gamma m_0 v_x) \quad (1.4.3)$$

The factor γ in Eq. (1.4.3) was known before the full relativistic effect was understood. Although relativity makes it abundantly clear that the result

is a space-time effect, it was historically interpreted as an increase in mass whereby the effective mass m is a function of speed:

$$m = \gamma m_0 \quad (1.4.4)$$

Even with relativity, the nomenclature remains and by definition the effective mass of a moving particle is equal to Eq. (1.4.4). Since the 4-momentum is a 4-vector, it is conserved between Lorentz frames. That is,

$$W_0^2 = W^2 - p^2 c^2 \quad (1.4.5)$$

The energy is related to momentum, in any given frame, as:

$$W^2 = m_0^2 c^4 + p^2 c^2 \quad (1.4.6)$$

Since W is second order in v/c , three-momentum is constant in low speed inertial frames. Energy is also nearly conserved. However, in high-energy systems neither energy nor momentum is conserved, only the combination. This example illustrates a general characteristic of 4-tensors that at low speeds the real and imaginary parts are separately conserved but at high speeds only the combined magnitude is conserved.

1.5. Electrodynamics

The three scalars defined so far are speed, c , time interval between events in a rest frame, τ , and mass, m_0 . A fourth is electric charge, q ; electric charge can have either sign. Just as an intrinsic part of any mass is the associated gravitational field, G , an intrinsic part of charge is the associated 4-vector potential field A_μ . Consider that the individual charges are much smaller than other dimensions and that there are many of them. For this case choose a differential volume, with dimensions (x_1, x_2, x_3) , in which each dimension is much less than any macroscopic dimension of interest but contains large numbers of charges. If both conditions are met, the tools of calculus apply. Charge density ρ is defined to be the charge per unit volume at a point. Charge density ρ_0 is defined in a frame in which the time-average position is at rest. Observers in fixed and moving frames see the same total charge but, because of the Lorentz contraction, the moving observer determines the volume containing it to be smaller by a factor of γ . Therefore, the charge density in a moving frame is increased by the factor:

$$\rho = \gamma \rho_0 \quad (1.5.1)$$

If the charge density moves with 4-velocity U_μ in a way similar to three dimensions, the 4-current density is defined to be:

$$\mathbf{J}_\mu = \rho_0 U_\mu = \{\gamma\rho_0\mathbf{v}, \gamma ic\rho_0\} = \{\mathbf{J}, ic\rho\} \quad (1.5.2)$$

The vector terms within the curly brackets, identified by bold font, indicate the first three dimensions, and the scalar term represents the fourth dimension. The 4-divergence of the current density is:

$$\frac{\partial \mathbf{J}_\mu}{\partial X_\mu} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (1.5.3)$$

The first equality of Eq. (1.5.3) follows from definition of terms and the second is true if and only if net charge is neither created nor destroyed. Pair production or annihilation may occur but there is no change in the total charge. The zero 4-divergence shows that the net change in the four-current is always equal to zero. Physically a net change in the total charge does not occur and charges are created and destroyed only in canceling pairs.

The 4-vector potential field $A_\mu(X_\gamma)$ is defined to be the potential that satisfies the differential equation:

$$\frac{\partial^2 A_\nu}{\partial X_\beta \partial X_\beta} = -\mu \mathbf{J}_\nu \quad (1.5.4)$$

Constant μ is defined to be the permeability of free space; it is a dimension-determining constant and defined to equal $4\pi/10^7$ Henrys/meter.

Taking the 4-divergence of Eq. (1.5.4) then combining with Eq. (1.5.3) gives:

$$\frac{\partial}{\partial X_\nu} \frac{\partial^2}{\partial X_\beta \partial X_\beta} A_\nu = \frac{\partial^2}{\partial X_\beta \partial X_\beta} \frac{\partial A_\nu}{\partial X_\nu} = -\mu \frac{\partial \mathbf{J}_\nu}{\partial X_\nu} = 0$$

Combining, it follows that:

$$\partial A_\nu / \partial X_\nu = 0 \quad (1.5.5)$$

Equation (1.5.5) shows that the divergence of A_ν is zero, from which it follows that, like charge, the total amount of 4-potential does not change. If transitions are made between different reference frames changes occur in the components of the potential but not in the sum over all four components.

The four-dimensional Laplacian of Eq. (1.5.4) may be integrated over all space to obtain an expression for the potential itself. By Eq. (A.6.2) the

potential of a moving charge is:

$$A_\alpha(X_\gamma) = \frac{\mu}{4\pi} \iiint \frac{J_\alpha(\mathbf{r}', t' - R/c)}{(R - \mathbf{R} \cdot \mathbf{v}/c)} dV' \quad (1.5.6)$$

The integral is over all source-bearing regions, dV' is differential volume, X_γ are the 4-coordinates of the field point, X'_γ are the 4-coordinates at the source point, \mathbf{R} is the vector from the source point to the field point. At low speeds Eq. (1.5.6) simplifies to:

$$A_\alpha(X_\gamma) = \frac{\mu}{4\pi} \iiint \frac{J_\alpha(\mathbf{r}', t' - R/c)}{R(X_\gamma, X'_\gamma)} dV' \quad (1.5.7)$$

Substituting in the three-dimensional values of J_α results in the three-dimensional potentials:

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}', t' - R/c)}{R(\mathbf{r} - \mathbf{r}', t')} dV' \\ \Phi(\mathbf{r}, t) &= -icA_4(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho(\mathbf{r}', t' - R/c)}{R(\mathbf{r} - \mathbf{r}', t')} dV' \end{aligned} \quad (1.5.8)$$

The constant ϵ is defined to be the permittivity of free space; it is a dimension determining constant and defined to be exactly equal to $1/(\mu c^2)$ Farads/meter.

For a point charge, instead of a charge distribution, the corresponding 4-potential is:

$$A_\alpha(X_\gamma) = \frac{\mu q}{4\pi} \frac{U_\alpha(\mathbf{r}', t' - R/c)}{(R - \mathbf{R} \cdot \mathbf{v}/c)} \quad (1.5.9)$$

The three-dimensional potentials are:

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu}{4\pi} \frac{q\mathbf{v}(\mathbf{r}', t' - R/c)}{(R - \mathbf{R} \cdot \mathbf{v}/c)} \\ \Phi(\mathbf{r}, t) &= \frac{\mu}{4\pi} \frac{q(R - \mathbf{R} \cdot \mathbf{v}/c)}{(R - \mathbf{R} \cdot \mathbf{v}/c)} \end{aligned} \quad (1.5.10)$$

If the charge moves at a speed much less than c Eq. (1.5.10) is the usual three-dimensional vector and scalar potential field of individual charges.

It is apparent from Eq. (1.5.10) that a charge moving towards or away from a field point generates potentials with magnitudes respectively larger or smaller than the low speed value.

1.6. The Field Equations

If ρ_0 is the charge density in an inertial reference frame in which the average speed of the charges is zero, then $\rho = \gamma\rho_0$ is the charge density in a moving frame. The charge density and the three-dimensional current density J_i were extended to form the 4-current density, as shown by Eq. (1.5.2), from which the Laplacian of the 4-potential was defined by Eq. (1.5.4). Other useful 4-tensors follow from four-dimensional operations on the 4-potential $A_\alpha(X_\gamma)$; some especially important ones follow.

A second rank antisymmetric tensor of interest follows from the potential by the equation:

$$f_{\alpha\beta} = \frac{\partial A_\beta}{\partial X_\alpha} - \frac{\partial A_\alpha}{\partial X_\beta} \quad (1.6.1)$$

Antisymmetric 4-tensors are spatial arrays of six numbers and, in common with all antisymmetric tensors, the trace is zero:

$$f_{\alpha\alpha} = 0 \quad (1.6.2)$$

Writing out the six values that appear in the upper right portion of the 4-tensor, and using the result to define the function Φ , gives:

$$\begin{aligned} f_{12} &= \frac{\partial A_2}{\partial X_1} - \frac{\partial A_1}{\partial X_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z \\ f_{23} &= \frac{\partial A_3}{\partial X_2} - \frac{\partial A_2}{\partial X_3} = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = B_x \\ f_{31} &= \frac{\partial A_1}{\partial X_3} - \frac{\partial A_3}{\partial X_1} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = B_y \\ f_{14} &= \frac{\partial A_4}{\partial X_1} - \frac{\partial A_1}{\partial X_4} = \frac{i}{c} \frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{ic\partial t} = -\frac{i}{c} E_x \\ f_{24} &= \frac{i}{c} \frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{ic\partial t} = -\frac{i}{c} E_y \\ f_{34} &= \frac{i}{c} \frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{ic\partial t} = -\frac{i}{c} E_z \end{aligned} \quad (1.6.3)$$

With the deductive approach to electromagnetism Eq. (1.6.3) are the defining terms for field vectors \mathbf{E} and \mathbf{B} . The result written in tensor form is:

$$(f) = \begin{pmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{pmatrix} \quad (1.6.4)$$

Differentiating $f_{\alpha\beta}$ with respect to X_β results in the equality chain:

$$\frac{\partial f_{\alpha\beta}}{\partial X_\beta} = \frac{\partial}{\partial X_\beta} \left(\frac{\partial A_\beta}{\partial X_\alpha} - \frac{\partial A_\alpha}{\partial X_\beta} \right) = \frac{\partial^2 A_\beta}{\partial X_\beta \partial X_\alpha} - \frac{\partial^2 A_\alpha}{\partial X_\beta \partial X_\beta} = \mu J_\alpha \quad (1.6.5)$$

Combining terms gives:

$$\frac{\partial f_{\alpha\beta}}{\partial X_\beta} = \mu J_\alpha \quad (1.6.6)$$

Evaluation of Eq. (1.6.6) gives:

$$\begin{aligned} \frac{\partial f_{1\beta}}{\partial X_\beta} &= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2} \frac{\partial E_x}{\partial t} = \mu J_x \\ \frac{\partial f_{2\beta}}{\partial X_\beta} &= \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} - \frac{1}{c^2} \frac{\partial E_y}{\partial t} = \mu J_y \\ \frac{\partial f_{3\beta}}{\partial X_\beta} &= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2} \frac{\partial E_z}{\partial t} = \mu J_z \\ \frac{c}{i} \frac{\partial f_{4\beta}}{\partial X_\beta} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon} \end{aligned} \quad (1.6.7)$$

These are the nonhomogeneous Maxwell equations and relate fields to sources. In three-dimensional notation:

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J}; \quad \epsilon \nabla \cdot \mathbf{E} = \rho \quad (1.6.8)$$

The nonhomogeneous Maxwell equations relate force field intensities \mathbf{E} and \mathbf{B} to sources ρ and \mathbf{J} . The first order terms of \mathbf{E} and \mathbf{B} are, respectively, independent of and proportional to the first power of the speed of the charge.

It follows from the definition of $f_{\alpha\beta}$ that:

$$\frac{\partial f_{\nu\sigma}}{\partial X_\alpha} + \frac{\partial f_{\sigma\alpha}}{\partial X_\nu} + \frac{\partial f_{\alpha\nu}}{\partial X_\sigma} = 0 \quad (1.6.9)$$

Evaluation of Eq. (1.6.9) for each tensor component gives:

$$\begin{aligned} \frac{\partial f_{12}}{\partial X_3} + \frac{\partial f_{23}}{\partial X_1} + \frac{\partial f_{31}}{\partial X_2} &= \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \\ \frac{\partial f_{24}}{\partial X_1} + \frac{\partial f_{41}}{\partial X_2} + \frac{\partial f_{12}}{\partial X_4} &= \frac{1}{ic} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} \right) = 0 \\ \frac{\partial f_{34}}{\partial X_2} + \frac{\partial f_{42}}{\partial X_3} + \frac{\partial f_{23}}{\partial X_4} &= \frac{1}{ic} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} \right) = 0 \\ \frac{\partial f_{14}}{\partial X_3} + \frac{\partial f_{43}}{\partial X_1} + \frac{\partial f_{31}}{\partial X_4} &= \frac{1}{ic} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t} \right) = 0 \end{aligned} \quad (1.6.10)$$

These are the homogeneous Maxwell equations and relate force field vectors \mathbf{E} and \mathbf{B} . In three-dimensional notation:

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0; \quad \nabla \cdot \mathbf{B} = 0 \quad (1.6.11)$$

Another useful 4-vector is the force intensity, defined by the equation:

$$F_\alpha^\nu = f_{\alpha\beta} J_\beta \quad (1.6.12)$$

Evaluation of each component of Eq. (1.6.12) gives:

$$\begin{aligned} F_1^\nu &= F_x^\nu = J_y B_z - J_z B_y + \rho E_x \\ F_2^\nu &= F_y^\nu = J_z B_x - J_x B_z + \rho E_y \\ F_3^\nu &= F_z^\nu = J_x B_y - J_y B_x + \rho E_z \\ F_4^\nu &= \frac{i}{c} (E_x J_x + E_y J_y + E_z J_z) \end{aligned} \quad (1.6.13)$$

These equations relate force and power to the interaction of the charges and the fields. In three-dimensional notation:

$$\mathbf{F}^\nu = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}; \quad -icF_4^\nu = \mathbf{E} \cdot \mathbf{J} \quad (1.6.14)$$

To assist in the interpretation of Eq. (1.6.12), consider the 4-scalar formed by taking the scalar product:

$$F_\alpha^\nu J_\alpha = f_{\alpha\beta} J_\alpha J_\beta = 0 \quad (1.6.15)$$

The second equality of Eq. (1.6.15) follows from the antisymmetric character of $f_{\alpha\beta}$ and shows that the 4-vector F_α^ν is perpendicular to the 4-current density. Since the 4-current density is proportional to the 4-velocity, it follows that F_α^ν is also perpendicular to the 4-velocity. Consider the differential with respect to proper time of the square of the 4-velocity:

$$\frac{d}{d\tau} (U_\alpha U_\alpha) = 2U_\alpha \frac{dU_\alpha}{d\tau} = \frac{d}{d\tau} (-c^2) = 0 \quad (1.6.16)$$

Therefore both the 4-acceleration and F_α^ν are perpendicular to the 4-velocity. This is a necessary but insufficient requirement for F_α^ν to be the force density.

This approach to the Maxwell equations is based upon the original axiom relating a charge to its accompanying potential. The form of the source shows that only charges produce a 4-curvature of the 4-potential field. The technique is a neat way both to package the electromagnetic

equations and to show that they take the same form in all inertial coordinate systems. The relationship between fields \mathbf{E} and \mathbf{B} and the potentials follows from Eq. (1.6.3). By direct comparison:

$$\begin{aligned} \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} &= B_k \Rightarrow \nabla \times \mathbf{A} = \mathbf{B} \\ -\frac{\partial \Phi}{\partial x_i} - \frac{\partial A_i}{\partial t} &= E_i \Rightarrow -\left(\nabla \Phi + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{E} \end{aligned} \quad (1.6.17)$$

1.7. Accelerating Charges

The potentials surrounding electric charges in uniform motion are given by Eq. (1.5.10) and the force fields are related to the potential by Eq. (1.6.3). The partial derivative operations of Eq. (1.6.3) take place at the field position and time, (\mathbf{r}, t) . The position and time at the source, (\mathbf{r}', t') , do not enter into the operations. To carry out the operations it is convenient to define S by the equation:

$$S = \left(\mathbf{R} - \frac{\mathbf{R} \cdot \mathbf{v}}{c} \right) \quad (1.7.1)$$

Operating upon the potential while keeping terms involving charge accelerations gives:

$$\begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon} \left\{ \frac{1}{\gamma^2 S^3} \left(\mathbf{R} - R \frac{\mathbf{v}}{c} \right) + \frac{1}{c^2 S^3} \mathbf{R} \times \left[\left(\mathbf{R} - R \frac{\mathbf{v}}{c} \right) \times \frac{\partial}{\partial t} \mathbf{v} \right] \right\} \\ \mathbf{B} &= \frac{1}{Rc} \mathbf{R} \times \mathbf{E} \end{aligned} \quad (1.7.2)$$

Keeping only first order terms in powers of v/c leads to:

$$\begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon R^3} \left\{ \left(\mathbf{R} - R \frac{\mathbf{v}}{c} \right) + \frac{1}{c^2} \mathbf{R} \times \left(\mathbf{R} \times \frac{\partial}{\partial t} \mathbf{v} \right) \right\} \\ \mathbf{B} &= -\frac{\mu q}{4\pi R^3} \mathbf{R} \times \left(\mathbf{v} + \frac{R}{c} \frac{\partial}{\partial t} \mathbf{v} \right) \end{aligned} \quad (1.7.3)$$

The equations show that: A stationary charge produces an electric field intensity that varies as the inverse square of the radius, but there is no magnetic field. If the charge is moving, both electric and magnetic field intensities exist that are proportional to the speed of the charge and varying as the inverse square of the radius. If the charge is accelerating, both electric and magnetic field intensities exist in proportion to the acceleration of the charge and the inverse radius. Where charge distributions are applicable Eq. (1.7.3) take the form of spatial integrals over charge bearing regions.

1.8. The Electromagnetic Stress Tensor

Another result of four-dimensional field analysis is the electromagnetic stress tensor. It is defined as the symmetric, second rank 4-tensor $T_{\alpha\beta}$:

$$\mu T_{\alpha\beta} = f_{\alpha\kappa} f_{\kappa\beta} + \frac{1}{4} \delta_{\alpha\beta} f_{\nu\sigma} f_{\nu\sigma} \quad (1.8.1)$$

A symmetric 4-tensor consists of an array of ten independent numbers. It may be shown, after some algebra, that the force density 4-vector of Eq. (1.6.12) is related to the electromagnetic stress tensor as:

$$F_{\alpha}^{\nu} = \partial T_{\alpha\beta} / \partial X_{\beta} \quad (1.8.2)$$

The independent components of $T_{\alpha\beta}$ follow from Eqs. (1.6.7) and (1.8.1). The result is:

$$\begin{aligned} T_{11} &= \frac{\varepsilon}{2} (E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu} (B_x^2 - B_y^2 - B_z^2) \\ T_{12} &= \varepsilon E_x E_y + \frac{1}{\mu} B_x B_y \\ T_{22} &= \frac{\varepsilon}{2} (E_y^2 - E_z^2 - E_x^2) + \frac{1}{2\mu} (B_y^2 - B_z^2 - B_x^2) \\ T_{23} &= \varepsilon E_y E_z + \frac{1}{\mu} B_y B_z \\ T_{33} &= \frac{\varepsilon}{2} (E_z^2 - E_x^2 - E_y^2) + \frac{1}{2\mu} (B_z^2 - B_x^2 - B_y^2) \\ T_{31} &= \varepsilon E_z E_x + \frac{1}{\mu} B_z B_x \\ T_{44} &= \frac{\varepsilon}{2} (E_x^2 + E_y^2 + E_z^2) + \frac{1}{2\mu} (B_x^2 + B_y^2 + B_z^2) \\ T_{14} &= \frac{1}{ic\mu} (E_y B_z - E_z B_y) \\ T_{24} &= \frac{1}{ic\mu} (E_z B_x - E_x B_z) \\ T_{34} &= \frac{1}{ic\mu} (E_x B_y - E_y B_x) \end{aligned} \quad (1.8.3)$$

The tensor may be written in the form:

$$(T) = \begin{pmatrix} T_{ij} & -\frac{i}{c} \mathbf{N} \\ -\frac{i}{c} \mathbf{N} & w \end{pmatrix} \quad (1.8.4)$$

By definition $w = T_{44}$ is equal to:

$$T_{44} = \frac{\epsilon}{2} E^2 + \frac{1}{2\mu} B^2 \tag{1.8.5}$$

T_{ij} is the three-dimensional electromagnetic stress tensor:

$$(T) = \begin{pmatrix} \frac{\epsilon}{2} [E_x^2 - E_y^2 - E_z^2] + \frac{1}{2\mu} [B_x^2 - B_y^2 - B_z^2] & \epsilon E_x E_y + \frac{1}{\mu} B_x B_y & \epsilon E_x E_z + \frac{1}{\mu} B_x B_z \\ \epsilon E_y E_x + \frac{1}{\mu} B_y B_x & \frac{\epsilon}{2} [E_y^2 - E_z^2 - E_x^2] + \frac{1}{2\mu} [B_y^2 - B_z^2 - B_x^2] & \epsilon E_y E_z + \frac{1}{\mu} B_y B_z \\ \epsilon E_z E_x + \frac{1}{\mu} B_z B_x & \epsilon E_z E_y + \frac{1}{\mu} B_z B_y & \frac{\epsilon}{2} [E_z^2 - E_x^2 - E_y^2] + \frac{1}{2\mu} [B_z^2 - B_x^2 - B_y^2] \end{pmatrix} \tag{1.8.6}$$

\mathbf{N} is the three-dimensional Poynting vector:

$$\mathbf{N} = (\mathbf{E} \times \mathbf{B})/\mu \tag{1.8.7}$$

Symmetric tensors of rank two in three dimensions reduce from six to three components by transforming to the principal axes and aligning one axis with the source field intensity. For example, if there is no magnetic field and if the electric field intensity is directed along the x -axis the tensor reduces to:

$$(T) = \frac{\epsilon}{2} \begin{pmatrix} E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & -E^2 \end{pmatrix} \tag{1.8.8}$$

To interpret the stress tensor, consider the four-dimensional spatial integral of Eq. (1.8.2). The equation may be written:

$$\iiint\iiint c'_{\sigma\alpha} F'^{\nu}_{\alpha} dX'_1 dX'_2 dX'_3 dX'_4 = \iiint\iiint c'_{\sigma\alpha} \frac{\partial T'_{\alpha\beta}}{\partial X'_\beta} dX'_1 dX'_2 dX'_3 dX'_4 \tag{1.8.9}$$

Working with the left side:

$$\begin{aligned} \iiint\iiint c'_{\sigma\alpha} F'^{\nu}_{\alpha} dX'_1 dX'_2 dX'_3 dX'_4 &= \iiint\iiint F'_{\sigma}{}^{\nu} dX'_1 dX'_2 dX'_3 dX'_4 \\ &= \iiint\iiint F'_{\sigma}{}^{\nu} dX_1 dX_2 dX_3 dX_4 \end{aligned}$$

Working with the right side:

$$\begin{aligned} \iiint c'_{\sigma\alpha} \frac{\partial T'_{\alpha\beta}}{\partial X'_\beta} dX'_1 dX'_2 dX'_3 dX'_4 &= \iiint \frac{\partial(c'_{\sigma\alpha} T'_{\alpha\beta})}{\partial X'_\beta} dX'_1 dX'_2 dX'_3 dX'_4 \\ &= \iiint c'_{\sigma\alpha} T'_{\alpha 4} dX'_1 dX'_2 dX'_3 \end{aligned}$$

The last equality results since the integral at the limits of the spatial integrals vanish. Working with the last integral, note that:

$$c'_{\alpha\beta} T_{\sigma\alpha} = c'_{\lambda\beta} c'_{\sigma\alpha} c'_{\lambda\gamma} T'_{\alpha\gamma} \quad (1.8.10)$$

Since $c'_{\lambda\beta} c'_{\lambda\gamma} = \delta_{\beta\gamma}$ it follows that $c'_{\alpha\beta} T_{\sigma\alpha} = c'_{\sigma\alpha} T'_{\alpha\beta}$ from which $c'_{\sigma\alpha} T'_{\alpha 4} = c'_{\alpha 4} T_{\sigma\alpha}$. This leaves the equality:

$$\iiint F'^{\nu} dX'_1 dX'_2 dX'_3 dX'_4 = \iiint c'_{\alpha 4} T_{\sigma\alpha} dX'_1 dX'_2 dX'_3 \quad (1.8.11)$$

Since $c'_{\alpha 4} = U_\alpha/ic$ this may be written:

$$\iiint F'^{\nu} dX'_1 dX'_2 dX'_3 dX'_4 = \frac{1}{ic} \iiint T_{\sigma\alpha} U_\alpha dX_1 dX_2 dX_3 \quad (1.8.12)$$

To change the 4-integral into a three-dimensional one, differentiate by (ict) to obtain:

$$\iiint F'^{\nu} dX'_1 dX'_2 dX'_3 = F_\sigma = -\frac{1}{c^2} \frac{\partial}{\partial t} \iiint T_{\sigma\alpha} U_\alpha dX_1 dX_2 dX_3 \quad (1.8.13)$$

Since all time integrals are zero at time $t = -\infty$, time integration has a value only at present time, t .

To examine results of these equations, consider a charge moving with low speed in the z -direction. With the axis in the direction of motion, the sum $T_{\sigma\alpha} U_\alpha$ takes the form:

$$T_{3\alpha} U_\alpha = \frac{\varepsilon}{2} E^2 \mathbf{v} \quad (1.8.14)$$

Combining gives:

$$\mathbf{F} = \int \mathbf{F}^\nu dV = \frac{d}{dt} \left\{ \frac{\mathbf{v}}{c^2} \int \left(\frac{\varepsilon}{2} E^2 \right) dV \right\} \quad (1.8.15)$$

The sign was changed to represent reaction of the field on its source, rather than *vice versa*. For a low speed particle undergoing differential acceleration

Eq. (1.8.15) takes the form:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} \tag{1.8.16}$$

The mass is calculated as:

$$m = \frac{1}{c^2} \int \left(\frac{\epsilon}{2} E^2 \right) dV \tag{1.8.17}$$

The interpretation accorded these equations is that Eq. (1.8.17) is Newton's law for electromagnetic mass, confirming that \mathbf{F} is a force. The expression for the mass shows that $(\epsilon E^2/2)$ is the energy density of an electric field.

1.9. Kinematic Properties of Fields

To further analyze the kinematic properties of fields, begin with the four-dimensional force equation, Eq. (1.6.14):

$$\mathbf{F}^\nu = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}; \quad -icF_4^\nu = \mathbf{E} \cdot \mathbf{J} \tag{1.9.1}$$

To express this equality in a way that depends upon the fields only, it is necessary to substitute for ρ and \mathbf{J} from the nonhomogeneous electromagnetic equations, Eq. (1.6.8):

$$\begin{aligned} \mathbf{F}^\nu &= \epsilon \mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{B} \times \left(\frac{1}{\mu} \nabla \times \mathbf{B} - \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \\ -icF_4^\nu &= \mathbf{E} \cdot \left(\frac{1}{\mu} \nabla \times \mathbf{B} - \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \end{aligned} \tag{1.9.2}$$

It is helpful to add zero to each equation in the form of terms proportional to the homogeneous Maxwell equations, Eq. (1.6.11). The added terms are:

$$\begin{aligned} &\frac{1}{\mu} \mathbf{B}(\nabla \cdot \mathbf{B}) - \epsilon \mathbf{E} \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \quad \text{and} \\ &-\mathbf{B} \cdot \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \end{aligned} \tag{1.9.3}$$

Combining gives:

$$\begin{aligned} \mathbf{F}^\nu &= \epsilon \{ \mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) \} \\ &+ \frac{1}{\mu} \{ \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{B}) \} - \frac{1}{c^2} \frac{\partial \mathbf{N}}{\partial t} \\ icF_4^\nu &= \frac{\partial}{\partial t} \left(\frac{\epsilon}{2} E^2 + \frac{1}{2\mu} B^2 \right) + \nabla \cdot \mathbf{N} \end{aligned} \tag{1.9.4}$$

Writing the first of Eq. (1.9.4) in tensor form gives:

$$F_i^v = \frac{\partial}{\partial x_j} \left\{ \varepsilon \left(E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \right) + \frac{1}{\mu} \left(B_i B_j - \frac{1}{2} \delta_{ij} B_k B_k \right) \right\} - \frac{1}{c^2} \frac{\partial}{\partial t} N_i \quad (1.9.5)$$

Integrating over a closed three-dimensional volume gives:

$$\begin{aligned} \oint \left\{ \varepsilon \left(E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \right) + \frac{1}{\mu} \left(B_i B_j - \frac{1}{2} \delta_{ij} B_k B_k \right) \right\} dS_j \\ = \int \left(\frac{1}{c^2} \frac{\partial N_i}{\partial t} + F_i^v \right) dV \end{aligned} \quad (1.9.6)$$

By Eq. (1.8.16) the last term on the right is the rate of change of momentum of all charges contained within the volume, $\mathbf{p}_{\text{charge}}$. Therefore, the first term on the right is the rate of change of field momentum, $\mathbf{p}_{\text{field}}$. It follows that the left side of the equation is equal to the force on the charges and fields within the volume of integration. The results may be written as:

$$\mathbf{p}_{\text{field}} = \frac{1}{c^2} \int \mathbf{N} dV; \quad \mathbf{F}^v = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \frac{d}{dt} \mathbf{p}_{\text{charge}} \quad (1.9.7)$$

Since \mathbf{F}^v is a force density, it follows from Eq. (1.9.7) that the electric field intensity is a force per unit charge. Since a wave travels at speed c , by the first of Eq. (1.9.7) the momentum passing through a planar surface is:

$$\mathbf{p}_{\text{field}} = \frac{1}{c} \int \mathbf{N} \cdot d\mathbf{S} \quad (1.9.8)$$

By definition $d\mathbf{S}$ is a differential vector area normally outward from the surface.

Integrating the second of Eqs. (1.9.1) and (1.9.4) over a three-dimensional volume gives:

$$\int (\mathbf{E} \cdot \mathbf{J}) dV = \frac{d}{dt} \int \left(\frac{\varepsilon}{2} E^2 + \frac{1}{2\mu} B^2 \right) dV + \oint \mathbf{N} \cdot d\mathbf{S} \quad (1.9.9)$$

Since the field intensity is a force per unit charge it follows that the left side of Eq. (1.9.9) is the rate at which energy enters the volume of integration. Therefore the volume integral on the right side must be the rate at which energy increases in the interior, and the surface integral must be the rate at which energy exits through the surface. It follows that the energy in the

electromagnetic fields is equal to:

$$W = \int \left(\frac{\epsilon}{2} \mathbf{E}^2 + \frac{1}{2\mu} \mathbf{B}^2 \right) dV \quad (1.9.10)$$

It also follows that the rate at which energy exits the volume through the surface is:

$$P = \oint \mathbf{N} \cdot d\mathbf{S} \quad (1.9.11)$$

A different formulation of Eq. (1.9.10) that is sometimes useful is by rewriting it in terms of the potentials. Combining Eq. (1.9.10) with Eqs. (1.6.8) and (1.6.17) results in:

$$\begin{aligned} W = \int [\rho\Phi + \mathbf{J} \cdot \mathbf{A}] dV + \oint \left[-\epsilon(\phi\mathbf{E}) + \frac{1}{\mu}(\mathbf{A} \times \mathbf{B}) \right] \cdot d\mathbf{S} \\ + \epsilon \int \left[-\mathbf{E} \cdot \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \frac{\partial \mathbf{E}}{\partial t} \right] dV \end{aligned} \quad (1.9.12)$$

For a charge moving at a constant speed, or if the charge acceleration is small enough so the energy escaping into the far field is negligible, only the first term of Eq. (1.9.12) is significant. For that case the total field energy may also be expressed as:

$$W = \int [\rho\Phi + \mathbf{J} \cdot \mathbf{A}] dV \quad (1.9.13)$$

1.10. A Lemma for Calculation of Electromagnetic Fields

A lemma is needed to assist in the unrestricted and systematic calculation of electromagnetic fields about known sources. To obtain it, begin with the general form for fields in a source-free region containing time-dependent fields:

$$\nabla \times \mathbf{B} - \epsilon\mu \frac{\partial \mathbf{E}}{\partial t} = 0 = \nabla \times \mathbf{E} + \epsilon\mu \frac{\partial \mathbf{B}}{\partial t} \quad (1.10.1)$$

Taking the curl of Eq. (1.10.1) and then substituting back and forth as needed gives:

$$\nabla \times (\nabla \times \mathbf{B}) + \epsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 = \nabla \times (\nabla \times \mathbf{E}) + \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (1.10.2)$$

This shows that, away from sources, \mathbf{E} and \mathbf{B} satisfy the same partial differential equation.

$$\nabla^2 \Psi - \epsilon\mu \frac{\partial^2 \Psi}{\partial t^2} = 0 \quad (1.10.3)$$

This is useful because of an associated lemma that begins with the vector field $\mathbf{F}(\mathbf{r}, t)$, defined by

$$\mathbf{F} = \nabla \times (\mathbf{r} \Psi) \quad (1.10.4)$$

The lemma is that if Ψ satisfies Eq. (1.10.3) then \mathbf{F} satisfies the differential equation:

$$\nabla \times (\nabla \times \mathbf{F}) + \varepsilon\mu \frac{\partial^2 \mathbf{F}}{\partial t^2} = 0 \quad (1.10.5)$$

To verify that Eq. (1.10.5) is correct, multiply Eq. (1.10.3) by $(-\mathbf{r})$, then take the curl:

$$-\nabla \times (\mathbf{r} \nabla^2 \Psi) + \varepsilon\mu \frac{\partial^2}{\partial t^2} [\nabla \times (\mathbf{r} \Psi)] = 0 \quad (1.10.6)$$

Comparing Eqs. (1.10.4) through (1.10.6) shows that Eq. (1.10.5) is satisfied if:

$$\nabla \times \{\nabla \times [\nabla \times (\mathbf{r} \Psi)]\} = -\nabla \times (\mathbf{r} \nabla^2 \Psi) \quad (1.10.7)$$

To confirm Eq. (1.10.7), begin with the identity for the curl of a scalar-vector product:

$$\nabla \times (\mathbf{r} \Psi) \equiv \Psi(\nabla \times \mathbf{r}) - \mathbf{r} \times \nabla \Psi \quad (1.10.8)$$

Since $\nabla \times \mathbf{r} \equiv 0$, it follows that:

$$\nabla \times [\nabla \times (\mathbf{r} \Psi)] = -\nabla \times (\mathbf{r} \times \nabla \Psi) \quad (1.10.9)$$

Combining Eqs. (1.10.7) and (1.10.9) gives:

$$\nabla \times [\nabla \times (\mathbf{r} \times \nabla \Psi)] - \nabla \times (\mathbf{r} \nabla^2 \Psi) = 0 \quad (1.10.10)$$

Two identities from vector analysis are:

$$\begin{aligned} \nabla(\mathbf{A} \cdot \mathbf{B}) &\equiv \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &\equiv \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \end{aligned} \quad (1.10.11)$$

Putting $\mathbf{A} = \mathbf{r}$ and $\mathbf{B} = \nabla \Psi$:

$$\begin{aligned} \nabla(\mathbf{r} \cdot \nabla \Psi) &\equiv (\mathbf{r} \cdot \nabla)\nabla \Psi + (\nabla \Psi \cdot \nabla)\mathbf{r} = (\mathbf{r} \cdot \nabla)\nabla \Psi + \nabla \Psi \\ \nabla \times (\mathbf{r} \times \nabla \Psi) &\equiv \mathbf{r} \nabla^2 \Psi - 2\nabla \Psi + (\mathbf{r} \cdot \nabla)\nabla \Psi \end{aligned} \quad (1.10.12)$$

Combining Eqs. (1.10.10) and (1.10.12):

$$\nabla \times (\mathbf{r} \times \nabla \Psi) - \mathbf{r} \nabla^2 \Psi + \nabla \Psi + \nabla(\mathbf{r} \cdot \nabla \Psi) = 0 \quad (1.10.13)$$

Since the curl of the gradient vanishes, taking the curl of Eq. (1.10.13) yields Eq. (1.10.10) and completes the proof.

1.11. The Scalar Differential Equation

To solve Eq. (1.10.3) it is useful to remove the time-dependent portion. For that purpose use the Fourier integral expansion:

$$\Psi(\mathbf{r}, t) = \int_{-\infty}^{\infty} \psi(\mathbf{r}, \omega) e^{i\omega t} d\omega \tag{1.11.1}$$

Substituting Eq. (1.11.1) into Eq. (1.10.3) leads to:

$$\int_{-\infty}^{\infty} (\nabla^2 \psi + k^2 \psi) e^{i\omega t} d\omega = 0 \tag{1.11.2}$$

By definition $k^2 = \omega^2 \epsilon \mu$. For this equation to be zero for all values of ω , the integrand of Eq. (1.11.2) must equal zero:

$$\nabla^2 \psi + k^2 \psi = 0 \tag{1.11.3}$$

This is the Helmholtz equation, solutions of which combine with Eqs. (1.10.3) to (1.10.5) to obtain the full solution for vector fields.

Certain differential vector operations in spherical coordinates are listed in Table 1.11.1. Using spherical coordinates with θ the polar angle from the z -axis, ϕ the azimuth angle from the x -axis, and r the radial distance from

Table 1.11.1. Differential vector operations, spherical coordinates.

Orthogonal line elements:	$dr, r d\theta, r \sin \theta d\phi$
Gradient	$\left\{ (\nabla \psi)_r = \frac{\partial \psi}{\partial r} \quad (\nabla \psi)_\theta = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad (\nabla \psi)_\phi = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \right\}$
Divergence of vector \mathbf{A} :	$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} A_\phi$
Components of curl \mathbf{A} :	$\left\{ \begin{aligned} (\nabla \times \mathbf{A})_r &= \frac{1}{r \sin \theta} \left[\frac{\partial(\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \\ (\nabla \times \mathbf{A})_\theta &= \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial(r A_\phi)}{\partial r} \\ (\nabla \times \mathbf{A})_\phi &= \frac{1}{r} \left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \end{aligned} \right\}$
Laplacian of $\Psi = \nabla^2 \Psi$:	$\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right\}$

the origin, by Table 1.11.1 the Helmholtz equation is given by:

$$\frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial \psi}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial \psi}{\partial r} \right] + k^2 \psi = 0 \quad (1.11.4)$$

Dividing the equation by k^2 shows that the radial dependence of the solution is a function only of the product $\sigma = kr$, and therefore ψ may be written as $\psi(\sigma, \theta, \phi)$. A theorem applicable to problems using spherical coordinates is that the complete solution of Eq. (1.11.4) is obtained by summing over all possible functions $\psi(\sigma, \theta, \phi)$ where:

$$\psi(r, \theta, \phi) = R(\sigma)\Theta(\theta)\Phi(\phi) \quad (1.11.5)$$

To obtain $\psi(\sigma, \theta, \phi)$, it is necessary to begin by solving for the solutions of Eq. (1.11.5) that involve only one independent variable. After obtaining the functional forms, all possible products are formed and weighted by a constant multiplying coefficient. The coefficient is determined by matching boundary conditions. Finally, all individual product functions with appropriate coefficients are summed.

Substituting Eq. (1.11.5) into Eq. (1.11.4) and multiplying by $r^2/\Psi(r, \theta, \phi)$ gives:

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{1}{R} \frac{d}{d\sigma} \left(\sigma^2 \frac{dR}{d\sigma} \right) + \sigma^2 = 0 \quad (1.11.6)$$

The first two terms are independent of the radius and the last two terms are independent of the angles, yet the two sets equal each other's negative, requiring both sets to be constant. The constant is known as the separation constant. A convenient choice of separation constant is for the radial terms to equal $\nu(\nu + 1)$ and the angular terms $-\nu(\nu + 1)$, and results in the separated, complete differential equations:

$$\frac{1}{\sigma^2} \frac{d}{d\sigma} \left(\sigma^2 \frac{dR}{d\sigma} \right) + \left(1 - \frac{\nu(\nu + 1)}{\sigma^2} \right) R = 0 \quad (1.11.7)$$

$$\frac{\Phi}{\sin \theta} \frac{d}{d\theta} \left[\sin \theta \frac{d\Theta}{d\theta} \right] + \frac{\Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} + \nu(\nu + 1)\Theta\Phi = 0 \quad (1.11.8)$$

The radial equation is a differential equation with one independent variable. The angular equation may be written as:

$$\frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left[\sin \theta \frac{d\Theta}{d\theta} \right] + \nu(\nu + 1) \sin^2 \theta + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \quad (1.11.9)$$

The first two terms of Eq. (1.11.9) are functions of θ only and the third is a function of ϕ only, yet the terms equal each other's negative. Again, both sets are constant. Putting the first two terms equal to m^2 , where m

is the second separation constant, results in two separated equations, each involving only one independent variable:

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left(\nu(\nu + 1) - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0 \quad (1.11.10)$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \quad (1.11.11)$$

Solutions of the separated differential equations and tabulated functions are in the Appendix.

Solutions of the radial equation are spherical Bessel, Neumann, and Hankel functions, respectively, $j_\nu(\sigma)$, $y_\nu(\sigma)$, and $h_\nu(\sigma)$. A particularly important linear combination is Hankel functions of the second kind and integer order: $h_\ell(\sigma)$ where “ ℓ ” represents any integer value of “ ν ”. Solutions of the zenith angle equation are associated Legendre functions; solutions are, in some instances, of integer order and in others of noninteger order. In all cases, the orders of the radial and zenith angle solutions are the same. Trigonometric functions form the solutions of the azimuth angle equation: $\sin \phi$, $\cos \phi$, and $\exp(\pm im\phi)$. Since all solutions to be considered extend over the full range of azimuth angle, zero through 2π , only integer values of degree m , are present. With exponential notation, the exponent may have either sign. With symbol $z_\nu(\sigma)$ representing a linear combination of possible radial solution forms, rather than writing the solution as two separate sums it is written as:

$$\psi_\nu^m(r, \theta, \phi) = z_\nu(\sigma)\Theta_\nu^m(\theta)e^{-im\phi} \quad (1.11.12)$$

With this notation, completeness requires m to include the full set of positive and negative integers, however the degree of the Legendre function is always positive.

1.12. Radiation Fields in Spherical Coordinates

Replacing \mathbf{B} by $\mu\mathbf{H}$ more closely matches common usage. For what lies ahead we are concerned only with free space and there μ is merely a unit-determining parameter that measures the magnetic field in amperes per meter instead of webers per square meter.

The field calculation procedure is due to Hansen, and begins with the vector theorem that a field with zero divergence is completely specified by its curl. It is, therefore, helpful to introduce the two independent field sets:

$$\eta\mathbf{H}_1 = \mathbf{r} \times \nabla\Psi_1 \quad \text{and} \quad \mathbf{E}_2 = \mathbf{r} \times \nabla\Psi_2 \quad (1.12.1)$$

Symbol $\eta = \sqrt{\mu/\varepsilon}$ indicates the wave impedance.

Since the free space divergences of both vectors are zero, solutions of Eq. (1.12.1) provide the complete set of possible values for vectors \mathbf{H}_1 and \mathbf{E}_2 . The remaining field solutions, \mathbf{H}_2 and \mathbf{E}_1 , may be obtained from Eq. (1.12.1) using the Maxwell curl equations. The total fields, $(\mathbf{E}_1 + \mathbf{E}_2)$ and $(\mathbf{H}_1 + \mathbf{H}_2)$, are then complete. If the boundary conditions are matched, the fields are also unique.

In what follows we use the notation that time dependence is $\exp(i\omega t)$ and azimuth angle dependence is $\exp(-jm\phi)$, where $i^2 = j^2 = -1$. The reasons for separate notation are that it permits separation of polarization and time dependencies and it permits restriction of separation constant m to the field of positive integers, without loss of generality. With Hansen's method the defining terms for phasor fields are, see Eq. (1.10.4):

$$\eta\tilde{\mathbf{H}}_1 = \mathbf{r} \times \nabla\psi_1 \quad \text{and} \quad \tilde{\mathbf{E}}_2 = \mathbf{r} \times \nabla\psi_2 \quad (1.12.2)$$

A tilde over a vector indicates that it is a phasor. It is required that the scalar functions satisfy the Helmholtz equation, Eq. (1.11.3). For integer modes, the results are solutions in the form of Eq. (1.11.12):

$$\begin{aligned} \psi_1 &= F(\ell, m)z_\ell(\sigma)\Theta_\ell^m e^{-jm\phi} \\ \psi_2 &= j G(\ell, m)z_\ell(\sigma)\Theta_\ell^m e^{-jm\phi} \end{aligned} \quad (1.12.3)$$

The order is not restricted to integer values and the radial function $z_\ell(\sigma)$ may be any linear combination of spherical Bessel and Neumann functions. The zenith angle function may be any linear combination of associated Legendre functions. Both the applicable functions and the constant multiplying coefficients $F(\ell, m)$ and $G(\ell, m)$ are determined by the boundary conditions.

Applying the operation of Eq. (1.12.2) to Eq. (1.12.3) gives the result:

$$\mathbf{r} \times \nabla\psi = -\frac{\hat{\theta}}{\sin\theta} \frac{\partial\psi}{\partial\phi} + \hat{\phi} \frac{\partial\psi}{\partial\theta} \quad (1.12.4)$$

Combining gives:

$$\begin{aligned} \eta\tilde{\mathbf{H}}_1 &= F(\ell, m)z_\ell(\sigma) \left[j\hat{\theta} \frac{m\Theta_\ell^m}{\sin\theta} + \hat{\phi} \frac{d\Theta_\ell^m}{d\theta} \right] e^{-jm\phi} \\ \tilde{\mathbf{E}}_2 &= jG(\ell, m)z_\ell(\sigma) \left[j\hat{\theta} \frac{m\Theta_\ell^m}{\sin\theta} + \hat{\phi} \frac{d\Theta_\ell^m}{d\theta} \right] e^{-jm\phi} \end{aligned} \quad (1.12.5)$$

Taking the curl of the second of Eq. (1.12.5) and then applying the Maxwell curl equation leads to:

$$\eta \tilde{\mathbf{H}}_2 = -iG(\ell, m)e^{-jm\phi} \left\{ j\ell(\ell + 1) \frac{z_\ell}{\sigma} \Theta_\ell^m \hat{\mathbf{r}} + z_\ell \left(j \frac{d\Theta_\ell^m}{d\theta} \hat{\theta} + \frac{m\Theta_\ell^m}{\sin\theta} \hat{\phi} \right) \right\} \quad (1.12.6)$$

The carat indicates a unit vector and a dot superscript indicates:

$$z_\ell^\bullet(\sigma) = \frac{1}{\sigma} \frac{d}{d\sigma} [\sigma z_\ell(\sigma)] \quad (1.12.7)$$

Taking the curl of the first of Eq. (1.12.5) and then applying the Maxwell curl equation leads to:

$$\tilde{\mathbf{E}}_1 = iF(\ell, m)e^{-jm\phi} \left\{ \ell(\ell + 1) \frac{z_\ell}{\sigma} \Theta_\ell^m \hat{\mathbf{r}} + z_\ell^\bullet \left(\frac{d\Theta_\ell^m}{d\theta} \hat{\theta} - j \frac{m\Theta_\ell^m}{\sin\theta} \hat{\phi} \right) \right\} \quad (1.12.8)$$

The total fields are the sum of Eqs. (1.12.5), (1.12.6) and (1.12.8). They may be written as:

$$\begin{aligned} \tilde{\mathbf{E}}_r &= i \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} F(\ell, m) \ell(\ell + 1) \frac{z_\ell(\sigma)}{\sigma} \Theta_\ell^m(\cos\theta) e^{-jm\phi} \\ \eta \tilde{\mathbf{H}}_r &= -ij \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} G(\ell, m) \ell(\ell + 1) \frac{z_\ell(\sigma)}{\sigma} \Theta_\ell^m(\cos\theta) e^{-jm\phi} \\ \tilde{\mathbf{E}}_\theta &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} \left[iF(\ell, m) z_\ell^\bullet \frac{d\Theta_\ell^m}{d\theta} - G(\ell, m) z_\ell \frac{m\Theta_\ell^m}{\sin\theta} \right] e^{-jm\phi} \\ \eta \tilde{\mathbf{H}}_\phi &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} \left[F(\ell, m) z_\ell \frac{d\Theta_\ell^m}{d\theta} - iG(\ell, m) z_\ell^\bullet \frac{m\Theta_\ell^m}{\sin\theta} \right] e^{-jm\phi} \\ \tilde{\mathbf{E}}_\phi &= -j \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} \left[iF(\ell, m) z_\ell^\bullet \frac{m\Theta_\ell^m}{\sin\theta} - G(\ell, m) z_\ell \frac{d\Theta_\ell^m}{d\theta} \right] e^{-jm\phi} \\ \eta \tilde{\mathbf{H}}_\theta &= j \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} i^{-\ell} \left[F(\ell, m) z_\ell \frac{m\Theta_\ell^m}{\sin\theta} - iG(\ell, m) z_\ell^\bullet \frac{d\Theta_\ell^m}{d\theta} \right] e^{-jm\phi} \end{aligned} \quad (1.12.9)$$

Without loss of generality, the phases of constants $F(\ell, m)$ and $G(\ell, m)$ and multiplying factor $i^{-\ell}$ have been picked for later convenience. Coefficients $F(\ell, m)$ multiply the radial component of the electric field terms and are TM (transverse magnetic) fields and modes, where ‘‘T’’ indicates transverse to the radial direction. Coefficients $G(\ell, m)$ multiply the radial component of the magnetic field and are TE (transverse electric) fields and modes. Terms with $\ell = m = 0$ have no radial fields and are the TEM (transverse

electric and magnetic) fields and mode. This result is valid for all possible electromagnetic field solutions.

Keeping only the real or only the imaginary part with respect to “ j ” provides, respectively, x or y polarization of the electric field intensity. The fields are right or left circularly polarized, respectively, with $j = i$ or $j = -i$. Since this result applies to all time-dependent outgoing waves, it follows that it also applies when the rate of change is arbitrarily small. Hence, it describes fields in the limit as the frequency goes to zero, a static charge distribution. Because of this general result, it is helpful to obtain a physical view of what constitutes field sources. The sources of coefficients $F(\ell, m)$ and $G(\ell, m)$ for static fields are discussed in the appendix, Secs. A.28 and A.29.

Consider a few special cases of Eq. (1.12.9). If the described fields are contained within a source-free region of space, and if that space is loss free, solutions have positive, integer values of orders and integer values of degrees. Spherical Bessel functions, which have no singularities, form the radial portion of the solution; spherical Neumann functions, which have singularities, are not present.

Associated Legendre functions of the first kind, and of integer order, which have no singularities, form the angular portion of the solution; fractional order associated Legendre functions and those of the second kind, which have singularities, are not present.

In the main, if the fields originate at a point and support an outward flow of energy from that point, the radial portion of the solution consists of spherical Hankel functions of the second kind. A solution within an enclosed space that excludes the z -axis, but has rotational symmetry, is described by associated Legendre functions of both the first and second kind, with noninteger, positive-real orders and integer degrees.

In all cases, if the medium in which the fields exist is “lossy”, the separation constants are complex numbers with a positive real part. Since all cases of interest in this book concern lossless media and a full 2π spatial rotation about the z -axis, both the order and degree are real and degrees have only integer values.

1.13. Electromagnetic Fields in a Box

It is helpful for the analysis of radiation problems that follow to know the possible modes in a rectangular cavity, the energy associated with the different modes, and the number of independent modes that can exist. To that

end consider all possible electric field modes that can exist inside an otherwise empty, rectangular cavity that is confined by walls of infinite conductivity. From Eqs. (1.11.2) and (1.11.3) the wave number k is, by definition:

$$k = \omega/c \quad (1.13.1)$$

Whatever time dependence a set of fields may have, it is most easily analyzed at a single frequency only. For each frequency, see Eqs. (1.6.8) and (1.6.11), the Maxwell equations in an empty hollow chamber are:

$$\begin{aligned} \eta \tilde{\mathbf{H}} &= \frac{i}{k} \nabla \times \tilde{\mathbf{E}}; & \tilde{\mathbf{E}} &= -\frac{i}{k} \nabla \times (\eta \tilde{\mathbf{H}}) \\ \nabla \cdot \tilde{\mathbf{H}} &= 0; & \nabla \cdot \tilde{\mathbf{E}} &= 0 \end{aligned} \quad (1.13.2)$$

Let the cavity be a rectangular box that extends from 0 to a along the x -axis, 0 to b along the y -axis, and 0 to d along the z -axis. Boundary conditions applied to perfectly conducting walls require all parallel electric field components to be zero at the surface. Since by Eq. (A.7.3) the fields are also spatial sinusoids the most general forms of possible electric field components are:

$$\begin{aligned} E_x &= E_1 \cos(k_x x) \sin(k_y y) \sin(k_z z) e^{i\omega t} \\ E_y &= E_2 \sin(k_x x) \cos(k_y y) \sin(k_z z) e^{i\omega t} \\ E_z &= E_3 \sin(k_x x) \sin(k_y y) \cos(k_z z) e^{i\omega t} \end{aligned} \quad (1.13.3)$$

Constants $E_1, E_2,$ and E_3 are specific to each particular problem. Since k satisfies Eq. (A.5.17) it is also a vector, and since by Eq. (1.13.2) the divergence is equal to zero, it follows that:

$$\mathbf{k} \cdot \tilde{\mathbf{E}} = 0 = k_x E_1 + k_y E_2 + k_z E_3 \quad (1.13.4)$$

Applying this condition shows that two of the field constants can be expressed as functions of the other. The electric field set is equal to:

$$\begin{aligned} E_x &= -\frac{k_x k_z}{k_x^2 + k_y^2} E_3 \cos(k_x x) \sin(k_y y) \sin(k_z z) \cos \omega t \\ E_y &= -\frac{k_y k_z}{k_x^2 + k_y^2} E_3 \sin(k_x x) \cos(k_y y) \sin(k_z z) \cos \omega t \\ E_z &= E_3 \sin(k_x x) \sin(k_y y) \cos(k_z z) \cos \omega t \end{aligned} \quad (1.13.5)$$

This leaves Eq. (1.13.5) with only one unknown field coefficient. Operating on Eq. (1.13.3) with the first curl equation of Eq. (1.13.2) gives the

accompanying set of magnetic field components:

$$\begin{aligned}\eta H_x &= \frac{i}{k} [k_y E_3 - k_z E_2] \sin(k_x x) \cos(k_y y) \cos(k_z z) e^{i\omega t} \\ \eta H_y &= \frac{i}{k} [k_z E_1 - k_x E_3] \cos(k_x x) \sin(k_y y) \cos(k_z z) e^{i\omega t} \\ \eta H_z &= \frac{i}{k} [k_x E_2 - k_y E_1] \cos(k_x x) \cos(k_y y) \sin(k_z z) e^{i\omega t}\end{aligned}\quad (1.13.6)$$

Substituting the field coefficients of Eq. (1.13.5) into (1.13.6) shows that $H_z = 0$ and the other field components are:

$$\begin{aligned}\eta H_x &= iE_3 \left(\frac{k_y k}{k_x^2 + k_y^2} \right) \sin(k_x x) \cos(k_y y) \cos(k_z z) e^{i\omega t} \\ \eta H_y &= iE_3 \left(\frac{k_x k}{k_x^2 + k_y^2} \right) \cos(k_x x) \sin(k_y y) \cos(k_z z) e^{i\omega t}\end{aligned}\quad (1.13.7)$$

Comparing Eqs. (1.13.5) and (1.13.7) shows that the electric and magnetic fields are out of time phase. Since the ideal cavity is lossless, the energy is constant and the total electric and magnetic energy is constant. That energy is, therefore, twice the magnetic energy. Integrating over the volume, $V = abd$, gives the total energy:

$$W = \frac{\varepsilon}{16} E_3^2 \left(1 + \frac{k_z^2}{k_x^2 + k_y^2} \right) V \quad (1.13.8)$$

With ℓ , m , n equal to integers, the conducting boundary condition is:

$$k_x = \frac{\ell\pi}{a}; \quad k_y = \frac{m\pi}{b}; \quad k_z = \frac{n\pi}{d} \quad (1.13.9)$$

Combining and introducing w as the energy per unit volume gives:

$$k_x = \frac{\ell\pi}{a}; \quad k_y = \frac{m\pi}{b}; \quad k_z = \frac{n\pi}{d} \quad (1.13.10)$$

$$w = \frac{\varepsilon}{16} E_3^2 \left(1 + \frac{(n+d)^2}{(\ell/a)^2 + (m/b)^2} \right) \quad (1.13.11)$$

The dual solution follows by starting with all possible magnetic field components then put $E_z = 0$. The two polarizations are independent and give dual results.

A related problem is to find the number of possible solutions within volume V . For this case let the cavity be cubic, from which $a = b = d$. The number of available states is equal to the number of spatial points in the positive quadrant of k -space. For integers ℓ , m , n much greater than one

the number of points is nearly equal to the volume of that quadrant and, in k -space, the unit length is π/a . The total number of points is therefore 1/4 the volume in phase space:

$$N = \frac{1}{8} \left(\frac{a}{\pi}\right)^3 \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^k k^2 dk = \frac{k^3 V}{6\pi^2} \quad (1.13.12)$$

The above argument follows from possible values of the electric field intensity inside the regions then obtaining the magnetic field and the condition $H_z = 0$ from it. The argument is equally valid starting with magnetic field intensity then obtaining the electric field, and the condition $E_z = 0$ from it, and gives an equal number of solutions. Therefore the total number of possible solutions is:

$$N = \frac{k^3 V}{3\pi^2} \quad (1.13.13)$$

The two solution types represent the two possible field polarizations.

The number of states between frequencies ω and $\omega + d\omega$ follows:

$$\frac{1}{V} dN = \frac{\omega^2}{\pi^2 c^3} d\omega \quad (1.13.14)$$

This expression may be used to evaluate the number of energy states available in free space by imagining all space to be in an enclosed system then letting the dimensions of the system become infinite.

1.14. From Energy to Electric Fields

The energy associated with an electric field is given by integral equations Eq. (1.9.10). Using it, it is commonly considered that the local energy density at each point in the field is

$$w(\mathbf{r}, t) = \varepsilon \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}^*(\mathbf{r}, t)/4 \quad (1.14.1)$$

It is often convenient to express this energy in terms of wave number \mathbf{k} . Since \mathbf{k} is a vector it may be used as a basis for dimensions, that is, in k -space. For this purpose it is convenient to express the field in coordinate space as an integral over all constituent parts in k -space:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathbf{E}_\omega(\mathbf{k}, \omega) e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} d\mathbf{k} d\omega \quad (1.14.2)$$

To evaluate the k -space field, $\mathbf{E}_\omega(\mathbf{k}, \omega)$, consider the integral expression drawn from Eq. (1.14.2):

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{-i(\omega' t - \mathbf{k}' \cdot \mathbf{r})} d\mathbf{r} dt \\ &= \int_{-\infty}^{\infty} d\mathbf{k} d\omega \int_{-\infty}^{\infty} \mathbf{E}_\omega(\mathbf{k}, \omega) e^{i([\omega - \omega'] t - [\mathbf{k} - \mathbf{k}'] \cdot \mathbf{r})} d\mathbf{r} dt \\ &= (2\pi)^2 \int_{-\infty}^{\infty} d\mathbf{k} d\omega \mathbf{E}_\omega(\mathbf{k}, \omega) \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}') = (2\pi)^2 \mathbf{E}_\omega(\mathbf{k}', \omega') \end{aligned}$$

It follows that:

$$\mathbf{E}_\omega(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} d\mathbf{r} dt \quad (1.14.3)$$

The two forms of electric field intensity, therefore, form a Fourier integral transform pair. It follows that the electric field energy in k -space is:

$$w(\mathbf{k}, \omega) = \varepsilon \mathbf{E}(\mathbf{k}, \omega) \cdot \mathbf{E}^*(\mathbf{k}, \omega) / 4 \quad (1.14.4)$$

It follows from Eqs. (1.14.1) and (1.14.4) that in both coordinate systems the field intensities are proportional to the square root of the energy density. Since only the scalar product between the field intensities is known, three-dimensional vectors are not completely specified by this argument. It is, however, complete for one-dimensional cases such as, for example, scalar fields.

References

- R. Becker, *Electromagnetic Fields and Interactions*, Blaisdell Publishing Co. and Blackie and Son (1964), reprinted by Dover Publications (1982).
- A. Einstein, B. Podolsky, N. Rosen, "Can a Quantum-Mechanical Description of Physical Reality be Considered Complete?" *Phys. Rev.* **47** (1935) 777–780.
- W.W. Hansen, "A New Type of Expansion in Radiation Problems," *Phys. Rev.* **47** (1935) 139–143.
- J.D. Jackson, *Classical Electrodynamics*, 2nd edition, John Wiley (1975).
- L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, trans. by H. Hamermesh, Addison-Wesley (1951).
- W.K.H. Panofsky, M. Phillips, *Classical Electricity and Magnetism*, 2nd edition, Addison-Wesley (1961).
- A. Sommerfeld, *Electrodynamics*, Academic Press (1952).
- J.A. Stratton, *Electromagnetic Theory*, McGraw-Hill (1941).
- J.B. Westgard, *Electrodynamics: A Concise Introduction*, Springer-Verlag (1997).