QUANTIZING TEICHMÜLLER SPACES USING GRAPHS

L. CHEKHOV
Gubkina 8,
119991, GSP-1,
Moscow, Russia
E-mail: chekhov@genesis.mi.ras.ru

We construct the mapping class group transformations that satisfy the pentagon relation for classical and quantum Teichmüller spaces coordinatized in terms of graphs. We derive classical and quantum geodesic algebras governed by the corresponding skein relations.

1. Introduction

This paper is based on the results of our joint papers with V. Fock [3]. The main goal of this activity was to describe a Hilbert space and observable algebra for 3D quantum gravity. For this, we use the following scheme. According to E. Verlinde and H. Verlinde [5], the classical phase space of Einstein gravity in a 3D manifold is the Teichmüller space of its boundary. (Analogously, the classical phase space for 3D Chern–Simons theory is the moduli space of flat connections on the boundary, which was quantized in [6,7].) The Teichmüller space possesses the canonical (Weil–Petersson) Poisson structure and the symmetry group, which is the mapping class group. According to the correspondence principle, (1) the observable algebra of the corresponding quantum theory is the noncommutative deformation of the $*$-algebra of functions on it governed by the Poisson structure, (2) the Hilbert spaces of the theory is the $*$-representation space of this algebra, and (3) the symmetry group acts on the algebra of observables by automorphisms. Assuming the quantization of a Poisson manifold exists and is unique we can solve this problem by constructing a family of $*$-algebras, which depend on the quantization parameter $\hbar$, and an action of the mapping class group on this family by outer automorphisms and showing that this algebra and the action reproduces the classical algebra, the classical action, and the Poisson structure in the limit $\hbar \to 0$.

In [3], a very closely related problem of describing *open* 2D surfaces was solved. The corresponding Teichmüller space has a degenerate Weil–Petersson Poisson structure, while the mapping class group is a symmetry group. We have managed to describe the deformation quantization of the corresponding Teichmüller space, the action of the mapping class group by outer automorphisms, the representations of the algebra, and the induced action of the mapping class group on the representation space. Following [5], the representation space of the observable algebra can be also interpreted as the space of conformal blocks of the Liouville conformal field theory. Our construction can be therefore interpreted as the construction of the conformal block spaces and the mapping class group actions for this CFT.

A mapping class group action can be conveniently described using a triangulation of the surface, which admits the description in terms of graphs. This construction is close to the cell decomposition of the moduli space a la Penner and Kontsevich [8,9].

The key point of the quantization procedure is constructing the quantum mapping class group transformation that consitently defines the morphisms between quantum $*$-algebras simultaneously preserving the quantum geodesic algebra.

The main mathematical ingredient of the construction is a version of the quantum dilogarithm by L. D. Faddeev [10]. We interpret the corresponding five-term relation as the only nontrivial relation in a certain groupoid having the mapping class group as the maximal subgroup. A similar construction has been made independently and simultaneously by R. M. Kashaev [11]. However, our construction seems to be simpler and more universal. It differs from the Kashaev construction by the number of variables in play. Our coordinates describing the Teichmüller spaces (see below) span a linear subspace in the space of the kashaev coordinates for the Liouville theory. Although being very similar technically (for instance, the spectrum of quantum Dehn twists [12] coincides in the both approaches), these two coordinate sets admit different interpretations. Say, our set seems to be more suitable for describing Teichmüller spaces and geodesic algebras as it matches the proper Teichmüller space dimension.

2. Classical Teichmüller spaces

Recall briefly a classical description of Teichmüller spaces of complex structures on Riemann surfaces with holes.

Teichmüller space T^h is a space of complex structures on a (possibly open) Riemann surface S modulo diffeomorphisms homotopy equivalent to identity. In the vicinity of a boundary component, the complex structure is isomorphic as a complex manifold either to an annulus (hole) or to a punctured disc (puncture).

For technical reasons, instead of the Teichmüller space $T^h(S)$ we consider its finite covering $T^H(S)$. A point of $T^H(S)$ is determined by a point of $T^h(S)$ and by the orientations of all holes (not punctures) of S. This covering is obviously ramified over the subspace of surfaces with punctures.

An oriented 2D surface can be continuously conformally transformed to the constant curvature surface. The Poincaré uniformization theorem claims that any complex surface S of a constant negative curvature (equal -1 in what follows) is a quotient of the upper half-plane $\mathbf{H}_+$ endowed with the hyperbolic metric $ds^2 = dzd\bar{z}/(\Im z)^2$ w.r.t. the action of a discrete Fuchsian subgroup $\Delta(S)$ of the automorphism group $PSL(2,\mathbf{R})$,

$$S = \mathbf{H}_+/\Delta(S).$$

In the hyperbolic metric, geodesics are either half-infinite circles with endpoints at the real line $\mathbf{R}$ or vertical half-lines; all points of the boundary $\mathbf{R}$ are at infinite distance from each other and from any interior point.

Any hyperbolic homotopy class of closed curves γ contains a unique *closed geodesic* of the length $l(\gamma) = \log|\lambda_1/\lambda_2|$, where λ_1 and λ_2 are (different) eigenvalues of the element of $PSL(2,\mathbf{R})$ that corresponds to γ. Recall that the mapping class group $D(S)$ is the group of homotopy classes of diffeomorphisms of the surface S. In this section, we give a simple combinatorial description of $D(S)$ for any open surface S.

A fat graph that is embedded into an oriented surface inherits the canonical fat structure from the surface orientation.

Denote by $|\Gamma|(S)$ the set of combinatorial types of three-valent graphs corresponding to a given surface. For any element of $|\Gamma|(S)$ we fix a *marking*, i.e., a numeration of the edges. Denote by $\Gamma(S)$ the set of isotopy classes of embeddings of marked fat graphs into S. The presence of the marking changes the set of embeddings since some graphs may have nontrivial symmetry group. Introducing the marking is a tool to remove this symmetry. (Here and below the vertical lines $|\cdot|$ indicate the diffeomorphism class.)

The mapping class group $D(S)$ obviously acts freely on the space of embedded marked graphs having the space of combinatorial graphs as a quotient,

$$\Gamma(S)/D(S) = |\Gamma|(S).$$

Recall that a group can be thought of to be a category with only one object and with all morphisms being invertible. Analogously, a *groupoid* is just a category such that all morphisms are invertible and such that any two objects are related by at least one morphism. Since the automorphism groups of different objects of a groupoid are obviously isomorphic to each other, we can associate a group to a groupoid in the canonical way. We are going to construct the groupoid giving the mapping class group and admitting a simpler description in terms of generators and relations than the mapping class group itself.

Definition 2.1. Let the set $|\Gamma|(S)$ be the set of objects. For any two graphs $|\Gamma|, |\Gamma_1| \in |\Gamma(S)|$ let a morphism from $|\Gamma|$ to $|\Gamma_1|$ be a homotopy class of marked embeddings of both $|\Gamma|$ and $|\Gamma_1|$ into S modulo the diagonal mapping class group action; we denote this morphism by $|\Gamma, \Gamma_1|$. If we have three embedded marked graphs $\Gamma, \Gamma_1, \Gamma_2$, then by definition the composition of $|\Gamma, \Gamma_1|$ and $|\Gamma_1, \Gamma_2|$ is $|\Gamma, \Gamma_2|$. The above described category is called the *modular groupoid.*

One can easily verify that (1) the multiplication of morphisms is unambiguously defined; (2) the class of the diagonal embedding $|\Gamma, \Gamma|$ is the identity morphism and the inverse of the morphism $|\Gamma, \Gamma_1|$ is $|\Gamma_1, \Gamma|$; (3) the group of automorphisms of an object is the mapping class group $D(S)$.

To give a description of the modular groupoid by generators and relations we need to introduce the distinguished sets of morphisms called *flips* and *graph symmetries.* We call a morphism $|\Gamma, \Gamma_\alpha|$ a flip if the embedding Γ_α is obtained from the embedding Γ by shrinking an edge α and blowing up the obtained four-valent vertex in the other direction (see Fig. 2 below). We use the notation Γ_α in order to emphasize the relation of this graph to the graph Γ. Note that for the given graph Γ, several marked embedded graphs may be denoted by Γ_α because no marking of Γ_α is indicated.

To each symmetry σ of a graph Γ we associate an automorphism, which is just $|\Gamma, \Gamma_\sigma|$.

There is no canonical identification of edges of different graphs even if a morphism between them is given. However, for two graphs related by a flip, we can introduce such an identification. It is especially transparent in the dual picture where a flip just replaces one edge by another. Hence, one can identify the set of edges of two graphs as far as a representation of a morphism between the graphs as a sequence of flips is given. We exploit this identification and denote the corresponding edges of different graphs by the same letter if it is clear which sequence of flips relating these graphs is considered. To avoid confusion, note that this identification has nothing to do with the marking.

In this notation, the graph $\Gamma_{\alpha_1\cdots\alpha_n}$ is the graph obtained as a result of consecutive flips α_n, …, α_1 of edges of a given graph Γ. There are three kinds of relations between flips, which are satisfied for any choice of marking for the graphs entering the relations.

Proposition 2.1. *A square of a flip is a graph symmetry: if* $|\Gamma_\alpha,\Gamma|$ *is a flip in an edge* α*, then* $|\Gamma,\Gamma_\alpha|$ *is also a flip and*[a] ***R.2.*** $|\Gamma,\Gamma_\alpha||\Gamma_\alpha,\Gamma| = 1$. *Flips in disjoint edges commute: if* α *and* β *are two edges having no common vertices, then* ***R.4.*** $|\Gamma_{\alpha\beta},\Gamma_\alpha||\Gamma_\alpha,\Gamma| = |\Gamma_{\alpha\beta},\Gamma_\beta||\Gamma_\beta,\Gamma|$. *Five consecutive flips in edges* α *and* β *having one common vertex is the identity: for such* α *and* β*, the graphs* $\Gamma_{\alpha\beta}$ *and* $\Gamma_{\beta\alpha}$ *are related by a flip and* ***R.5.*** $|\Gamma,\Gamma_\alpha||\Gamma_\alpha,\Gamma_{\beta\alpha}||\Gamma_{\beta\alpha},\Gamma_{\alpha\beta}||\Gamma_{\alpha\beta},\Gamma_\beta||\Gamma_\beta,\Gamma| = 1$.

The proofs of relations **R.2** and **R.4** are obvious. Relation **R.5** can be seen more transparently in the dual graph picture. Indeed, a graph dual to a three-valent graph is a graph having triangular faces. A flip of the original graph corresponds to removing an edge on the dual graph and inserting another diagonal of the appearing quadrilateral. Figure 1 shows that the combination of the five flips is the identity. **q.e.d.**

Theorem 2.1. *1. Flips and graph symmetries generate the modular groupoid. 2. The only relations between the generators are* ***R.2, R.4, R.5****, and the natural relations between flips and graph symmetries.*

Replacing the mapping class group by the modular groupoid, we can simply express the latter through generators and relations.

Note that a graph symmetry can be represented as a ratio of two flips in a given edge and the modular groupoid is therefore generated only by the flips. We do not describe relations between flips and graph symmetries in details because they are quite obvious. In fact, the symmetry groups of Γ and Γ_α act transitively on on the set of flips $|\Gamma,\Gamma_\alpha|$, and this action can be considered as relations between flips and graph symmetries.

Theorem 1 can be proved using direct combinatorial methods of the simplicial geometry (cf. Viro [13]). However, we give the main idea of another proof, which is more specific for the 2D situation.

Proof of Theorem 1.

To any connected simplicial complex S we can associate a groupoid by taking a point in each top-dimensional simplex for objects and the homotopy classes of oriented paths connecting the chosen points as morphisms.

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[a] The notation **R.n** indicates the number n of graphs entering this relation.

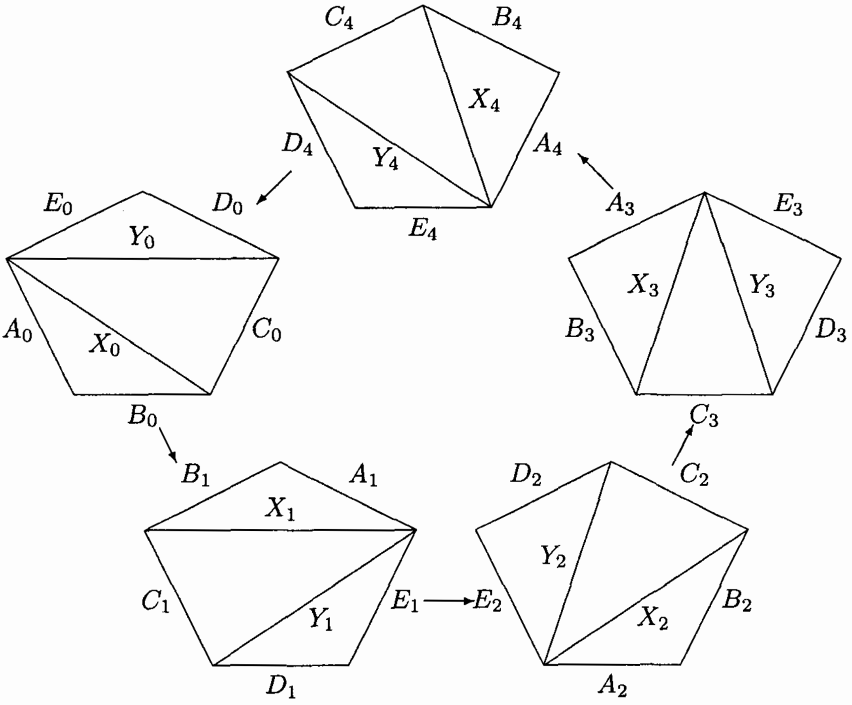

Figure 1. The combination of the five flips is the identity.

The corresponding group is the fundamental group of the topological space given by the complex.

To any codimension one simplex we can associate two classes of paths (differing by orientation and having the identity morphism as their product) connecting adjacent top-dimensional simplices. It is natural to call them *flips.* We can associate a relation between the flips to any codimension two simplex. It is obvious that this set of flips generates the groupoid and that the only relation between the flips are given by codimension two simplices.

The same is true for an orbifold simplicial complex, where we replace simplices by quotients of simplices by finite groups. In this case, we must choose one generic point per each top-dimensional simplex as an object and orbifold homotopy classes of paths as morphisms. The corresponding group is the orbifold fundamental group of the orbifold given by the complex. The groupoid is now generated by flips and groups of top dimension simplices and still the only nontrivial relations are those given by codimension two simplices.

Let us consider the Strebel [14] orbifold simplicial decomposition of the moduli space of complex structures on S. We preserve the notation $\Gamma(S)$ for the (open domain of) Strebel simplicial complex with the cells of lower dimensions corresponding to fat graphs with the vertices of higher valences allowed. The true modular group (not the groupoid) $D(S)$ is then the group of all nontrivial morphisms $|\Gamma,\Gamma|$ for a given graph Γ. It is then given by a representation of the group $\pi_1(\Gamma(S))$. The obstacle for this group to be trivial lies just in reductions of the surface S: the closure of the Strebel graph complex is provided by additional cells that correspond to reductions of the Riemann surface S and are not described by fat graphs. Thus the orbifold fundamental group of the moduli space M is just the mapping class group $D(S)$. Recall that Strebel orbisimplices are enumerated by fat graphs corresponding to S and the dimension of a simplex is equal to the number of its edges. One can easily see that the groupoid of the Strebel complex coincides with the modular groupoid. Moreover, the flips of the former correspond to the flips of the latter. The relations between flips are given by codimension two cells, which correspond either to graphs with two four-valent vertices (which produces relation **R.4**) or to graphs with one five-valent vertex (which produces relation **R.5**). Relation **R.2** holds true for any simplicial complex. **q.e.d.**

2.0.1. *Teichmüller space description using graphs.*

Since Strebel [14], the fat graphs have been used to coordinatize the moduli space. After Penner [8], the fat graphs are used for describing not only moduli, but also Teichmüller spaces. We use a version of this description, which is rather explicit and simple.

Theorem 2.2. *Given a three-valent graph $\Gamma \in \Gamma(S)$ of genus g and number of punctures n, there exists a one-to-one correspondence between the set of points of $T^H(S)$ and the set $\mathbf{R}^{\#\,edges}$ of edges of this graph endowed with real numbers (lengths).*

We propose an explicit way for constructing the Fuchsian group $\Delta(S) \subset PSL(2,\mathbf{R})$ corresponding to a given set of numbers on edges of a graph

$\Gamma \in \Gamma(S)$ such that $S = \mathbf{H}_+/\Delta(S)$.[b] For this, we must associate an element $P_\gamma \in PSL(2, \mathbf{R})$ to any element of the fundamental group $\gamma \in \pi_1(S)$. We associate the matrix $X_{Z_\alpha} \in PSL(2, \mathbf{R})$ of the Möbius transformation to each edge α,

$$X_{Z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix}. \tag{1}$$

In order to parametrize a *path* over edges of the graph, we introduce the matrices of the "right" and "left" turns

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L \equiv R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \tag{2}$$

and the operators R_Z and L_Z of the respective "right" and "left" turns,

$$R_Z \equiv RX_Z = \begin{pmatrix} e^{-Z/2} & -e^{Z/2} \\ 0 & e^{Z/2} \end{pmatrix}, \tag{3}$$

$$L_Z \equiv LX_Z = \begin{pmatrix} e^{-Z/2} & 0 \\ -e^{-Z/2} & e^{Z/2} \end{pmatrix}. \tag{4}$$

We introduce now the notion of *geodesic* (closed geodesic curve) on the graph. Let a *closed path* in the graph Γ be any oriented path, which starts and terminates at an oriented edge of the corresponding graph. To each such path, we set into the correspondence the product of matrices

$$P_{Z_1 \dots Z_n} = L_{Z_n} L_{Z_{n-1}} R_{Z_{n-2}} \cdots R_{Z_2} L_{Z_1}, \tag{5}$$

where the matrices L_{Z_i} or R_{Z_i} are inserted depending on which turn—left or right—the path is going on the corresponding step (paths with turnings back not at the terminate point are equivalent to the corresponding paths (5) with these turnings removed). If path (5) has the turning back at the terminate point, we must add additional matrices L or R to it, say, $P_{Z_1 \dots Z_n} \to LP_{Z_1 \dots Z_n} L$. Then, $\operatorname{tr} P_{Z_1 \dots Z_n}$ becomes independent on this turning back as well.

Proposition 2.2. [15] *There is a one-to-one correspondence between the set of conjugate classes of oriented paths* $\{P_{Z_1 \dots Z_n}\}$ *and closed (oriented) geodesics* $\{\gamma\}$ *on the moduli space; the length* L_γ *of a geodesic is determined from the relation*

$$G_\gamma \equiv 2\cosh(L_\gamma/2) = \operatorname{tr} P_{Z_1 \dots Z_n}, \tag{6}$$

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[b] Note that $\pi_1(S)$ is isomorphic to $\pi_1(\Gamma)$.

where $P_{Z_1 \dots Z_n}$ is a unique closed path without turning backs (it is not necessarily a representative of $\pi_1(\Gamma)$ as far as the starting point is erased) that corresponds to a given conjugate class of oriented paths in Γ.

In what follows, we call G_γ the *geodesic function.*

Example 2.1. Consider the geodesic γ_j surrounding the jth hole (round-the-face geodesic). Then, obviously, it has the form $\operatorname{tr} R_{Z_1} R_{Z_2} \dots R_{Z_n}$, or $\operatorname{tr} L_{Z_1} L_{Z_2} \dots L_{Z_n}$ depending on the orientation. In this case, all of the matrices R_x (L_x) are upper (lower) triangular and formula (6) implies that

$$l_{\gamma_j} = \left| \sum_{i=1}^{n} Z_i \right| .$$

The sign of this sum gives the orientation of the hole.

For each embedded graph $\Gamma \in \Gamma(S)$ we therefore obtain a global coordinate system. Hence, there exists a transition map between coordinate systems corresponding to different embedded graphs, which, in other words, is a morphism of the modular groupoid. It follows from Theorem 2.1 that any transition map can be expressed as a composition of the transition maps corresponding to flips.

For the flip, the transition map is Eq. (8) below. According to Theorem 2.1, these transition maps give the coordinate expression for the action of the mapping class group.

2.0.2. *Weil–Petersson forms*

A canonical Poisson structure called the Weil–Petersson structure exists on $T^H(S)$. This structure is degenerate, and its Casimir functions are just the lengths of geodesics surrounding holes.

In the coordinates $\{Z_\alpha\}$, the Weil–Petersson bracket B_{WP} has a very simple form

Theorem 2.3.

$$B_{WP} = \sum_{v} \sum_{i=1}^{3} \frac{\partial}{\partial Z_{v_i}} \wedge \frac{\partial}{\partial Z_{v_{i+1}}}, \tag{7}$$

where the sum is taken over all vertices v, while v_i, $i = 1, 2, 3 \bmod 3$, are labels of cyclically ordered edges incident to this vertex.

The proof [15] relies on the independence of this form on the choice of the embedded graph and of the set of Casimir functions.

The graph transformation that preserves Poisson structure (7) under the *flip* operation and becomes an identity after the series of flip transformations depicted in Fig. 1 for the dual graph (i.e., satisfies the *pentagon identity*) has the form depicted in Fig. 2 where

$$M_Z : \{A,B,C,D,Z\} \to \{A+\phi(Z), B-\phi(-Z), C+\phi(Z), D-\phi(-Z), -Z\} \tag{8}$$

and, in the classical case [15],

$$\phi(Z) = \log(e^Z + 1). \tag{9}$$

Lemma 2.1. *Transformation (8) preserves the traces of products over paths (6), so the classical geodesic lengths are invariant under the action of the mapping class group.*

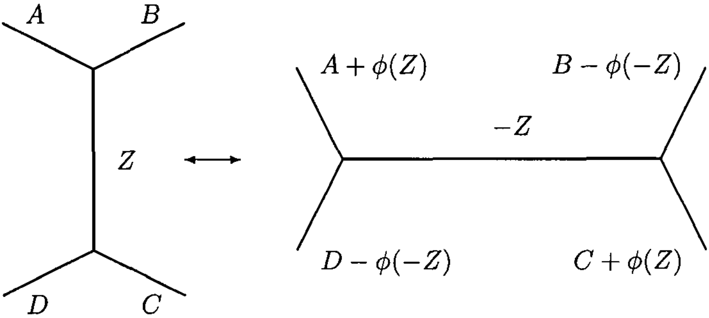


Fig. 2

Although it tautologically follows from geometry (see [15]), it can be verified straightforwardly that transformation (8) with the function $\phi(Z)$ from (9) gives $M_Z^2 = I$ and satisfies the pentagon relation depicted in Fig. 1.

2.0.3. *Geodesic laminations*

In what follows, geodesics that have no self intersections are important. In fact, these geodesics constitute a basis both in classical and quantum geodesic spaces.

Definition 2.2. Let a *geodesic lamination* (GL) on a two-dimensional surface be a homotopy class of a collection of finite number of self- and mutually nonintersecting unoriented closed curves (closed geodesics). We set into the correspondence to each GL the product $G_{\gamma_1} \dots G_{\gamma_n}$ of quantities (6) of all geodesics constituting a GL. Then, a GL containing a contractible curve of length zero is twice the GL with this curve removed.

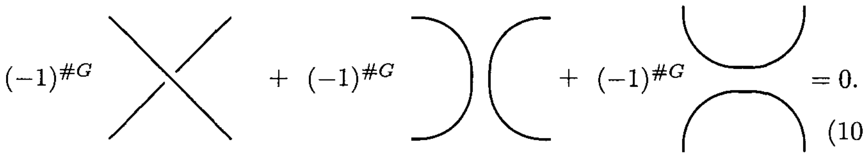
We call a GL *simple* if it consists of a single (nonselfintersecting) geodesic. We call a geodesic *graph simple* if it does not pass the same edge of the graph more than once. The set of all graph simple geodesics obviously depends on the choice of a graph and is always finite.

2.1. *Poisson algebra of geodesics and classical skein relations*

The functions $\{G_\gamma\}$ (6) were studied in [16]. They generate an algebra (w.r.t. both the multiplication and the Weil–Petersson Poisson bracket) over **Z**.

2.1.1. *Classical skein relation*

Using the relation $\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) - \operatorname{tr} A \cdot \operatorname{tr} B = 0$, which holds for arbitrary 2×2 matrices A and B with unit determinants, we can "disentangle" any product of geodesic functions, i.e., express it as a unique finite linear combination of GLs. Let us consider a part of the graph and introduce the additional factor $\#G$—the total number of geodesics in the graph that is a union of the corresponding subgraph and the rest of the graph, which is assumed to be the same for all subgraphs below,—we can uniformly present the classical skein relation as



(10)

2.1.2. *Elementary Poisson algebra*

Consider now the Poisson structure of geodesics. Two nonintersecting geodesics have trivial bracket and, by Lemma 2.1, the same holds for arbitrary two geodesics entering one and the same GL. Because a Poisson bracket $\{A, B\}$ can be always presented as a sum of the Poisson brackets of constituents of A and B (the Leibnitz rule), it suffices to consider only "simple" intersections of two geodesics with the respective geodesic functions G^1 and G^2 of the form

$$G^1 = \operatorname{tr}{}^1 \ldots X^1_C R^1 X^1_Z L^1 X^1_A \ldots, \tag{11}$$

$$G^2 = \operatorname{tr}{}^2 \ldots X^2_B L^2 X^2_Z R^2 X^2_D \ldots, \tag{12}$$

where the superscripts 1 and 2 pertain to the matrix spaces.

A bracket between X^1_C and X^2_B possesses a simple r-matrix structure,

$$\{X^1_C, X^2_B\} = \frac{1}{4}(-1)^{i+j}(e^1_{ii} \otimes e^2_{jj}) X^1_C \otimes X^2_B, \tag{13}$$

where the matrix e_{ij} has the unity at the crossing of ith line and jth column and is zero otherwise. Then, direct calculations give

$$\{G_1, G_2\} = \frac{1}{2}(G_{\mathrm{H}} - G_{\mathrm{I}}), \tag{14}$$

where G_{I} corresponds to the geodesic that is obtained by erasing the edge Z and joining together the edges "A" and "D" as well as "B" and "C" in a natural way (the second term in (10)); G_{H} corresponds to the geodesic that passes over the edge Z twice, so it has the form $\operatorname{tr} \ldots X_C R_Z R_D \ldots$ $\ldots X_B L_Z L_A \ldots$ (the third term in (10)). These are the relations obtained by Goldman [16] in the continuous parameterization (the classical Turaev–Viro algebra).

Remark 2.1. Because the Poisson bracket satisfies the Leibnitz rule, relation (14) can be naturally generalized to the case of multiple intersections (which may occur at the same edge as well).

Example 2.2. For the torus $(T^h_{1,1})$, we have three generators X, Y, Z such that

$$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 2$$

(the Casimir element is $X + Y + Z$). Then, the geodesic functions for three graph simple geodesics are

$$\begin{aligned} G_X &= \operatorname{tr} L X_Y R X_Z = e^{-Y/2-Z/2} + e^{-Y/2+Z/2} + e^{Y/2+Z/2}, \\ G_Y &= \operatorname{tr} L X_Z R X_X = e^{-Z/2-X/2} + e^{-Z/2+X/2} + e^{Z/2+X/2}, \\ G_Z &= \operatorname{tr} L X_X R X_Y = e^{-X/2-Y/2} + e^{-X/2+Y/2} + e^{X/2+Y/2}. \end{aligned} \tag{15}$$

Introducing the geodesic function

$$\widetilde{G_Z} = \operatorname{tr} R_Z R_X L_Z L_Y = e^{-X/2-Y/2-Z} + e^{X/2-Y/2}(e^{-Z} + e^{Z} + 2) + e^{X/2+Y/2+Z},$$

obtained from G_Z by the flip transformation, we have $\{G_X, G_Y\} = \widetilde{G}_Z/2 - G_Z/2$, and because relation (10) implies that $G_X G_Y = G_Z + \widetilde{G}_Z$, we obtain

$$\{G_X, G_Y\} = \frac{1}{2} G_X G_Y + G_Z, \tag{16}$$

i.e., the classical Poisson algebra becomes closed on the subset $\{G_X, G_Y, G_Z\}$ of geodesic functions (other relations are obtained by cyclic permutations in (16)) for the price of introducing elements of the second order in the r.h.s. Algebra (16) is the semiclassical limit of the Poisson $so_q(3)$ algebra, which is the deformation quantization of (16).

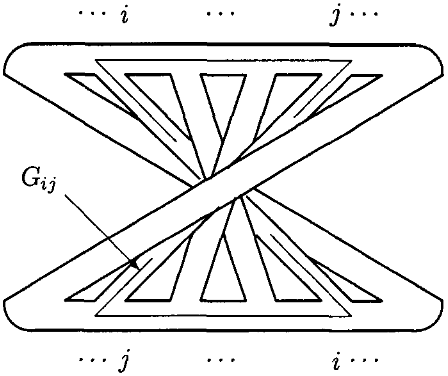
2.2. ***General Poisson algebras of geodesics***

In order to generalize Example 1, we must find the graph on which graph simple geodesics constitute a convenient algebraic basis. This is the graph in Fig. 3 in which m edges connect crosswise two horizontal line subgraphs; two-point vertices at the ends of these subgraphs are fictitious. Simple closed geodesics in this picture are those and only those that pass exactly two different "vertical" edges of the graph (see the example in Fig. 3); they can be therefore enumerated by the numbers of these edges, and we denote them G_{ij} where $i < j$. The geodesic Poisson algebra for G_{ij} is

$$\{G_{ij}, G_{kl}\} = \begin{cases} 0, & j < k, \\ 0, & k < i,\ j < l, \\ G_{ik} G_{jl} - G_{kj}\ G_{il}, & i < k < j < l, \\ \frac{1}{2}\ G_{ij}\ G_{jl} -\ G_{il}, & j = k, \\ G_{il} - \frac{1}{2}\ G_{ij}\ G_{il}, & i = k,\ j < l \\ G_{ik} - \frac{1}{2}\ G_{ij}\ G_{kj}, & j = l,\ i < k. \end{cases} \tag{17}$$

The graph in Fig. 3 has genus $\frac{m}{2} - 1$ and *two* holes if m is even and genus $(m-1)/2$ and *one* hole if m is odd. Such geodesic bases for m even and *smooth* Riemann surfaces were considered in [17]. The Poisson algebras of geodesics obtained there *coincide exactly* with (17). These are the so-called $so_q(m)$ algebras whose representations were constructed in [19].

In mathematical literature, this algebra have also appeared as the Poisson algebra of the monodromy data (Stokes matrices) of some matrix differential equation [20] and on the symplectic groupoid of upper-triangular matrices A [21]. For the $m \times m$-matrices, there are $\left[\frac{m}{2}\right]$ central elements of this algebra generated by the polynomial invariants $f_G(\lambda) \equiv \det(G + \lambda G^T) = \sum f_i(G)\lambda^i$. The total Poisson dimension d of algebra (17) is $\frac{m(m-1)}{2} - \left[\frac{m}{2}\right]$, and for $m = 3, 4, 5, 6, \ldots$ we have $d = 2, 4, 8, 12, \ldots$. The dimensions of the corresponding Teichmüller spaces are $D = 2, 4, 8, 10, \ldots$, so we see that the Teichmüller spaces are embedded as the *Poisson leaves* into algebra (17); the dimensions of these special leaves become lesser than the highest dimensions of the Poisson representations of algebras (17) starting with the genus 2 surface with two holes.



3. Quantization

A quantization of a Poisson manifold, which is eqiuvariant under a discrete group D action, is a family of $*$-algebras $A^{\hbar}$ depending on a positive real parameter $\hbar$ with D acting by outer automorphisms and having the following properties. **1.** (Flatness.) All algebras are isomorphic (noncanonically) as linear spaces. **2.** (Correspondence.) For $\hbar = 0$, the algebra is isomorphic as a D-module to the $*$-algebra of complex-valued functions on the Poisson manifold. **3.** The Poisson bracket on A^0 given by $\{a_1, a_2\} = \lim_{\hbar\to 0} \frac{[a_1,a_2]}{\hbar}$ coincides with the Poisson bracket given by the Poisson structure of the manifold. Here we construct a quantization of a Teichmüller space $T^{\hbar}(S)$ that is equivariant w.r.t. the mapping class group. Let $T^{\hbar}(\Gamma)$, where $\Gamma \in \Gamma_0(S)$, be an algebra generated by the generators $Z^{\hbar}_{\alpha}$ (one generator per one (unoriented) edge α) with the relations

$$[Z^{\hbar}_{\alpha}, Z^{\hbar}_{\beta}] = 2\pi i\hbar\{z_{\alpha}, z_{\beta}\} \tag{18}$$

(cf. (7)) and the $*$-structure

$$(Z^{\hbar}_{\alpha})^{*} = Z^{\hbar}_{\alpha}. \tag{19}$$

Here z_{α} and $\{\cdot,\cdot\}$ stand for the respective coordinate function and the Poisson bracket on the classical Teichmüller space. Because of (7), the right-hand side of (18) is a constant taking only five values 0, $\pm 2\pi i\hbar$, and $\pm 4\pi i\hbar$.

On the modular groupoid language, we have constructed one $*$-algebra per object. Now, in order to describe the equvariance we must associate a homomorphism of the correponding $*$-algebras to any morphism of the modular groupoid. For this, we must associate a morphism of algebras to any flip and verify that the relation $M_Z^2 = I$ (Fig. 2) and the five-term relation depicted in Fig. 1 for the dual graph are satisfied by the algebra morphisms.

We now define the flip morphisms by Eq. (8) with the (quantum) function

$$\phi(z) \equiv \phi^{\hbar}(z) = -\frac{\pi\hbar}{2}\int_{\Omega}\frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp, \tag{20}$$

where the contour Ω goes along the real axis bypassing the origin from above.

Function (20) is the Faddeev [10] generalizations of the quantum dilogarithm. The described algebras and morphisms between them must satisfy the following properties.

Theorem 3.1. *1. The family of algebras* $T^{\hbar}(\Gamma)$ *is a quantization of* $T(S)$*, where* $\Gamma \in \Gamma_0(S)$*. 2. The center of the algebra is generated by the sums* $\sum_{\alpha\in I} Z_\alpha^{\hbar}$ *over all edges* $\alpha \in I$ *surrounding a given face* I*. 3. In the limit* $\hbar \mapsto 0$*, morphism (8), (20) coincides with the classical morphism (8), (9). 4. Morphism (8), (20) is indeed the $*$-algebra morphism. 5. The morphisms* $T^{\hbar}(\Gamma) \to T^{1/\hbar}(\Gamma)$ *given by* $Z_\alpha^{\hbar} \mapsto Z_\alpha^{1/\hbar}$ *commute with morphisms (8). 6. The flip morphisms satisfy the two-term relation* $M_Z^2 = I$ *(cf. (8)). 7. The flip morphisms satisfy the pentagon identity in Fig. 1.*[c]

Proofs. The proofs of Properties 1, 2, and 6 are obvious. The proof of Property 3 is provided if

$$\lim_{\hbar\to 0}\phi^{\hbar}(z) = \log(e^{z}+1). \tag{21}$$

Taking into account that $\lim_{z\to-\infty}\phi^{\hbar}(z) = 0$, it suffices to prove that $\lim_{\hbar\to 0}\frac{\partial}{\partial z}\phi^{\hbar}(z) = \frac{1}{e^{-z}+1}$. The left-hand side of the latter equality can be

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[c] This result was independently obtained by R. M. Kashaev [11].

easily computed using residues:

$$\lim_{\hbar\to 0}\frac{\partial}{\partial z}\phi^{\hbar}(z)=\lim_{\hbar\to 0}-\frac{\pi\hbar}{2}\int_{\Omega}\frac{-ipe^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=\frac{i}{2}\int_{\Omega}\frac{e^{-ipz}}{\sinh(\pi p)}dp$$
$$=\frac{i}{2}\frac{1}{1+e^{z}}\left(\int_{\Omega}-\int_{\Omega+i}\right)\frac{e^{-ipz}}{\sinh(\pi p)}dp=\frac{-1}{1+e^{z}}\mathrm{Res}_{z=i\pi}\frac{e^{-ipz}}{\sinh(\pi p)}$$
$$=\frac{1}{e^{-z}+1}.\ \square \tag{22}$$

In order to prove Property 4 we must verify that $[A+\phi^{\hbar}(Z),B-\phi^{\hbar}(-Z)]=0$ and $[A+\phi^{\hbar}(Z),D-\phi^{\hbar}(-Z)]=-2\pi i\hbar$ (other relations are obviously satisfied). For this, we first verify the identity

$$\phi^{\hbar}(z)-\phi^{\hbar}(-z)=z. \tag{23}$$

Indeed,

$$\phi^{\hbar}(z)-\phi^{\hbar}(-z)=-\frac{\pi\hbar}{2}\int_{\Omega}\frac{e^{-ipz}-e^{ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=$$
$$-\frac{\pi\hbar}{2}\left(\int_{\Omega}+\int_{-\Omega}\right)\frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=\frac{\pi\hbar}{2}2\pi i\mathrm{Res}_{z=0}\frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}=z.$$

Using this property, we can transform the commutators,

$$[A+\phi^{\hbar}(Z),B-\phi^{\hbar}(-Z)]=[A,B]-[A,\phi^{\hbar}(-Z)]+[\phi^{\hbar}(Z),B]=$$
$$=2\pi i\hbar\left(-1-\frac{\partial}{\partial Z}(-\phi^{\hbar}(-Z))+\frac{\partial}{\partial Z}\phi^{\hbar}(Z)\right)=0.$$

Analogously,

$$[A+\phi^{\hbar}(Z),D-\phi^{\hbar}(-Z)]=[A,D]-[A,\phi^{\hbar}(-Z)]+[\phi^{\hbar}(Z),D]=$$
$$=2\pi i\hbar\left(\frac{\partial}{\partial}(-\phi^{\hbar}(-Z))+\frac{\partial}{\partial Z}\phi^{\hbar}(Z)\right)=2\pi i\hbar.$$

We need to prove also that the morphism preserves the real structure what is obviously equivalent to the realness condition for the function $\phi^{\hbar}(z)$. We have

$$\overline{\phi^{\hbar}(z)}=-\frac{\pi\hbar}{2}\int_{\overline{\Omega}}\frac{e^{ip\bar{z}}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=$$
$$=\frac{\pi\hbar}{2}\int_{-\Omega}\frac{e^{ip\bar{z}}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=-\frac{\pi\hbar}{2}\int_{\Omega}\frac{e^{-ip\bar{z}}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp=\phi^{\hbar}(\bar{z}).\ \square$$

For checking Property 5 we must verify that the morphism $T^{\hbar}(\Gamma)\to T^{1/\hbar}(\Gamma)$ commutes with a flip. It means that $(A+\phi^{\hbar}(Z))/\hbar=A/\hbar+$

$\phi^\hbar(Z/\hbar)$, $(B-\phi^{1/\hbar}(-Z))/\hbar = A/\hbar - \phi^\hbar(-Z/\hbar)$, etc. Therefore, it suffices to prove that

$$\phi^\hbar(z)/\hbar = \phi^{1/\hbar}(z/\hbar). \tag{24}$$

Indeed,

$$\phi^{1/\hbar}(z/\hbar) = -\frac{\pi}{2\hbar}\int_\Omega \frac{e^{-ipz/\hbar}}{\sinh(\pi p)\sinh(\pi p/\hbar)}dp =$$
$$= -\frac{\pi}{2\hbar}\int_\Omega \frac{e^{-iqz}}{\sinh(\pi\hbar q)\sinh(\pi q)}d(q\hbar) = \phi^\hbar(z)/\hbar. \ \square \tag{25}$$

Now let us pass to the proof of most nontrivial Property 7. The property can be reformulated as follows. There are seven generators involved in the sequence of flips depicted in Fig. 1 for the dual graph, where they are denoted as A, B, C, D, E, X, and Y. During a flip, the piece of graph shown in Fig. 1 just gets cyclically rotated. Denote by $A_i, B_i, C_i, D_i, E_i, X_i$, and Y_i the algebra elements associated to the edges of this piece of graph after i flips are performed. From (8), (20), these elements are changed as follows:

$$\begin{pmatrix} X_{i+1} \\ Y_{i+1} \\ A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \\ E_{i+1} \end{pmatrix} = \begin{pmatrix} Y_i - \phi^\hbar(-X_i) \\ -X_i \\ D_i \\ E_i \\ A_i + \phi^\hbar(X_i) \\ B_i - \phi^\hbar(-X_i) \\ C_i + \phi^\hbar(X_i) \end{pmatrix} \tag{26}$$

We are going to prove that this operator transformation is five-periodic.

Assume for a moment that the five-periodicity of X_i is proven. Then, the five-periodicity of Y_i is obvious, since $Y_{i+1} = -X_i$. The five-periodicity of, say, A_i follows from the calculation

$$X_{i+1} = Y_i - \phi^\hbar(X_i) = -X_{i-1} - \phi^\hbar(X_i)$$

Therefore,

$$\phi^\hbar(-X_i) = -X_{i+1} - X_{i-1}$$

Taking into account (23), we have

$$\phi^\hbar(X_i) = X_i - X_{i+1} - X_{i-1}$$

Now we can use these identities to transform A_{i+5}:

$$\begin{aligned}
&A_{i+5} = D_{i+4} = B_{i+3} - \phi^{\hbar}(-X_{i+3}) = E_{i+2} - \phi^{\hbar}(-X_{i+3}) = \\
&= C_{i+1} + \phi^{\hbar}(X_{i+1}) - \phi^{\hbar}(-X_{i+3}) = A_i + \phi^{\hbar}(X_i) + \phi^{\hbar}(X_{i+1}) - \phi^{\hbar}(-X_{i+3}) = \\
&= A_i + (X_i - X_{i-1} - X_{i+1}) + (X_{i+1} - X_i - X_{i+2}) + (X_{i+4} + X_{i+2}) = \\
&= A_i + X_{i+4} - X_{i-1} = A_i.
\end{aligned}$$

We have shown that the five-periodicity of A_i (and therefore the one of B_i, C_i, D_i, E_i, and Y_i) follows form the five-periodicity of X_i.

Now we need to prove the five-periodicity of X_i. Let us introduce four new algebra elements

$$U = e^{X_i}; \quad V = e^{Y_i}; \quad \tilde{U} = e^{X_i/\hbar}; \quad \tilde{V} = e^{Y_i/\hbar}.$$

They satisfy the following commutation relations

$$\begin{aligned}
&U_i V_i = q V_i U_i, \qquad \tilde{U}_i \tilde{V}_i = \tilde{q} \tilde{V}_i \tilde{U}_i, \\
&U_i \tilde{V}_i = \tilde{V}_i U_i, \qquad V_i \tilde{U}_i = \tilde{U}_i V_i, \qquad \tilde{U}_i = (U_i)^{\hbar}, \qquad \tilde{V}_i = (V_i)^{\hbar},
\end{aligned} \tag{27}$$

where

$$q = e^{2\pi i \hbar}, \qquad \tilde{q} = e^{2\pi i \hbar}.$$

These variables are transformed in an especially simple way,

$$U_{i-1} = V_i^{-1} \tag{28}$$

$$V_{i-1} = U_i(1 + qV_i) \tag{29}$$

$$\tilde{U}_{i-1} = \tilde{V}_i^{-1} \tag{30}$$

$$\tilde{V}_{i-1} = V_i(1 + \tilde{q}\tilde{V}_i). \tag{31}$$

As the first step of the proof, we consider the inverse transformation laws for X_i and Y_i:

$$X_{i-1} = -Y_i; \qquad Y_{i-1} = X_i + \phi^{\hbar}(Y_i).$$

Equations (28) and (30) are obvious. Then, using the standard formula

$$e^{A+F(B)} = e^{A} e^{\int_B^{B+[A,B]} F(z)dz},$$

we obtain

$$\begin{aligned}
&V_{i-1} = e^{Y_{i-1}} = e^{X_i + \phi^{\hbar}(Y_i)} = e^{X_i} e^{\int_Y^{Y+2\pi i\hbar} \phi^{\hbar}(z)dz} = \\
&= U_i \exp\left(-\frac{\pi\hbar}{2} \int_{\Omega} \frac{e^{-ipz}(e^{-2p\pi\hbar} - 1)}{(-ip)\sinh(\pi p)\sinh(\pi\hbar p)} dp\right) = U_i \exp\left(\frac{\pi\hbar}{i} \int_{\Omega} \frac{e^{-ip(z-\pi i\hbar)}}{p\sinh(\pi p)} dp\right) = \\
&= U_i(1 + qV_i).
\end{aligned}$$

The proof of (31) is analogous.

Now in order to prove that X_i is five-periodic, it suffices to verify that both U_i and $\tilde{U}_i$ are five-periodic. Indeed, if only the operator U_i is five-periodic, it does not suffice because the logarithm of an operator is ambiguously defined. However, if we have two families of operators U and $\tilde{U}$, which depend continuously on $\hbar$, then, assuming the existence of an operator X (depending on $\hbar$ continuously) such that $U = e^X$ and $\tilde{U} = e^{X/\hbar}$, this operator appears to be unique. (It can be found as $\lim_{(m+n/\hbar)\to 0}(U^m\tilde{U}^n)/(m+n/\hbar)$ for any irrational value of $\hbar$.) The five-periodicity of sequence (29) (and (31)) is a direct calculation using (27).

Corollary 3.1. *1. Let K be an operator acting in the Hilbert space $L^2(\mathbf{R})$ and having the integral kernel*

$$K(x,z) = F^\hbar(z)e^{-\frac{zx}{2\pi i\hbar}}, \tag{32}$$

where

$$F^\hbar(z) = \exp\left(-\frac{1}{4}\int_\Omega \frac{e^{-ipz}}{p\sinh(\pi p)\sinh(\pi\hbar p)}dp\right) \tag{33}$$

Then the operator K is unitary up to a multiplicative constant and satisfies the identity

$$K^5 = \text{const}. \tag{34}$$

2. Let $\hbar = m/n$ be a rational number and assume that both m and n are odd. Introduce a linear operator $L(u)$ acting in the space $\mathbf{C}^n$ and depending on one positive real parameter u through its matrix

$$L(u)^i_j = F^\hbar(j,u)q^{2ij}, \tag{35}$$

where

$$F^\hbar(j,u) = (1+u)^{j/n}\prod_{k=0}^{j-1}(1+q^{2k-1}u^{1/n})^{-1}.$$

Then the following identity holds:

$$L(u)L(v+uv)L(v+vu^{-1}+u^{-1})L(u^{-1}v^{-1}+u^{-1})L(v^{-1}) = 1. \tag{36}$$

3.1. *Geodesic length operators*

The aim of this paragraph is to imbed algebra of geodesic functions (6) into a suitable completion of the constructed algebra $T^{\hbar}(S)$.

For any γ, function G_γ (6) can be expressed in terms of the graph coordinates on T^H,

$$G_\gamma \equiv \operatorname{tr} P_{Z_1\cdots Z_n} = \sum_{j\in J} \exp\left\{ \frac{1}{2} \sum_{\alpha\in E(\Gamma)} m_j(\gamma,\alpha) z_\alpha \right\}, \qquad (37)$$

where $m_j(\gamma,\alpha)$ are *integer* numbers and J is a finite set of indices. In order to find the quantum analogues of these functions, we denote by $\widehat{T}^{\hbar}$ a completion of the algebra $T^{\hbar}$ containing e^{xZ_α} for any real x. Let for any closed path γ on S, the *quantum geodesic* operator $G^{\hbar}_\gamma \in \widehat{T}^{\hbar}$ be

$$G^{\hbar}_\gamma \equiv {}^{\times}_{\times} \operatorname{tr} P_{Z_1\ldots Z_n} {}^{\times}_{\times} \equiv \sum_{\substack{j\in J\\ \kappa\in\{j\}}} \exp\left\{ \frac{1}{2} \sum_{\alpha\in E(\Gamma)} \left(m_j(\gamma,\alpha) Z^{\hbar}_\alpha + 2\pi i\hbar c^{\kappa}_j(\gamma,\alpha)\right) \right\}, \qquad (38)$$

where the *quantum ordering* ${}^{\times}_{\times}\cdot{}^{\times}_{\times}$ implies that we vary classical expression (37) by introducing additional integer coefficients $c^{\kappa}_j(\gamma,\alpha)$, which must be determined from the conditions below.

Note that the operators $\{G^{\frac{1}{\hbar}}_\gamma\}$ can be considered as belonging to the algebra $\overset{\frown}{T}^{\hbar}$. In terms of the generators of $\widehat{T}^{\hbar}$, they are

$$G^{\frac{1}{\hbar}}_\gamma = \sum_{j\in J} \exp\left\{ \frac{1}{2\hbar} \sum_{\alpha\in E(\Gamma)} \left(m_j(\gamma,\alpha) Z^{\hbar}_\alpha + 2\pi i c^{\kappa}_j(\gamma,\alpha)\right) \right\}, \qquad (39)$$

Now let us formulate the defining properties of quantum geodesics.

1. *The mapping class group* action $\Delta(S)$ (8) preserves the set $\{G^{\hbar}_\gamma\}$, i.e., for any $\delta \in \Delta(S)$ and any closed path γ, we have $\delta(G^{\hbar}_\gamma) = G^{\hbar}_{\delta\gamma}$.

2. *Geodesic algebra.* The product of two quantum geodesics is a linear combination of *quantum geodesic laminations* (QGLs) governed by the skein relation [22]. Analogously to the classical case, a QGL is a set of self- and mutually nonintersecting quantum geodesics.

3. *Unorientness.* Quantum traces of direct and inverse geodesic operators coincide.

4. *Exponents of geodesics.*

A quantum geodesic $G^{\hbar}_{n\gamma}$ with nonnegative winding number n is expressed via $G^{\hbar}_\gamma$ exactly as in the classical case,

$$G^{\hbar}_{n\gamma} = 2T_n\left(G^{\hbar}_\gamma/2\right), \qquad (40)$$

where $T_n(x)$ are Chebyshev's polynomials. **5.** For any γ and γ', the operators $G^\hbar_\gamma$ and $G^{\frac{1}{\hbar}}_{\gamma'}$ commute. **6.** If closed paths γ and γ' do not intersect, then the operators $G^\hbar_\gamma$ and $G^\hbar_{\gamma'}$ commute. Property 6 implies that *all* quantum geodesics constituting a QGL mutually commute. We denote the Weyl ordering by a usual normal ordering symbol $:\cdot:$, i.e.,

$$:e^{a_1}\,e^{a_2}\cdots e^{a_n}: \equiv e^{a_1+\cdots+a_n} \quad \text{for any set} \quad \{a_i : a_i \neq -a_j\}.$$

Proposition 3.1. *If a (quantum) mapping class group transformations (8) transform a graph simple geodesic γ into the graph simple geodesic $\widetilde{\gamma}$, then, for the both corresponding quantum geodesic functions, all $c^{\kappa}_{j}(\gamma,\alpha) \equiv 0$ in (38).*

Example 3.1. Quantum geodesics for torus. For the torus, there are three graph simple quantum geodesics, which are exactly (15) in the Weyl-ordered form. Then, the quantum geodesics $\widetilde{G}^\hbar_Z$ obtained from $G^\hbar_Z$ by the flip transformation is

$$\widetilde{G}^\hbar_Z = e^{-X/2-Y/2-Z} + e^{X/2-Y/2-Z} + e^{X/2-Y/2}\cdot 2\cos(\pi\hbar) + e^{X/2-Y/2+Z} + e^{X/2+Y/2+Z}. \tag{41}$$

The *product* of two graph simple geodesic functions is

$$G^\hbar_X G^\hbar_Y = e^{i\pi\hbar/2}\widetilde{G}^\hbar_Z + e^{-i\pi\hbar/2}G^\hbar_Z. \tag{42}$$

This algebra is exactly the $so_q(3)$ quantum algebra studied in [19], i.e., it is a finitely generated algebra with the lexicographic basis. Indeed, denoting $q \equiv e^{-i\pi\hbar}$, $[A,B]_q \equiv q^{1/2}AB - q^{-1/2}BA$, and $\xi \equiv q - q^{-1}$, we obtain from (42)

$$[G^\hbar_X, G^\hbar_Y]_q = \xi G^\hbar_Z, \quad [G^\hbar_Y, G^\hbar_Z]_q = \xi G^\hbar_X, \quad [G^\hbar_Z, G^\hbar_X]_q = \xi G^\hbar_Y, \tag{43}$$

with the only central element (the quantum Markov relation)

$$M = G^\hbar_X G^\hbar_Y G^\hbar_Z - q^{1/2}\big((G^\hbar_X)^2 + q^{-2}(G^\hbar_Y)^2 + (G^\hbar_Z)^2\big). \tag{44}$$

3.2. *Algebra of quantum geodesics*

Let γ_1 and γ_2 be two graph simple geodesics with one nontrivial intersection. So, for $G^\hbar_1$ and $G^\hbar_2$, formula (38) implies, by virtue of Lemma 3.1, the mere Weyl ordering.

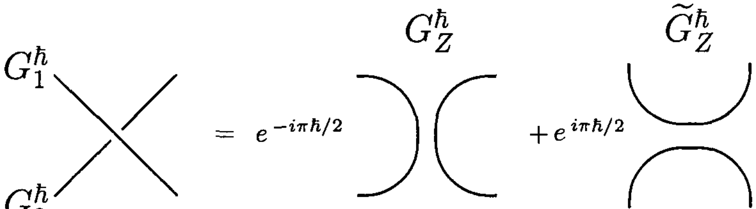
After some algebra, we obtain (cf. (14))

$$G^\hbar_1 G^\hbar_2 = e^{-i\pi\hbar/2}G^\hbar_Z + e^{i\pi\hbar/2}\widetilde{G}^\hbar_Z, \tag{45}$$

where $G^\hbar_Z$ literally coincides with the Weyl-ordered G_I in the classical case (cf. (14)), whereas $\widetilde{G}^\hbar_Z$ contains the quantum correction term,

$$\begin{aligned}\widetilde{G}^\hbar_Z &= \genfrac{}{}{0pt}{}{\times}{\times}\, \mathrm{tr}_1\, \mathrm{tr}_2 \dots (e^1_{ij} \otimes e^2_{ji})[X^1_Z \otimes X^2_Z] \dots \genfrac{}{}{0pt}{}{\times}{\times} \\ &= :\mathrm{tr}_1\, \mathrm{tr}_2 \dots (e^1_{ij} \otimes e^2_{ji}) \left[X^1_Z \otimes X^2_Z + 2(1 - \cos \pi\hbar) e^1_{11} \otimes e^2_{22}\right] \dots : .\end{aligned}$$

Here $e^1_{ij} \otimes e^2_{ji}$ is the standard r-matrix that permutes the spaces "1" and "2," and as a result, the "skein" relation of form (14) appears. Locally, this relation has exactly the form proposed by Turaev [22], i.e., for intersecting graph simple geodesics, we have the defining relation



(46)

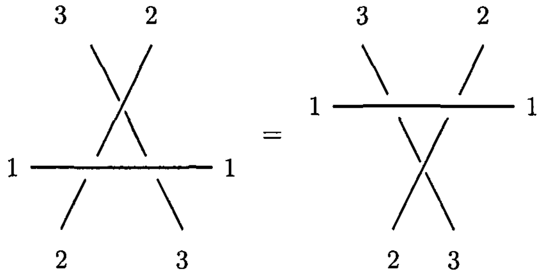
(The order of crossing $G^\hbar_1$ and $G^\hbar_2$ depends on which geodesic function occupies the first place in the product; the rest of the graph remains unchanged for all items in (46)) Note, however, that if the corresponding geodesics γ_1 and γ_2 are graph simple, we may turn the geodesic $\widetilde{G}^\hbar_Z$ again into the simple geodesic $\widetilde{G}^{\hbar\prime}_Z$ by performing the quantum flip w.r.t. the edge Z.

If we now compare two unambiguously determined expressions: $\widetilde{G}^{\hbar\prime}_Z$, which must be Weyl ordered, and $G^\hbar_Z$ obtained from the geodesic algebra, we find that $G^\hbar_Z = \widetilde{G}^{\hbar\prime}_Z$. This is a hint, which eventually results in the following lemma [3].

Lemma 3.1. *There exists quantum ordering* $\genfrac{}{}{0pt}{}{\times}{\times} \dots \genfrac{}{}{0pt}{}{\times}{\times}$ *(38), generated by the geodesic algebra, that is consistent with quantum mapping class group transformations (8), i.e., the quantum geodesic algebra must be invariant under the action of the quantum mapping class group. For a single crossing of two simple QGLs, the relation has form (46).*

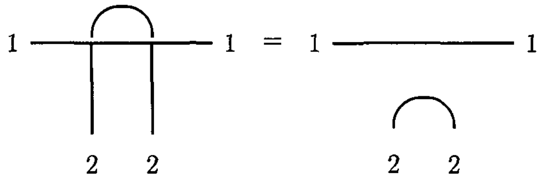
Lemma 3.2. *If more than one intersection of two QGLs occur, the quantum skein relations must be applied simultaneously at all intersection points.*

Lemma 3.2 implies the Riedemeister moves for a graph as soon as we set the empty loop to be $-e^{-i\pi\hbar} - e^{i\pi\hbar}$. That is, for arbitrary three intersecting geodesics, applying (46) simultaneously at all intersection points, we obtain



(47)

and



(48)

Example 3.2. Quantum algebra $M_{1,2}$. Let us consider the quantum geodesic algebra of torus with two punctures and the basic graph in Fig. 4, which is a particular case of the graph in Fig. 3.

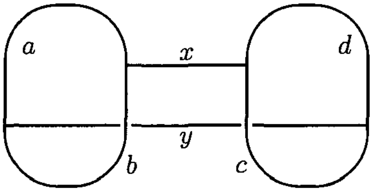


Fig. 4

There are six graph simple geodesics in Fig. 4: $X_1 = \operatorname{tr} R_y R_a L_x L_d$, $X_2 = \operatorname{tr} L_y L_b R_x R_c$, $Y_1 = \operatorname{tr} L_a R_b$, $Y_2 = \operatorname{tr} R_d L_c$, $Z_1 = \operatorname{tr} L_y R_a L_x R_c$, and $Z_2 = \operatorname{tr} R_y L_b R_x L_d$. (The Weyl normal ordering is assumed.) Then it is straightforward to verify using the skein relations (or directly using formulas (3) and (4)) that the corresponding algebra has the following structure (cf. [23]). Let $C(t_1, t_2, t_3)$ denote the set of cyclic commutators of $so_q(3)$ algebra, $[t_i, t_{i+1}]_q = \xi t_{i+2}$, $i = 1, 2, 3 \bmod 3$ (cf. (43)). Then, the algebra

$M_{1,2}$ is

$$\begin{gathered} C(X_1,Z_2,Y_1); \quad C(X_2,Y_1,Z_1); \quad C(X_1,Y_2,Z_1); \quad C(X_2,Z_2,Y_2); \\ [X_1,X_2]=[Y_1,Y_2]=0, \qquad [Z_1,Z_2]=\xi(Y_1Y_2-X_1X_2). \end{gathered} \tag{49}$$

This algebra was studied in [23] in relation with the Kauffman bracket skein quantizations.

Algebra (49) possesses two central elements (related to geodesics around the holes),

$$Z_1Z_2-qY_1Y_2-q^{-1}X_1X_2, \tag{50}$$

and

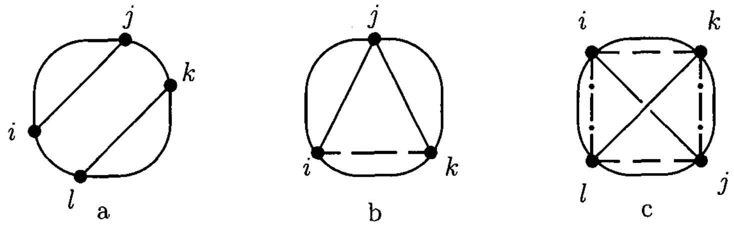
$$\begin{gathered} X_1X_2Y_1Y_2-q^{3/2}X_2Y_1Z_1-q^{-1/2}X_2Z_2Y_2-q^{-3/2}X_1Y_1Z_2-q^{1/2}X_1Z_1Y_2 \\ +Y_1^2+Y_2^2+q^2Z_1^2+q^{-2}Z_2^2+q^{-2}X_1^2+q^2X_2^2, \end{gathered} \tag{51}$$

and admits a lexicographic ordering, as follows from (49).

3.3. *Quantizing the Nelson–Regge algebras*

Algebra (17) was quantized by the deformation quantization method in [17,25]. We are now able to implement quantization conditions (18). It is convenient to represent the elements a_{ij} as chords connecting the points of the cyclically ordered set of indices $i,j\in\{\ldots,m,(1,2,\ldots,m),1\ldots\}$. Then, three variants are possible: if two chords do not intersect, then the corresponding geodesics do not intersect as well and the quantum geodesics commute (Fig. 5a); if two chords have a common vertex, then the corresponding geodesics intersect at one point and the three quantum geodesics a_{ij}, a_{jk}, and a_{ki} (as in Fig. 5b) constitute the quantum subalgebra $so_q(3)$; if two chords intersect in the middle point (Fig. 5c), then the corresponding geodesics a_{ij} and a_{kl}, $i<k<j<l$, have double intersection and satisfy the commutation relation

$$[a_{ij},a_{kl}]=\xi(a_{ik}a_{jl}-a_{il}a_{jk}). \tag{52}$$



4. Conclusion

In this paper, we briefly reviewed the action of the mapping class group on the classical and quantum Teichmller spaces. We considered algebras of classical and quantum geodesics, which are parameterized by the edge lengths of graphs; these lengths coordinatize the Teichmüller space. Elements of the quantum mapping class group satisfy the pentagon relation and preserve the classical and quantum geodesic structures; these elements establish automorphisms between quantum geodesic algebras corresponding to different graphs representing the Riemann surface.

In higher dimensions ($g > 2$ for $n = 1$), addition restrictions on quantum algebras $M_{g,n}$ corresponding to moduli spaces must appear (likewise the Schottky problem concerning the period matrix structure.) Using the similar approach, R. Kashaev have found the quantum Liouville central charge using the modular and mapping class group transformations acting in the space $M_{3,1}$ [26].

The author expresses his gratitude to the organizers and speakers for the hospitality and the atmosphere of creativity during the Woods Hole meeting.

The paper was supported by the Russian Program "Nonlinear Dynamics and Solitons."

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