

Chapter 1

Introduction

1.1 Preliminaries

We shall start with a brief preliminary to explain why we have come to the study of white noise functionals.

The authors are sure that everybody who is interested in probability theory, more generally in science, is quite familiar with Brownian motion and knows how it is involved in the theory of stochastic processes and stochastic analysis. It is therefore, natural to provide a survey of Brownian motion first of all and to explain a theory of analysis of its functionals (Brownian functionals).

Indeed, the analysis of them has been extensively developed. While we are studying such an analysis, we have the idea of taking *a white noise*, the time derivative $\dot{B}(t)$ of a Brownian motion $B(t)$, to be the variables of functionals in question, instead of a Brownian motion itself. The main reason is that the $\{\dot{B}(t), t \in R\}$ forms a system of i.i.d. (independent identically distributed) random variables. This is an essential part for the analysis of functionals of white noise. In particular, if we meet nonlinear functionals of a Brownian motion, its expression in terms of a white noise shows a big advantage to analyze them. The i.i.d. property of the variables makes the analysis simpler and efficient.

We would like to emphasize the significant characteristics of white noise analysis. The characteristics are also considered as advantages. They are now in order.

(1) The space of generalized white noise functionals is one of the main subjects of the study. They are naturally defined since $\dot{B}(t)$'s are taken

to be the variables of functionals. The space of generalized white noise functionals is very much *bigger than the classical L^2 -space* of functionals of Brownian motion. As a result, we can carry on the calculus on wider class of random complex systems, including causal calculus where time development is explicitly involved.

(2) Infinite dimensional rotation group is introduced. It describes invariance of the probability distribution of white noise. We are therefore led to a harmonic analysis arising from the rotation group. Complexification of the group has close connection with quantum dynamics and other fields of application.

(3) Innovation approach is our favorable method of analysis. White noise $\dot{B}(t)$ is typical and Poisson noise can be discussed in a similar manner since both are systems of idealized elemental random variables. Interesting remark is that important dissimilarities are found and we can even discuss duality between two noises.

More details will be discussed in Section 1.5.

1.2 Our idea of establishing white noise analysis

We shall deal with random complex systems expressed as functionals of white noise. What we shall discuss are of Brownian motion or mostly concerned with white noise and their functionals. Those of Poisson noise will also be discussed. Some of the results of the latter case are obtained in a similar manner to the Gaussian case, but dissimilarity between them is sufficient.

Our interest is focused on those systems of functionals which are evolutionary, namely those systems change and develop as time or space-time parameter goes by.

As is noted in Carleton Mathematical Notes⁴¹, we want to establish a basic theory so that it enables us to discuss random systems in a mathematically systematic manner and comprehensively. There have so far been many directions on stochastic analysis, hence it would be fine if those approaches can be set up in a unified manner.

Another motivation is that a stochastic differential equation can be

dealt with as if it were the case of ordinary differential equations, although the new analysis to be proposed would be beyond the classical functional analysis. The new stochastic analysis is based on white noise, either Gaussian or Poisson noises or even compound cases, depending on the choice of variables of functionals in question.

However, it is not natural to assume that the given system is driven by a white noise that is given in advance. Generally, starting with a given random system, we have to construct a white noise, in terms of which the given system is expressed as its functional. The white noise thus obtained is nothing but the *innovation* of the system. The notion of an innovation for an evolutionary random system will be prescribed later, but at present, we roughly understand that it is a system of independent random variables, may be idealized variables, which contain, up to each time, just the same information as what the given random system gains.

1. Our idea starts, therefore, with the step of “**Reductionism**” for random complex systems. Practically, we are to meet a huge building blocks of a random system to be investigated, the constituent elements of which should be elemental, and in fact atomic. They represent basic units of randomness, something like an atom in physics, where the collection of atoms constitute a matter.

We are mainly concerned with evolutionary random complex systems, so that actual implementation is to construct the *innovation* which is an elemental random system by extracting necessary and sufficient information from the given random complex system. This is the step of reduction and in fact, the first and the most important step of our mathematical approach to the study of the given random system.

The standard innovation can be expressed as the time derivative of a Lévy process. This choice is reasonable, since a Lévy process has independent increments and satisfies some continuity in time which may naturally be assumed.

2. Then follows the second step “**Synthesis**”. There the given random system should be expressed as a functional (which is non-random and, in general, nonlinear) of the innovation that has just been obtained in step **1.** the reduction. Thus, we have an analytic representation of the random complex phenomenon in question by choosing suitable functionals of the innovation that have been established.

Since those functionals are non-random, we can appeal to the known

theory of functional analysis, although the variables are random.

3. Finally, we are ready to study the “**Analysis**” of those functionals, in fact, nonlinear functionals of the innovation. It may be proceeded by having been suggested by the ordinary functional analysis, however it is noted that the variable is innovation, namely elemental random variables that are mutually independent. The differential and integral calculus should be newly established. This can be done by overcoming difficulties that arise there.

It is specifically noted that we can find an infinite dimensional rotation group and we can carry on an infinite dimensional harmonic analysis arising from the group (note that we do not specify the group).

Further various applications can be discussed, and one can even see beautiful interplay between our theory and the studies of actual problems in various fields of science.

After those steps, there naturally follow interesting applications in various fields of science, some of which are going to be presented in this volume.

If we follow these three steps, it may be said that we are influenced by the historically famous literature by J. Bernoulli’s article “*Ars Conjectandi*” appeared in 1713. In fact, such a way of thinking is closely related to the definition of a stochastic process, in reality the term *stochastice* has first appeared in that literature. Also, we would like to recommend the reader the most famous monograph by P. Lévy¹⁰², in particular, Chapter II for definition of a stochastic process. Incidentally, the reductionism in the present terminology did start, in reality, much earlier in the book, Chapter VI, although he did not use the term innovation.

It would be meaningful to recall quickly a brief history of defining a stochastic process. However we are afraid that, when such a history is mentioned, one is usually suggested to remind the traditional method, where basic random quantity is given in advance. We should first study from our viewpoint; namely first we investigate basic elemental random systems, then we see suitable combination of those elemental systems such that they can be an innovation. Thus, follow the steps **1**, **2**, **3**, discussed above.

At present, we are sure that the *innovation approach* is one of the most efficient and legitimate directions to the study of stochastic processes and

random fields, or more generally to evolutionary random complex systems. Indeed, having had many attempts to investigate various random functions, we have come to recognize the significance of the classical idea of defining a stochastic process due to J. Bernoulli, P. Lévy, A.N. Kolmogorov, and others. We shall be back to this topic in Section 4.

We can tell some more concrete story from somewhat different viewpoint. An additive process appears, in an intuitive level, from the theory of innovation of a stochastic process. For an additive process (Lévy process), a most ground breaking theory is the so-called *Lévy's decomposition theorem*¹⁰² (it is also called the Lévy–Itô decomposition theorem). Under some additional and in fact, mild assumptions, we are given a Lévy process, the sample functions of which are ruled functions almost surely, to squeeze the innovation out of the given stochastic process, for which an explicit decomposition formula has been given. Another interesting course of the decomposition comes from the determination of the infinitely divisible probability distribution established by A. Ja. Khinchin and P. Lévy, but we do not go further in this direction.

The time derivative of an additive process is a general *white noise*. We are, therefore, led to the Lévy decomposition of a general white noise, as a result. It claims that a general *stationary* white noise can be decomposed (up to constant) into the following two parts:

- 1) (Gaussian) white noise,
- 2) compound Poisson noise.

A compound Poisson noise is a sum (may be a continuous sum) of independent Poisson noises with different scales of jump. Thus, we may choose a Gaussian noise and single Poisson noise with unit scale to be the representatives of the set of general white noises which are *elemental*.

Some brief interpretations to elemental noises will be given in the following chapter.

One of the significant traditional problems to be discussed related to our framework on innovation is the *prediction theory*, in particular the non-linear prediction theory. The theory has stimulated the innovation theory. Heuristic approach can be seen in the work by N. Wiener and P. Masani. We refer to the monograph¹¹⁴, where white noise is playing a dominant role.

Another direction is the theory of stochastic differential equations (SDE); more generally theory of stochastic partial differential equations (SPDE) and stochastic variational equation (SVE) involving white noise. We can understand that these theories can be discussed from our viewpoint of white noise analysis.

Generalizations of these directions, some concrete problems on the theory of stochastic processes and various interplay with other fields of science have led us to a general theory of white noise analysis.

Poisson noise, which is another elemental noise, can be dealt with in the similar manner as the Gaussian case, however a Poisson noise has its proper characteristics, so that we can discuss the analysis of Poisson noise functionals with much emphasis on the difference from Gaussian case. Discovering intrinsic properties of Poisson noise is an interesting and in fact significant problem.

1.3 A brief synopsis of the book

This section gives a brief synopsis of the actual content of this monograph.

Originally we planned to write this monograph so as to be self-contained, and much has been done. Unfortunately this idea is not quite successful, for one thing the white noise theory is rapidly expanding having connections with many other fields in science.

The notion of generalized white noise functionals (white noise distributions) is one of the main topics of this monograph, Chapter 3 and part of Chapter 4 will be devoted to the study of those functionals.

The main part of Chapter 2 involves the notion of generalized white noise functionals. We take systems of idealized elemental random variables, in particular (Gaussian) white noise and Poisson noise. To fix the idea, we take white noise, a representation of which is $\dot{B}(t)$. We understand that variables are given, so natural class of functions is to be defined. Our idea is to start with polynomials in the given variables, then to introduce a general class of functions. Further we shall go to the differential and integral calculus.

These steps could be accepted by everybody, and various phenomena

are expressed as functions of white noise, and they are analyzed so that the given phenomena can be identified and clarified. Since the variables are independent, we may expect the stochastic calculus can be done nicely. This is true, however, we have to pay a price. First we have to give a correct meaning to $\dot{B}(t)$'s as well as to their functions. We note that it is better to call them functionals, since they depend on $\dot{B}(t), t \in R$, which is a function of t . In addition, we require to introduce a good functional representation, because original functionals are, in general, functions of generalized functions, the sample functions of $\dot{B}(t)$. This can be done; indeed, the so-called \mathcal{T} - or \mathcal{S} -transform serves this need, and does even more.

Needless to say, Gaussian systems are sitting in the center of systems of random variables because of their very significant properties, one of which is linearity. Concerning the linearity, Poisson noise comes right after. Linear combinations of white noise and Poisson noise lead us to discuss probabilistic properties by applying linear operations acting on them.

Advantages of rotation group $O(E)$

We have the infinite dimensional rotation group $O(E)$, where E is a nuclear space. This group can characterize the (Gaussian) white noise measure. It is natural that there arises an infinite dimensional harmonic analysis from the group. It is not locally compact under the usual compact-open topology, but we may still try analogous approach. While, essentially infinite dimensional analysis can be seen with the help of the infinite dimensional rotation group. These will be discussed in Chapter 5.

Like the finite dimensional case, complexification of the rotation group enables us to have more fruitful results. Those may be said to have come from the viewpoint of harmonic analysis. In particular we can find interesting relationship between complex white noise and quantum dynamics.

Poisson noise is interesting and significant as much as Gaussian white noise. Despite the appearances of sample functions, a Poisson process or Poisson noise has significant characteristics, although they often appear in an intrinsic manner. Details will be discussed in Chapter 7. We compare Brownian motion, which is Gaussian, and Poisson process in Chapter 2 briefly before discussing concrete results in Chapter 7 and after.

Innovation theory will support our approach that starts with reduction

of random complex systems (Chapter 8). Applications to variational calculus will be seen in Chapter 9, although to some extent we have discussed in another monograph⁷¹ by us.

We would like to emphasize intimate connections with quantum dynamics. Chapter 10 is devoted to explain some notable roads to this direction.

The Appendix involves some necessary formulas, basic notions which are not in the main stream of this book, and discrete parameter white noise, which serves a note for an invitation to white noise. Certainly it is simpler than the main stream of our story of white noise depending on a continuous parameter. For instance, measurability of functionals depending on continuously many variables, structure of a nuclear space of functions, continuously many independent random variables and so forth. It should be noted that infinite dimensional rotation group and unitary group enjoy more profound and rich structure in the continuous parameter case. In any case, discrete parameter white noise is much easier to deal with compared to the continuous parameter case.

1.4 Some general background

One may ask oneself what does a random complex system mean, the answer may be, in particular, a stochastic process $X(t)$ parametrized by the time parameter t .

We are recommended to follow the method of defining a stochastic process introduced by J. Bernoulli first and later by A.N. Kolmogorov, as will be discussed in Chapter 2. We essentially follow this idea, but in our case, characteristic functional is often used in order to define and identify a stochastic process as well as a generalized stochastic process. A characteristic functional can be defined even for generalized stochastic processes, the sample functions of which are generalized functions almost surely. A generalization to the case where t is multi-dimensional, namely to the case of a random field, is almost straightforward. To fix the idea, the parameter space T , where t runs, is taken to be R^d , $d \geq 1$, or its finite or infinite interval.

a) A well-known traditional method of defining a stochastic process starts with a consistent family of finite dimensional probability distributions. Then, the Kolmogorov extension theorem guarantees the existence

of a probability measure μ on R^T , or more precisely, existence and uniqueness of a probability measure μ on a measurable space (R^T, \mathcal{B}) , where \mathcal{B} is the sigma-field generated by cylinder subsets of R^T . The measure space (R^T, \mathcal{B}, μ) is a stochastic process. The μ -almost all x in R^T is a sample function of the stochastic process. We often use the traditional notation such as $X(t), t \in T$, defining $X(t) = X(t, x) = x(t)$.

b) In fact, another more familiar and traditional method is to give a function $X(t, \omega), t \in T, \omega \in \Omega$, where (Ω, \mathbf{B}, P) , with the sigma-field \mathbf{B} of subsets of Ω , is a probability measure space. The function $X(t, \omega)$ satisfies the condition that $X(t, \omega)$ is \mathbf{B} -measurable for every t .

The condition implies that any random vector $(X(t_1), X(t_2), \dots, X(t_n))$ has a probability distribution and the requirements in a) are satisfied. Thus we are given a stochastic process which is in agreement with $X(t, x)$ in a). Almost all x are sample functions of the process $X(t)$,

c) Suppose that $X(t, \omega)$ is given as in b) and is a measurable function of t . It may be a generalized function of t for each ω . In any case, taking a suitable space E of test functions, a continuous bilinear form, denoted by $\langle X(\cdot, \omega), \xi \rangle$ with $\xi \in E$ is well defined.

Hence, the characteristic functional $C_X(\xi)$ of $X(t)$ is defined:

$$C_X(\xi) = E(\exp[i\langle X(\cdot, \omega), \xi \rangle]). \quad (1.4.1)$$

It is easy to prove that $C_X(\xi)$ satisfies the following properties.

- 1) $C_X(\xi)$ is continuous in ξ ,
- 2) $C_X(0) = 1$,
- 3) $C_X(\xi)$ is positive definite.

Conversely, suppose a functional $C(\xi)$, satisfying the conditions 1)–3), is given, then we can form a probability measure μ on E^* . Then, as in a), a stochastic process exists. Its characteristic functional is equal to the given $C(\xi)$. This is the characteristic functional method of defining a stochastic process.

Existence and uniqueness of a measure will be discussed in Chapter 2.

Indeed, the above result is an infinite dimensional generalization of the

S. Bochner theorem that guarantees the relationship between probability distribution on finite dimensional Euclidean space and characteristic function satisfying the conditions 1), 2) and 3) for finite dimensional case.

We understand a stochastic process, the same for a random field, from the standpoint c), however a) and b) are also referred occasionally depending on the situation.

1.4.1 *Characteristics of white noise analysis*

Generalized white noise functionals are introduced in Chapter 2. The introduction of classes of generalized white noise functionals and their effective use are one of the main advantages of the white noise analysis. It is noted that a realization of a white noise is the time derivative, denoted by $\dot{B}(t)$ of a Brownian motion $B(t)$. The collection $\{\dot{B}(t), t \in R\}$ has been understood to be a generalized stochastic process with independent values at every t in the sense of Gel'fand. Formally, it has been viewed as a system of idealized elemental random variables. We wish to form nonlinear functionals of $\dot{B}(t), t \in R$, however each single $\dot{B}(t)$ has not been rigorously defined, although only smeared variable $\dot{B}(\xi)$ was defined. We often hear that $\dot{B}(t)$ is understood as a Gaussian process with mean 0 and covariance $\delta(t - s)$; this is no more than a formal understanding.

Key point 1

Under such a circumstance, we propose to give the *identity* to each $\dot{B}(t)$ for every t . This is rigorously done, and we can proceed to define function(al)s of $\dot{B}(t)$'s. Having been given the system of variables, it is reasonable to come to the definitions of polynomials in those variables. The trick to do this is not so easy as is imagined. The so-called *renormalization* technique is used and we naturally introduce generalized white noise functionals. Starting with the usual Hilbert space of functionals of Brownian motion with finite variance, two typical classes of generalized white noise functionals are introduced; one uses an integral representation of those functionals, where Hilbert space structure remains and the other is an infinite dimensional analogue of the space of the Schwartz distributions. Both spaces involve important examples in white noise analysis, however they are slightly different and share the roles.

Chapter 4 discusses Gaussian systems which are classes of Gaussian

random variables satisfying a condition under which those variables exist consistently. If permitted, we may say that a Gaussian system has linear structure, which determines the system. Roughly speaking, a Gaussian process can be represented by a system of additive Gaussian processes, in fact those processes are modified Brownian motions. Linear analysis and Brownian motions can completely determine the given Gaussian process. The Brownian motion may be replaced by white noise.

We note that linearity and white noise are involved. Keeping the linearity, we may try to replace white noise with Poisson noise (or by compound Poisson noise). For one thing, each of these two noises forms i.e.r.v. and they are atomic system, i.e. cannot be decomposed into independent non-trivial systems. So far as linear operators are concerned, we can deal with linear processes involving linear functionals of compound Poisson process in the similar manner to the Gaussian case. These will be seen in Chapter 4.

Key point 2

We then come to the second advantage of white noise analysis; namely the use of the *infinite dimensional rotation group*. As is easily seen from the expression of its probability density function, the n dimensional standard Gaussian distribution is invariant under the rotation group $SO(n)$. We expect similar situation in the infinite dimensional case, i.e. for the case of white noise measure μ . But this is not the case. In fact, we take the group $O(E)$ of rotations of a nuclear space E , due to H. Yoshizawa ¹⁷¹. It contains not only the projective limit $G(\infty)$ of $SO(n)$, but involves other rotations which have probabilistic meanings. Indeed, the subgroup $G(\infty)$ occupies, intuitively speaking, only narrow part of $O(E)$.

Starting from the introduction of $O(E)$, we can carry out an infinite dimensional harmonic analysis arising from the infinite dimensional rotation group. It is noted that a subgroup isomorphic to the conformal group plays interesting roles in probability theory.

Complexification of the rotation group, that is the *infinite dimensional unitary group*, is denoted by $U(E)$. There appear more interesting subgroups, where significant roles are played by the ordinary Fourier transform.

We shall come back to Poisson noise in Chapter 7. Invariance of Poisson

noise, in particular symmetric group and a characterization of Poisson noise will be discussed. We shall even speak of an infinite symmetric group in connection with its unitary representation.

Key point 3

Although we have discussed innovation of a stochastic process and a random field in the monograph⁷¹, we shall discuss again in line with the idea *reduction* in Chapter 8. Weak sense innovation is also discussed briefly.

Part of the theory may be a rephrasing of the classical theory.

Chapter 9 contains variational calculus, which has also been discussed in the monograph⁷¹. We shall further discuss illustrative examples and operator fields defined by creation and annihilation operators. It is our hope that our approach to operator fields would contribute to quantum field theory.

The last chapter is devoted to four typical applications to quantum dynamics. They are the path integral with a development to the Chern–Simons action integral, infinite dimensional Dirichlet forms, the time operators and Euclidean field which is touched upon briefly.

The appendix contains some necessary formulas and notions in analysis and quick review of discrete parameter white noise, which will serve to good understanding of the basic ideas of white noise theory.

Having followed the chapters successively, the readers will see what is the white noise analysis and may answer the question on what are the advantages of this analysis practically. We repeat the characteristics and advantages which were briefly mentioned in the preface.