

# Preface

These notes are based on a series of lectures delivered by one of the authors for the infinite dimensional analysis seminar at Meijo University in the last five years. They are presented here as lectures on white noise functionals. The aim of the lectures are twofold; one is to explain the basic idea of the white noise analysis to those who are interested in this area and the second is to propose some new directions of our theory to the experts who are working on stochastic analysis. For these aims, a perspective of the white noise theory here is presented in this book.

Thus, the reader can see that effective use of our favorite generalized white noise functionals does meet the needs of the time. Indeed, it is now the time to use *generalized white noise functionals* which has been introduced as a natural development of the stochastic analysis. They serve profound roles also in quantum as well as statistical mechanics. Another basic concept of our analysis is the *infinite dimensional rotation group*. With the use of this group we can carry out a harmonic infinite dimensional analysis.

There are several aspects of white noise analysis. Among them,

1) Analysis of stochastic processes (historical reference: J. Bernoulli, *Ars Conjectandi* (1713) CAP II, where one can see the term *stochastic*). We are stimulated by such a classical idea.

2) The method of the theory of functional analysis developed in the second half of 20th century. This has a close connection with what we are going to discuss, although in a naive level.

3) Harmonic analysis arising from

- i) the infinite dimensional *rotation group*, and
- ii) the infinite symmetric group.

There are interesting relationships between the two groups.

4) Following the standard roadmap of analysis, first comes determination of basic variables to define functions (in fact, functionals) to be dealt with. Then follow the analysis and applications.

One of the *characteristics* of what we are studying is that the space of variables is *infinite dimensional*, so that we discuss an infinite dimensional analysis. As for the infinite dimensionality with which we are concerned, one should not be simple minded. The reason why we have to be so careful to deal with this concept will be explained in many places in these notes by taking actual objects, however we now explain in an intuitive manner.

In order to analyze random complex systems, which are the object to be investigated, the basic variables are preferable to be independent, stationary in time and atomic. Good examples are (Gaussian) white noise  $\dot{B}(t)$  and Poisson noise  $\dot{P}(t)$ . They are the time derivatives of a Brownian motion  $B(t)$  and a Poisson process  $P(t)$ , respectively. Both are systems of *idealized elemental random variables*. Independence at every  $t$  comes from the fact that each of those two processes has independent increments.

To fix the idea, let us take white noise  $\dot{B}(t)$ . The time parameter  $t$  runs through a continuum, say  $R^1$ . Since white noise  $\{\dot{B}(t), t \in R\}$  is a system of continuously many independent random variables, although each member is an infinitesimal random variables, the probability distribution might be thought of as a direct product of continuously many Gaussian distributions. Separability might not be expected. Obviously this is not a correct understanding as we shall see in Chapter 2. In fact, sample functions of  $\dot{B}(t), t \in R$ , are generalized functions, so that the probability distribution of white noise is given on the space of generalized functions. The measure space, thus obtained, is certainly separable. These two facts seem to be a contradiction, but in reality, it is not so.

An interpretation of these facts will be given in Chapter 2, where an identity, as it were, of  $\dot{B}(t)$  is given as a generalized white noise functional (indeed, it is a linear generalized functional).

Note that white noise  $\dot{B}(t)$  has been understood as a generalized stochas-

tic process, from a view point of classical stochastic analysis, where smeared variables such that

$$\dot{B}(\xi) = - \int B(t)\xi'(t)dt$$

have meaning.

We now propose to take  $\dot{B}(t)$  without smearing. Indeed, white noise analysis does not want to have white noise smeared, since the time variable  $t$  should not disappear. Having fixed the system of variables  $\{\dot{B}(t), t \in R\}$ , we come to nonlinear functionals, starting from polynomials in  $\dot{B}(t)$ 's. Unfortunately, bare polynomials (with degree  $\geq 2$ ) cannot be in any acceptable class of white noise functionals. We have therefore found a most significant trick, which is called *renormalization*, to make polynomials to be in a class of acceptable functionals. The class is nothing but the space of generalized white noise functionals that has been emphasized at the beginning of this preface. This class will be discussed in details in Chapter 2. Indeed, with the use of generalized white noise functionals, we have done more than classical stochastic analysis.

As the next step, we are led to introduce a partial differential operator in a generalized sense such that

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)},$$

(see Kubo-Takenaka<sup>89</sup>).

Other notions and operators like Laplacians that are necessary for our analysis can naturally be introduced successively.

We should note that Poisson noise and its functionals can be discussed in the same idea, but important fact is that dissimilarity between two noises is quite interesting and important. Furthermore, we have discovered some *duality* that provide connections between them.

Thus, a roadmap is ready and a new stochastic analysis proceeds. Needless to say, we can establish intimate relations with other fields, not only within mathematics but wider area of sciences.

These notes have been prepared so as to be self-contained. As a result,

some elementary description is included. Beginners would find it easy to follow the edifice of white noise theory.

The present work, though far from claiming completeness, aims at giving an outline of white noise analysis responding to growing interest in this approach. It is our hope that the present notes would contribute to the future development of stochastic analysis and of infinite dimensional analysis.

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