PART 1

The Affine and Linear Structures of $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$

Introduction

Starting from intuitively geometric objects, we treat

1. a point as a zero vector,
2. a directed segment (along a line) as a vector, and
3. two directed segments along the same or parallel lines as the same vector if both have the same length and direction.

And hence, we define two vector operations: scalar multiplication $\alpha\vec{x}$ and addition $\vec{x} + \vec{y}$ and develop their operational properties. In the process, we single out the linear combination, dependence and independence among vectors as the main tools and establish the affine structures on a line, a plane and a space, respectively. Then, we extract the essence of concepts obtained and formulate, via rough ideas of linear isomorphism, the abstract sets $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$ as the standard one-dimensional, two-dimensional and three-dimensional vector spaces over the real field, respectively. So far, changes of coordinates in the same space are the most prominent results among all, which indicates implicitly the concepts of affine and linear transformations.

Then, we focus our attention to these mappings between spaces that preserve the ratios of signed lengths of segments along the same or parallel lines. They are affine transformations (see Secs. 1.4, 2.7 and 2.8), in particular, linear transformations if they map the zero vector into the zero vector when the spaces concerned are considered as vector spaces.

The main themes will be topics on linear transformations or, equivalently, real matrices of order $m \times n$, where $m, n = 1, 2, 3$, such as:

1. Eigenvalues and eigenvectors (Secs. 2.7.1, 2.7.2, 3.7.1 and 3.7.2).
2. Various decompositions, for example, elementary matrix factorizations, LU, LDU, LDL* and LPU, etc. (Secs. 2.7.5 and 3.7.5).
3. Rank, Sylvester's law of inertia (Secs. 2.7.1, 2.7.5, 3.7.1 and 3.7.5).

4. Diagonalizability (Secs. 2.7.6 and 3.7.6).
5. Jordan and rational canonical forms (Secs. 2.7.7, 2.7.8, 3.7.7 and 3.7.8).

A suitable choice of basis for the kernel or/and the image of a linear transformation will play a central role in handling these problems.

Based on results about linear transformations, affine transformations are composed of linear transformations followed by translations. We discuss topics such as:

1. Stretch, reflection, shearing, rotation and orthogonal reflection and their matrix representations and geometric mapping properties (Secs. 2.8.2 and 3.8.2).
2. Affine invariants (Secs. 2.8.3 and 3.8.3).
3. Affine geometries, including sketches of projective plane $P^2(\mathbb{R})$ and projective spaces $P^3(\mathbb{R})$ (Ex. <B> of Sec. 2.6, Sec. 2.8.4, Ex. <B> of Sec. 2.8.5 and Sec. 3.8.4).
4. Quadratic curves (Sec. 2.8.5) and quadrics (Sec. 3.8.5).

Chapter 1 is trivial in content but is necessary in the inductive process. Chapter 2 is crucial both in content and in method. Methods in Chap. 2 are essentially the extensions of geometric and algebraic ones which are learned from the middle school mathematical courses, in particular, the ways of solving simultaneous linear equations and their geometric interpretations. Hence, methods adopted in Chap. 2 play a transitive role from the middle school ones to the more sophisticated and well-established ones used in nowadays linear algebra, as will be seen and formulated in Chap. 3. In short, methods and contents in Chap. 2 can be considered as buffer zones between classical middle school algebra and modern linear algebra.

Based on our inductive construction and description about linear algebras on $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, we hope that readers will possess enough solid foundations, both in geometric intuition and in algebraic manipulation. Thus, they can foresee, realize and construct what the n-dimensional vector space $\mathbb{R}^n$ $(n \geq 4)$ and the linear algebras on it are, even on the more abstract vector spaces over fields. For this purpose, we have arranged intensively problems in Exercises <B> and <C> for minded readers to practice, and Appendix B for reference.

The use of matrices (of order $m \times n, m, n = 1, 2, 3$) and determinants (of order $m, m = 2, 3$) comes to surface naturally as we proceed without introducing them beforehand in a particularly selected section. Here in

Part 1, we emphasize the computational aspects of matrices and determinants. So, the needed readers should consult Sec. B.4 for matrices and Sec. B.6 for determinants. Sections 4.3 and 5.3 will formally introduce the theory of determinants of order 2 and 3, respectively, via geometric considerations.

CHAPTER 1

The One-Dimensional Real Vector Space $\mathbb{R}$ (or $\mathbb{R}^1$)

Introduction

Our theory starts from the following simple geometric

Postulate *A single point determines a unique zero-dimensional* (*vector*) *space.*

Usually, a little black point or spot is used as an intuitively geometric model of zero-dimensional space. Notice that "point" is an undefined term without length, width and height.

In the physical world, it is reasonable to imagine that there exits two different points. Hence, one has the

Postulate *Any two different points determine one and only one straight line.*

A straightened loop, extended beyond any finite limit in both directions, is a lively geometric model of a straight line. Mathematically, pick up two different points O and A on a flat paper, imagining extended beyond any limit in any direction, and then, connect O and A by a ruler. Now, we have a geometric model of an unlimited straight line L (see Fig. 1.1).

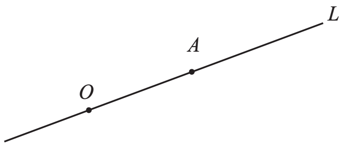

Fig. 1.1

As far as the basic concepts of straight lines are concerned, one should know the following facts (1)–(6).

(1) There are uncountably infinite points on L.

(2) The straight line determined by any two different points on L coincides with L.

(3) Any two points P and Q on L decide a *segment*, denoted by PQ:

1. If $P = Q$ (i.e. P and Q coincide, and represent the same point), then the segment PQ degenerates into a single P (or Q);
2. If $P \neq Q$ (i.e. P and Q are different points), then PQ consists of those points on L lying between P and Q (included).

(4) If one starts from point P, walking along L toward point Q, then one gets the *directed segment* $\overrightarrow{PQ}$; if from Q to P, reversing the direction, one has the *directed segment* $\overrightarrow{QP}$ (see Fig. 1.2).

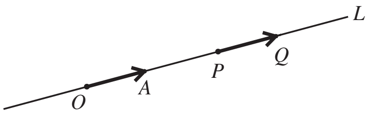

Fig. 1.2

(5) Arbitrarily fix two different points O and A on line L. Consider the segment OA as one unit in length. Then, one should be able to measure the *distance* between any two points P and Q on line L or the *length* of the segment PQ. As usual, distance and length are always non-negative real numbers.

In order to extend the mathematical knowledge we have up to now, here we introduce the *signed length* of a segment PQ as follows:

1. Let $P = Q$, then PQ has length zero;
2. Let $P \neq Q$,

$$PQ \text{ has length} > 0 \Leftrightarrow \overrightarrow{PQ} \text{ has the same direction as } \overrightarrow{OA};$$
$$PQ \text{ has length} < 0 \Leftrightarrow \overrightarrow{PQ} \text{ has the opposite direction as } \overrightarrow{OA}.$$

Therefore, the direction of $\overrightarrow{OA}$ is designated as the *positive direction* of the line L with O as its origin, while $\overrightarrow{AO}$ the *negative direction*.

Remark For convenience, we endow PQ with two meanings: one represents the segment with endpoints P and Q, the other represents the length of that segment. Similarly, $\overrightarrow{PQ}$ has two meanings too: the directed segment from P to Q and the signed length of that segment.

Therefore, finally we have

(6) For any three points P, Q and R on line L, their signed lengths $\overrightarrow{PQ}, \overrightarrow{QR}$ and $\overrightarrow{PR}$ always satisfy the following identity:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} \quad \text{(see Fig. 1.3)}.$$

Fig. 1.3

Sketch of the Content

Based on these facts, this chapter contains four sections, trying to vectorize (Sec. 1.1) and coordinatize (Sec. 1.2) the straight line, and studying the linear changes between different coordinate systems (Sec. 1.3). Invariants under affine transformation are discussed in Sec. 1.4.

The main result is that, under coordinatization, a straight line can be considered as a concrete geometric model of the real number system (field) $\mathbb{R}$. Hence, $\mathbb{R}$ is an abstract representation of the *one-dimensional vector space over the real field* $\mathbb{R}$.

Our introduction to two-dimensional (Chap. 2) and three-dimensional (Chap. 3) vector spaces will be modeled after the way we have treated here in Chap. 1.

1.1 Vectorization of a Straight Line: Affine Structure

Fix a straight line L.

We provide a directed segment $\overrightarrow{PQ}$ on line L as a (line) *vector*. If $P = Q, \overrightarrow{PQ}$ is called a *zero vector*, denoted by $\vec{0}$. On the contrary, $\overrightarrow{PQ}$ is a *nonzero vector* if $P \neq Q$.

Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ are *identical*, i.e. $\overrightarrow{PQ} = \overrightarrow{P'Q'}$.

$\Leftrightarrow$ 1. $PQ = P'Q'$ (equal in length),

2. "the direction from P to Q (along L)" is the same as "the direction from P' to Q'". (1.1.1)

We call properties 1 and 2 as the *parallel invariance of vector* (see Fig. 1.4).

Fig. 1.4

In particular, for any points P and Q on L, one has

$$\overrightarrow{PP} = \overrightarrow{QQ} = \overrightarrow{0}. \tag{1.1.2}$$

Hence, zero vector is uniquely defined.

Now, fix any two different points O and X on L. For simplicity, denote the vector $\overrightarrow{OX}$ by $\overrightarrow{x}$, i.e.

$$\overrightarrow{x} = \overrightarrow{OX}.$$

Note that $\overrightarrow{x} \neq \overrightarrow{0}$.

For any fixed point P on L, the ratio of the signed length $\overrightarrow{OP}$ with respect to $\overrightarrow{OX}$ is

$$\frac{\overrightarrow{OP}}{\overrightarrow{OX}} = \alpha,$$

where the real number α has the following properties:

$\alpha = 0 \Leftrightarrow P = O$;
$0 < \alpha < 1 \Leftrightarrow P$ lies on the segment OX $(P \neq O, X)$;
$\alpha > 1 \Leftrightarrow P$ and X lie on the same side of O and $OP > OX$;
$\alpha = 1 \Leftrightarrow P = X$; and
$\alpha < 0 \Leftrightarrow P$ and X lie on the different sides of O.

In all cases, designate the vector

$$\overrightarrow{OP} = \alpha\overrightarrow{OX} = \alpha\overrightarrow{x}.$$

On the other hand, for any given $\alpha \in \mathbb{R}$, there corresponds one and only one point P on line L such that $\overrightarrow{OP} = \alpha\overrightarrow{x}$ holds (see Fig. 1.5).

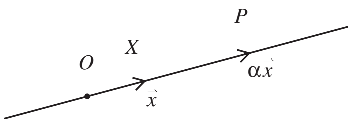

Fig. 1.5

Summarize as

The Vectorization of a straight line

Fix any two different points O and X on a straight line L and denote the vector $\overrightarrow{OX}$ by $\overrightarrow{x}$. Then the *scalar product* $\alpha\overrightarrow{x}$ of an arbitrary real number

$\alpha \in \mathbb{R}$ with the fixed vector $\vec{x}$ is suitable for describing any point P on L (i.e. the position vector $\overrightarrow{OP}$). Call the set

$$L(O;X) = \{\alpha\vec{x} \mid \alpha \in \mathbb{R}\}$$

the *vectorized space* of the line L with O as the *origin*, $\overrightarrow{OO} = \vec{0}$ as *zero vector* and $\vec{x}$ as the *base vector*. Elements in $L(O;X)$ are called (line) *vectors* which have the following algebraic operation properties: $\alpha, \beta \in \mathbb{R}$,

1. $(\alpha+\beta)\vec{x} = \alpha\vec{x} + \beta\vec{x} = \beta\vec{x} + \alpha\vec{x}$;
2. $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x}) = \beta(\alpha\vec{x})$;
3. $1\vec{x} = \vec{x}$;
4. Let $\vec{0} = \overrightarrow{OO}$, then $\alpha\vec{x} + \vec{0} = \vec{0} + \alpha\vec{x} = \alpha\vec{x}$;
5. $(-\alpha)\vec{x} = -\alpha\vec{x}$; $\vec{x} + (-\vec{x}) = \vec{0} = \vec{x} - \vec{x}$;
6. $0\vec{x} = \alpha\vec{0} = \vec{0}$.

In short, via the position vector $\overrightarrow{OP} = \alpha\vec{x}$ for any α, points P on L have the above algebraic operation properties. (1.1.3)

Using the concept of (1.1.3), one can establish the algebraic characterization for three points lying on the same line.

Suppose that points O, X and Y are collinear. Let $\vec{x} = \overrightarrow{OX}$ and $\vec{y} = \overrightarrow{OY}$.

In case $O = X = Y$: then $\vec{x} = \vec{y} = \vec{0}$, and hence $\vec{x} = \alpha\vec{y}$ or $\vec{y} = \alpha\vec{x}$ holds for any $\alpha \in \mathbb{R}$.

In case two of O, X and Y coincide, say $X \neq O = Y$: then $\vec{y} = \vec{0}$ and $\vec{y} = 0\vec{x}$ holds.

If O, X and Y are different from each other, owing to the fact that Y lies on the line determined by O and X, $\vec{y}$ belongs to $L(O;X)$. Hence, there exists $\alpha \in \mathbb{R}$ such that $\vec{y} = \alpha\vec{x}$.

We summarize these results as

Linear dependence of line vectors
Let $\vec{x} = \overrightarrow{OX}$ and $\vec{y} = \overrightarrow{OY}$. Then

(1) (geometric) Points O, X and Y are collinear.
$\Leftrightarrow$ (2) (algebraic) There exists $\alpha \in \mathbb{R}$ such that $\vec{y} = \alpha\vec{x}$ or $\vec{x} = \alpha\vec{y}$.
$\Leftrightarrow$ (3) (algebraic) There exist scalars $\alpha, \beta \in \mathbb{R}$, not all equal to zero, such that $\alpha\vec{x} + \beta\vec{y} = \vec{0}$.

In any one of these three cases, vectors $\vec{x}$ and $\vec{y}$ are said to be *linear dependent* (on each other). (1.1.4)

As contrast to linear dependence, one has

$$\alpha\vec{x} = \vec{0} \Leftrightarrow 1^\circ \alpha = 0 \ (\vec{x} \text{ may not be } \vec{0}); \quad \text{or}$$
$$2^\circ \vec{x} = \vec{0} \ (\alpha \text{ may not be } 0).$$

Therefore, we have

Linear independence of a nonzero vector
Let $OX = \vec{x}$. Then

(1) (geometric) Points O and X are different.
$\Leftrightarrow$ (2) (algebraic) If there exists $\alpha \in \mathbb{R}$ such that $\alpha\vec{x} = \vec{0}$, then it is necessarily that $\alpha = 0$.

In any of these situations, vector $\vec{x}$ is said to be *linear independent* (from any other vector, whatsoever!). (1.1.5)

That is to say, a single nonzero vector must be linearly independent.

Exercises
<A>

1. Interpret geometrically 1 to 6 in (1.1.3).
2. Finish the incomplete proofs of (1.1.4), for example, (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1).

1.2 Coordinatization of a Straight Line: $\mathbb{R}^1$ (or $\mathbb{R}$)

Suppose the straight line L is provided with a fixed vectorized space $L(O;X)$, where $\vec{x} = \overrightarrow{OX}$ is the base vector.

We call the set

$$\mathcal{B} = \{\vec{x}\}$$

a *basis* of $L(O;X)$.

For any given point $P \in L$, the fact that $\overrightarrow{OP} \in L(O;X)$ induces a unique $\alpha \in \mathbb{R}$ such that $\overrightarrow{OP} = \alpha\vec{x}$ holds. Then, this unique scalar, denoted by

$$\alpha = [\overrightarrow{OP}]_{\mathcal{B}} = [P]_{\mathcal{B}}$$

is defined to be the *coordinate* of the point P or the vector $\overrightarrow{OP}$ *with respect to the basis* $\mathcal{B}$. In particular,

$$[O]_{\mathcal{B}} = 0, \quad [X]_{\mathcal{B}} = 1.$$

For example:

$$[P]_{\mathcal{B}} = -2 \Leftrightarrow \overrightarrow{OP} = -2\vec{x};$$
$$[Q]_{\mathcal{B}} = \frac{3}{2} \Leftrightarrow \overrightarrow{OQ} = \frac{3}{2}\vec{x}.$$

See Fig. 1.6.

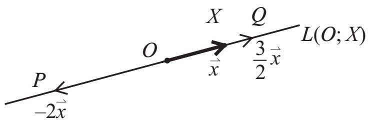

Fig. 1.6

Now we summarize as

The coordinatization of a straight line

Let $L(O;X)$ be an arbitrary vectorized space of line L, with $\mathcal{B} = \{\vec{x}\}$, $\vec{x} = \overrightarrow{OX}$, as a basis. The set

$$\mathbb{R}_{L(O;X)} = \{[P]_{\mathcal{B}} \mid P \in L\}$$

is called the *coordinatized space* of L with respect to $\mathcal{B}$. Explain further as follows.

(1) There is a one-to-one correspondence from any point P on line L onto corresponding number $[P]_{\mathcal{B}}$ of the real number system $\mathbb{R}$.

(2) Define a mapping Φ: $L(O;X) \to \mathbb{R}$ by

$$\Phi(\alpha\vec{x}) = \alpha \quad (\text{or } \Phi(\overrightarrow{OP}) = [P]_{\mathcal{B}}, P \in L).$$

Then Φ is one-to-one, onto and preserves algebraic operations, i.e. for any $\alpha, \beta \in \mathbb{R}$,

1. $\Phi(\beta(\alpha\vec{x})) = \beta\alpha$,
2. $\Phi(\alpha\vec{x} + \beta\vec{x}) = \alpha + \beta$. (1.2.1)

According to Sec. B.7 of Appendix B, mappings like Φ here are called *linear isomorphisms* and therefore, conceptually, $L(O;X)$, $\mathbb{R}_{L(O;X)}$ and $\mathbb{R}$ are considered being identical (see Fig. 1.7).

We have already known that the following are equivalent.

(1) Only two different points, needless a third one, are enough to determine a unique line.

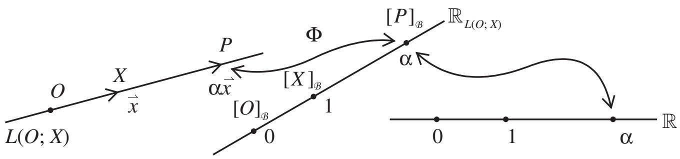

Fig. 1.7

(2) Only one nonzero vector is enough to generate the whole space $L(O;X)$ (refer to (1.1.4) and (1.1.5)).

Hence, we say that $L(O;X)$ is a *one-dimensional vector space* with zero vector $\vec{0}$. Accurately speaking, $L(O;X)$ is a *one-dimensional affine space* (see Sec. 2.8 or Fig. B.2) of the line L.

Owing to arbitrariness of O and X, the line L can be endowed with infinitely many vectorized spaces $L(O;X)$. But according to (1.2.1), no matter how O and X are chosen, we always have that

$$L(O;X) \underset{\Phi}{\simeq} \mathbb{R}_{L(O;X)} = \mathbb{R}. \tag{1.2.2}$$

So, we assign $\mathbb{R}$ another role, representing the *standard one-dimensional real vector space* and denoted by

$$\mathbb{R}^1. \tag{1.2.3}$$

A number α in $\mathbb{R}$ is identical with the position vector in $\mathbb{R}^1$, starting from 0 and pointing toward α, and is still denoted by α (see Fig. 1.8).

For the sake of reference and comparison, we summarize as

The real number system $\mathbb{R}$ and the standard one-dimensional vector space $\mathbb{R}^1$

(1) $\mathbb{R}$ (simply called the real field, refer to Sec. A.3)

(a) *Addition* For any $x, y \in \mathbb{R}$,

$$x + y \in \mathbb{R}$$

satisfies the following properties.

1. (commutative) $x + y = y + x$.
2. (associative) $(x + y) + z = x + (y + z)$.
3. (zero element) 0: $0 + x = x$.
4. (inverse element) For each $x \in \mathbb{R}$, there exists a unique element in $\mathbb{R}$, denoted as $-x$, such that

$$x + (-x) = x - x = 0.$$

(b) *Multiplication* For any x, $y \in \mathbb{R}$,

$$xy \in \mathbb{R}$$

satisfies the following properties.

1. (commutative) $xy = yx$.
2. (associative) $(xy)z = x(yz)$.
3. (unit element) 1: $1x = x$.
4. (inverse element) For each nonzero $x \in \mathbb{R}$, there exists a unique element in $\mathbb{R}$, denoted by x^{-1} or $\frac{1}{x}$, such that

$$xx^{-1} = 1.$$

(c) The addition and multiplication satisfy the distributive law

$$x(y+z) = xy + xz.$$

(2) $\mathbb{R}^1$ (see (1.2.3) and refer to Sec. B.1)

(a) *Addition* To every pair of elements x and y in $\mathbb{R}^1$, there is a unique element

$$x + y \in \mathbb{R}^1$$

satisfying the following properties.

1. $x + y = y + x$.
2. $(x + y) + z = x + (y + z)$.
3. (zero vector 0) $x + 0 = x$.
4. (inverse vector) For each $x \in \mathbb{R}^1$, there exists a unique element in $\mathbb{R}^1$, denoted as $-x$, such that

$$x + (-x) = x - x = 0.$$

(b) *Scalar multiplication* To each $x \in \mathbb{R}^1$ and every real number $\alpha \in \mathbb{R}$, there exists a unique element

$$\alpha x \in \mathbb{R}^1$$

satisfying the following properties.

1. $1x = x$.
2. $\alpha(\beta x) = (\alpha\beta)x, \alpha, \beta \in \mathbb{R}$.

(c) The addition and scalar multiplication satisfy the distributive laws

$$\begin{aligned}(\alpha + \beta)x &= \alpha x + \beta x;\\ \alpha(x + y) &= \alpha x + \alpha y.\end{aligned} \tag{1.2.4}$$

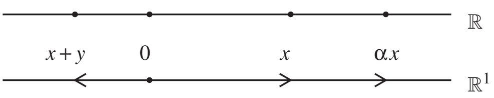

Fig. 1.8

See Fig. 1.8.

Remark On many occasions, no distinction between $\mathbb{R}$ and $\mathbb{R}^1$, in notation, will be specified and we simply denote $\mathbb{R}^1$ by $\mathbb{R}$.

In this sense, element in $\mathbb{R}$ has double meanings. One is to represent a number treated as a point on the real line. The other is to represent a position vector, pointing from 0 toward itself on the real line. One should remember that any element in $\mathbb{R}$, either as a number or as a vector, somewhat enjoys different algebraic operation properties as indicated in (1.2.4).

From now on, if necessarily or conveniently, we do not hesitate to use $\mathbb{R}$ to replace $\mathbb{R}^1$, both in notation and in meaning. Any straight line L, endowed with a vectorized space $L(O;X)$, is nothing but a concrete geometric model of $\mathbb{R}$ (i.e. $\mathbb{R}^1$).

1.3 Changes of Coordinates: Affine and Linear Transformations (or Mappings)

Let L be a straight line with two vectorized spaces $L(O;X)$ and $L(O';Y)$ on it.

The same point P on L has different coordinates $[P]_{\mathcal{B}}$ and $[P]_{\mathcal{B}'}$, respectively, with respect to the different bases

$$\mathcal{B} = \{\vec{x}\}, \quad \vec{x} = \overrightarrow{OX}, \quad \text{and}$$
$$\mathcal{B}' = \{\vec{y}\}, \quad \vec{y} = \overrightarrow{O'Y}$$

(see Fig. 1.9).

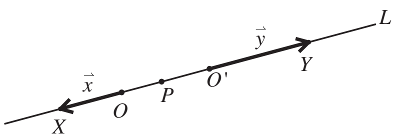

Fig. 1.9

Our purpose here is to find out the relation between $[P]_{\mathcal{B}}$ and $[P]_{\mathcal{B}'}$.

Suppose that, temporarily,

$$[P]_{\mathcal{B}} = \mu \ (\Leftrightarrow \overrightarrow{OP} = \mu\vec{x}), \quad [O']_{\mathcal{B}} = \alpha_0;$$
$$[P]_{\mathcal{B}'} = \nu \ (\Leftrightarrow \overrightarrow{O'P} = \nu\vec{y}), \quad [O]_{\mathcal{B}'} = \beta_0.$$

Since $\vec{x}$ and $\vec{y}$ are collinear, there exist constants α and β such that $\vec{y} = \alpha\vec{x}$ and $\vec{x} = \beta\vec{y}$. Hence $\vec{x} = \alpha\beta\vec{x}$ implies $(\alpha\beta - 1)\vec{x} = \vec{0}$. The linear independence of $\vec{x}$ shows that $\alpha\beta - 1 = 0$ should hold, i.e.

$$\alpha\beta = 1.$$

Now, owing to the fact that $\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$,

$$\begin{aligned} &\mu\vec{x} = \alpha_0\vec{x} + \nu\vec{y} = \alpha_0\vec{x} + \nu\alpha\vec{x} = (\alpha_0 + \nu\alpha)\vec{x} \\ &\Rightarrow \mu = \alpha_0 + \nu\alpha, \quad \text{or} \\ &[P]_{\mathcal{B}} = [O']_{\mathcal{B}} + \alpha[P]_{\mathcal{B}'}. \end{aligned} \tag{1.3.1}$$

Similarly, by using $\overrightarrow{O'P} = \overrightarrow{O'O} + \overrightarrow{OP}$, one has

$$\begin{aligned} \nu &= \beta_0 + \beta\mu, \quad \text{or} \\ [P]_{\mathcal{B}'} &= [O]_{\mathcal{B}'} + \beta[P]_{\mathcal{B}}. \end{aligned} \tag{1.3.2}$$

Remark (1.3.1) and (1.3.2) are reversible.

Suppose that (1.3.1) is true. Then one has (why $\alpha \neq 0$?)

$$\nu = -\frac{\alpha_0}{\alpha} + \frac{1}{\alpha}\mu.$$

But $\overrightarrow{OO'} = -\overrightarrow{O'O} \Rightarrow \alpha_0\vec{x} = -\beta_0\vec{y} = -\beta_0\alpha\vec{x} \Rightarrow \alpha_0 = -\beta_0\alpha$ or $\beta_0 = -\frac{\alpha_0}{\alpha}$. Since $\alpha\beta = 1$, therefore

$$\nu = \beta_0 + \beta\mu.$$

This is (1.3.2).

Similarly, (1.3.1) is deduced from (1.3.2) with the same process.

Summarize the above results as

Coordinate changes of two vectorized spaces on the same line
Let

$$L(O;X) \text{ with basis } \mathcal{B}=\{\vec{x}\}, \vec{x}=\overrightarrow{OX}, \text{ and}$$
$$L(O';Y) \text{ with basis } \mathcal{B}'=\{\vec{y}\}, \vec{y}=\overrightarrow{O'Y}$$

be two vectorized spaces of the line L. Suppose that

$$\vec{y}=\alpha\vec{x}, \quad \vec{x}=\beta\vec{y} \qquad (\text{therefore, } \alpha\beta=1).$$

Then the coordinates $[P]_{\mathcal{B}}$ and $[P]_{\mathcal{B}'}$ with respect to bases $\mathcal{B}$ and $\mathcal{B}'$, of the same point P on L, satisfy the following reversible linear relations:

$$[P]_{\mathcal{B}}=[O']_{\mathcal{B}}+\alpha[P]_{\mathcal{B}'}, \quad \text{and}$$
$$[P]_{\mathcal{B}'}=[O]_{\mathcal{B}'}+\beta[P]_{\mathcal{B}}.$$

In particular, if $O=O'$, then $[O']_{\mathcal{B}}=[O]_{\mathcal{B}'}=0$. (1.3.3)

In specific terminology, Eqs. such as (1.3.1) and (1.3.2) are called *affine transformations* or *mappings* between *affine spaces* $L(O;X)$ and $L(O';Y)$; in case $O=O'$, called (*invertible*) *linear transformation* (see Sec. B.7).

Finally, here is an example.

Example Determine the relation between the Centigrade (°C) and the Fahrenheit (°F) on the thermometer.

Solution (see Fig. 1.10) Suppose O and X are 0°C and 1°C respectively. Also, O' and Y are 0°F and 1°F. Then, we have

$$\overrightarrow{O'O}=32\overrightarrow{O'Y} \quad \text{and} \quad \overrightarrow{OX}=\frac{9}{5}\overrightarrow{O'Y}.$$

Let $C=\{\overrightarrow{OX}\}$ and $F=\{\overrightarrow{O'Y}\}$. Now, for any given point P on the thermometer, we have

$$[P]_{\mathrm{C}}=\text{the Centigrade degree of } P, \text{ and}$$
$$[P]_{\mathrm{F}}=\text{the Fahrenheit degree of } P.$$

By using the fact that $\overrightarrow{O'P}=\overrightarrow{O'O}+\overrightarrow{OP}$, one has the following relation

$$[P]_{\mathrm{F}}=[O]_{\mathrm{F}}+\frac{9}{5}[P]_{\mathrm{C}}=32+\frac{9}{5}[P]_{\mathrm{F}}, \quad \text{or}$$
$$[P]_{\mathrm{C}}=\frac{5}{9}\{[P]_{\mathrm{F}}-32\}$$

between $[P]_{\mathrm{F}}$ and $[P]_{\mathrm{C}}$. □

Fig. 1.10

Exercises

<A> (Adopt the notations in (1.3.3).)

1. Suppose that $L(O;X)$ and $L(O';Y)$ are two vectorized spaces on the same line L. Let

$$[O]_{\mathcal{B}'} = -5 \quad \text{and} \quad \vec{y} = \frac{1}{3}\vec{x}.$$

   (a) Locate the points O, X, O' and Y on the line L.
   (b) If a point $P \in L$ and $[P]_{\mathcal{B}} = 0.2$, find $[P]_{\mathcal{B}'}$? If $[P]_{\mathcal{B}'} = 15$, what is $[P]_{\mathcal{B}}$?

2. Construct two vectorized spaces $L(O;X)$ and $L(O';Y)$ on the same line L, and explain graphically the following equations as changes of coordinates with

$$[P]_{\mathcal{B}} = x \quad \text{and} \quad [P]_{\mathcal{B}'} = y, \quad P \in L.$$

   (a) $y = -2x$.
   (b) $y = \sqrt{3}x - \frac{5}{3}$.
   (c) $x = 6y$.
   (d) $x = -15y + 32$.

1.4 Affine Invariants

Construct a vectorized space $L(O;X)$ on the line L and a vectorized space $L'(O';X')$ on the line L', where L and L' may not be coincident. Let $\mathcal{B} = \{\overrightarrow{OX}\}$ and $\mathcal{B}' = \{\overrightarrow{O'X'}\}$ be the respective basis on L and L'. Also, let

$$[P]_{\mathcal{B}} = x \quad \text{for } P \in L, \qquad \text{and}$$
$$[P']_{\mathcal{B}'} = y \quad \text{for } P' \in L'.$$

A mapping or transformation T from L onto L' (see Sec. A.2) is called an *affine mapping* or *transformation* if there exist constants a and $b \neq 0$ such that

$$T(x) = y = a + bx \tag{1.4.1}$$

holds for all $P \in L$ and the corresponding $P' \in L'$. Note that $y = T(x)$ is one-to-one. In case $a = 0$, $y = T(x) = bx$ is called a *linear transformation* (isomorphism) from the vector space $L(O; X)$ onto the vector space $L'(O'; X')$. In this sense, change of coordinates on the same line as introduced in (1.3.3) is a special kind of affine mapping.

For any two fixed different points P_1 and P_2 with $[P_1]_{\mathcal{B}} = x_1$ and $[P_2]_{\mathcal{B}} = x_2$, the whole line L has *coordinate representation*

$$x = (1-t)x_1 + tx_2, \quad t \in \mathbb{R} \tag{1.4.2}$$

with respect to the basis $\mathcal{B}$. The *directed segment* $\overrightarrow{P_1P_2}$ or $\overrightarrow{x_1x_2}$ with P_1 as *initial point* and P_2 as *terminal point* is the set of points

$$x = (1-t)x_1 + tx_2, \quad 0 \leq t \leq 1. \tag{1.4.3}$$

In case $0 < t < 1$, the corresponding point x is called an *interior point* of $\overrightarrow{x_1x_2}$, otherwise (i.e. $t < 0$ or $t > 1$) an *exterior point*. See Fig. 1.11.

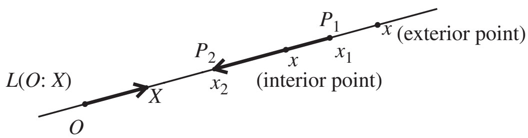

Fig. 1.11

Applying (1.4.1) and (1.4.3), we see that an affine mapping maps a (directed) segment $\overrightarrow{x_1x_2}$ onto a (directed) segment $\overrightarrow{y_1y_2}$, preserving end points, interior points and exterior points. In fact, a point $x = (1-t)x_1 + tx_2$ is mapped into the point

$$y = (1-t)y_1 + ty_2 \tag{1.4.4}$$

with $y_1 = a + bx_1$ and $y_2 = a + bx_2$.

Orient the line L by the basis vector $\overrightarrow{OX}$ in $L(O;X)$ and let $x_2 - x_1$ be the signed length of the segment $\overrightarrow{x_1x_2}$ as we did in Sec. 1.1. For convenience, we also use $\overrightarrow{x_1x_2}$ to denote its signed length. Then, by (1.4.2), we see that

$$\begin{aligned}(1-t)(x-x_1) &= t(x_2-x) \\ \Rightarrow \frac{\overrightarrow{x_1x}}{\overrightarrow{xx_2}} &= \frac{t}{1-t}, \quad t \neq 0, 1 \end{aligned} \tag{1.4.5}$$

which is equal to $\frac{\overrightarrow{y_1y}}{\overrightarrow{yy_2}}$, by using (1.4.4). This means that an affine mapping preserves the ratio of two line segments along the line (see also (1.4.6) and Ex. <A> 2).

Finally, $y_2 - y_1 = a + bx_2 - (a + bx_1) = b(x_2 - x_1)$ means that

$$\overrightarrow{y_1y_2} = b\overrightarrow{x_1x_2}. \tag{1.4.6}$$

Then, an affine mapping does not preserve the signed length except $b = 1$, and does not preserve the orientation of directed segment except $b > 0$.

We summarize as

Affine invariants

An affine transformation between straight lines preserves

1. (directed) line segment along with end points, interior points and exterior points, and
2. ratio of (signed) lengths of two line segments (along the same line) which are called *affine invariants*. It does not necessarily preserve
3. (signed) length, and
4. orientation. (1.4.7)

Remark Affine coordinate system and affine (or barycentric) coordinate.

Let a straight line L be vectorized as $L(O;X)$, and P_1 and P_2 be two arbitrarily fixed different points. As usual, denote the basis $\mathcal{B} = \{\overrightarrow{OX}\}$ and $[P_1]_\mathcal{B} = x_1$, $[P_2]_\mathcal{B} = x_2$.

(1.4.2) can be rewritten as

$$x = \lambda_1 x_1 + \lambda_2 x_2, \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{and} \quad \lambda_1 + \lambda_2 = 1 \tag{1.4.8}$$

for any point P on L with $[P]_\mathcal{B} = x$. Then, we call the order pair

$$(\lambda_1, \lambda_2) \tag{1.4.9}$$

the *affine* or *barycentric coordinate* of the point P or x with respect to the *affine coordinate system* $\{P_1, P_2\}$ or $\{\overrightarrow{OP_1}, \overrightarrow{OP_2}\}$. In particular, $(\frac{1}{2}, \frac{1}{2})$ is the *barycenter* or the *middle point* of the segment P_1P_2 or x_1x_2.

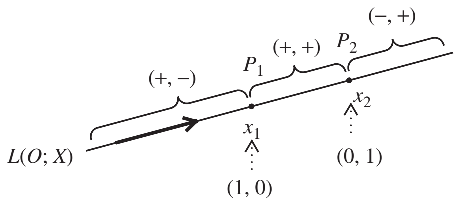

Fig. 1.12

x is an interior point of P_1P_2 if and only if it has affine coordinate (λ_1, λ_2) with components $\lambda_1 > 0$ and $\lambda_2 > 0$. See Fig. 1.12. As a trivial consequence, the points P_1 and P_2 divide the whole line L into three different parts: $(+, -)$, $(+, +)$, and $(-, +)$.

Exercises

<A>

1. For each pair P_1, P_2 of different points on $L(O; X)$ and each pair P'_1, P'_2, of different points on $L'(O'; X')$, show that there exists a unique affine mapping T from $L(O; X)$ onto $L'(O'; X')$ such that
$$T(P_1) = P'_1 \quad \text{and} \quad T(P_2) = P'_2.$$
2. A one-to-one and onto mapping $T \colon L(O; X) \to L'(O'; X')$ is affine if and only if T preserves ratio of signed lengths of any two segments.