3

FUNCTIONAL INTEGRALS IN QUANTUM FIELD THEORY

3-1 INTRODUCTION

In this section we will briefly describe the procedure by which one deduces that in quantum field theory the main quantities of interest, the Green functions, are the analogues of the correlation functions discussed in the previous chapter, in the context of statistical mechanics. These Green functions can be expressed in terms of the variations with respect to sources of a generating functional analogous to the one defined in Sec. 2-2. There are, of course, major differences hidden beyond the analogies.

There is no cutoff in the relativistic quantum field theory and thus, strictly speaking, the theory is not well defined in four dimensions. This difficulty is handled by treating the perturbation theory as a formal structure, which gives rise to diagrams, etc., and then defining the physical quantities by a procedure which extracts the finite part of every term in the perturbation series. This process is the regularization and renormalization program, associated with the names of Dyson, Gell-Man, Low and others.

The relativistic Lagrangian has a part quadratic in the derivatives of the fields, but its form is

$$\mathscr{L}_0 = -\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi + \mu^2\phi^2) \qquad (3\text{-}1)$$

where ∂^μ are 4-derivatives. The square of the time derivative ∂^0 enters with the opposite sign to the spatial derivatives. Thus one encounters singularities in the free theory.

Furthermore, as we shall see, $\mathscr{L}$ enters in the exponential of the generating functional as

$$\mathrm{i}\int \mathscr{L}(\phi)\,\mathrm{d}^4x$$

If one replaces t by $-\mathrm{i}\tau$, the derivative part of $\mathscr{L}$ becomes a sum of squares, i.e. it is Euclidean, and the i is replaced by a minus sign, completing the analogy with statistical mechanics presented in Sec. 2-2. This procedure is called a Wick rotation. The physical quantities which are calculated for real times are obtained as analytic continuations of the Euclidean ones. We will start by illustrating the process in the framework of the quantum mechanics of a system with a single degree of freedom, following Abers and Lee.[1]

3-2 FUNCTIONAL INTEGRALS FOR A QUANTUM-MECHANICAL SYSTEM WITH ONE DEGREE OF FREEDOM

3.2.1 Schwinger's Transformation Function

Denote by $Q(t)$ the position operator in the Heisenberg picture, and by $|q, t\rangle$ its eigenstates:

$$Q(t)\,|q,t\rangle = q\,|q,t\rangle \qquad (3\text{-}2)$$

The probability amplitude that a particle which was at q at time t will be at q' at time t', also called the Schwinger transformation function, is

$$F(q't';qt) = \langle q't'\,|qt\rangle \qquad (3\text{-}3)$$

In the Schrödinger picture, Q_s is time-independent, and is related to $Q(t)$ by

$$Q(t) = \exp(\mathrm{i}Ht)Q_s\exp(-\mathrm{i}Ht) \qquad (3\text{-}4)$$

where we have taken the Hamiltonian H to be time-independent. Q_s has time-independent eigenstates

$$Q_s\,|q\rangle = q\,|q\rangle \qquad (3\text{-}5)$$

The relation between the eigenstates is

$$|q\rangle = \exp(-iHt)\,|q, t\rangle \qquad (3\text{-}6)$$

and in terms of the states $|q\rangle$

$$F(q't'; qt) = \langle q'\,|\exp[iH(t-t')]\,|q\rangle \qquad (3\text{-}7)$$

In order to express F as a path integral we divide the time interval into $n+1$ intervals:

$$t = t_0, t_1, \ldots, t_{n+1} = t'; \qquad t_k = t_0 + k\epsilon$$

Then

$$\begin{aligned} F(q't'; qt) = F(q't + (n+1)\epsilon; qt) &= \langle q't + (n+1)\epsilon \,|\, qt\rangle \\ &= \int \mathrm{d}q_1 \ldots \mathrm{d}q_n \,\langle q't' \,|\, q_n t_n\rangle \langle q_n t_n \,|\, q_{n-1} t_{n-1}\rangle \\ &\times \langle q_{n-1} t_{n-1} \,|\, q_{n-2} t_{n-2}\rangle \ldots \langle q_1 t_1 \,|\, qt\rangle \end{aligned} \qquad (3\text{-}8)$$

ϵ can be made arbitrarily small by increasing n, thus

$$\begin{aligned} \langle q_l t_l \,|\, q_{l-1} t_{l-1}\rangle &= \langle q_l \,|\, \exp(-i\epsilon H) \,|\, q_{l-1}\rangle \\ &= \delta(q_l - q_{l-1}) - i\epsilon \langle q_l \,|\, H \,|\, q_{l-1}\rangle + 0(\epsilon^2) \end{aligned} \qquad (3\text{-}9)$$

If

$$H = \frac{p^2}{2} + V(Q) \qquad (3\text{-}10)$$

then the matrix element on the r.h.s. of (3-9) can be written as

$$\begin{aligned} \langle q_l \,|\, H \,|\, q_{l-1}\rangle &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp[ip(q_l - q_{l-1})] \left[\frac{p^2}{2} + V(q_l)\right] \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp[ip(q_l - q_{l-1})]\, H\left(p, \frac{q_l + q_{l-1}}{2}\right) \end{aligned} \qquad (3\text{-}11)$$

The δ-function in Eq. (3-9) can also be Fourier transformed, giving for the matrix element:

$$\langle q_l t_l \,|\, q_{l-1} t_{l-1}\rangle = \int \frac{\mathrm{d}p}{2\pi} \exp\left\{ i\left[p(q_l - q_{l-1}) - \epsilon H\left(p, \frac{q_l + q_{l-1}}{2}\right)\right]\right\} + 0(\epsilon^2) \qquad (3\text{-}12)$$

and Schwinger's function becomes:

$$F(q't';qt) = \lim_{n\to\infty} \int \prod_{i=1}^{n} \mathrm{d}q_i \int_{-\infty}^{\infty} \prod_{i=1}^{n+1} \frac{\mathrm{d}p_i}{2\pi}$$
$$\exp\left\{i \sum_{j=1}^{n+1} \left[p_j(q_j - q_{j-1}) - \frac{t'-t}{n+1} H\left(p_j, \frac{q_j + q_{j-1}}{2}\right)\right]\right\} \qquad (3\text{-}13)$$

which in the limit $n \to \infty$ is an operational definition of the path integral:

$$F = \int \frac{\mathcal{D}p\,\mathcal{D}q}{2\pi} \exp\left\{ \mathrm{i} \int_t^{t'} [p\dot{q} - H(p, q)]\; \mathrm{d}t \right\} \qquad (3\text{-}14)$$

where one sums over all $p(t), q(t)$, such that

$$q(t) = q; \quad q(t') = q'$$

If, furthermore, H is quadratic in the momenta, as we assumed in (3-10), then the integral over p in (3-13) can be performed using the Fresnel integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp\left[\mathrm{i}\epsilon \left(p\dot{q} - \frac{p^2}{2}\right)\right] = \frac{1}{\sqrt{2\pi \mathrm{i}\epsilon}} \exp\left(\frac{1}{2} \mathrm{i}\epsilon \dot{q}^2\right) \qquad (3\text{-}15)$$

leading to

$$F = \lim_{n\to\infty} \int \prod \frac{\mathrm{d}q_i}{\sqrt{2\pi \mathrm{i}\epsilon}} \exp\left[\mathrm{i}\epsilon \sum_{i=1}^{n+1} \frac{1}{2}\dot{q}_i^2 - V\left(\frac{q_i + q_{i-1}}{2}\right)\right]$$
$$\equiv \int \frac{\mathcal{D}q}{\sqrt{2\pi \mathrm{i}\epsilon}} \exp\left[\mathrm{i} \int_t^{t'} \mathcal{L}(q, \dot{q})\, \mathrm{d}t \right] \qquad (3\text{-}16)$$

and $\mathcal{L}$ is the Lagrangian.

This procedure has a heuristic value but is far from rigorous. For the type of system discussed above, rigorous formulations have been developed by Nelson, Simon and others.[2] Furthermore, if the potentials are velocity dependent, matters become even more complicated, and one has to devise effective Lagrangians case by case.[3]

The main advantage is that in (3-16) there are no more operators, and $\mathcal{L}$ is the classical Lagrangian. Hence using this representation, one can study the effects of the symmetries of the classical Lagrangian on quantum-mechanical objects.

3-2-2 Matrix Elements – Green Functions

If we choose t_0 such that $t \leqslant t_0 \leqslant t'$, we can identify t_0 with an end of one of the intervals t_i, denoted by t_{i0}. We have

$$Q(t_0)\,|q_{i0}t_{i0}\rangle = q_{i0}\,|q_{i0}t_{i0}\rangle$$

Thus, in the decomposition (3-8) we can insert $Q(t_0)$ just in front of $|q_{i0}t_{i0}\rangle$, and obtain

$$\langle q't'\,|Q(t_0)\,|qt\rangle = \int \frac{\mathscr{D}p\,\mathscr{D}q}{2\pi}\, q(t_0)\exp\left\{ \mathrm{i}\int_t^{t'} [p\dot{q} - H(p,q)]\,\mathrm{d}t \right\} \tag{3-17}$$

where $q(t_0) = q_{i0}$. When H is quadratic in p, the integration over p proceeds as before.

Next we consider two times $t_1 t_2$, such that

$$t < t_2 < t_1 < t'$$

Identifying $t_1 = t_{i1}$ and $t_2 = t_{i2}$ we can proceed to write

$$\langle q't'\,|\,Q(t_1)Q(t_2)\,|\,qt\rangle = \int \frac{\mathscr{D}q\,\mathscr{D}p}{2\pi}\, q(t_1)q(t_2)\exp\left\{ \mathrm{i}\int_t^{t'} (p\dot{q} - H)\,\mathrm{d}t \right\} \tag{3-18}$$

On the right-hand side of the last equation the order of the two q's is irrelevant. But this expression is equal to a matrix element of the operators with the indicated order. The only thing which is important on the right-hand side is the fact that $t_1 > t_2$. For $t_1 < t_2$ the right-hand side is equal to the matrix element of the two operators multiplied in the opposite order.

If we define a time-ordering operator T by:

$$TQ(t_1)Q(t_2) = \begin{cases} Q(t_1)Q(t_2) & (t_1 > t_2) \\ Q(t_2)Q(t_1) & (t_2 > t_1) \end{cases} \tag{3-19}$$

then for $t < t_i < t'$:

$$\langle q't'\,|T[Q(t_1)\ldots Q(t_n)]\,\,|qt\rangle = \int \frac{\mathscr{D}p\,\mathscr{D}q}{2\pi}\, q(t_1)\ldots q(t_n)\exp\left\{ \mathrm{i}\int_t^{t'} (p\dot{q} - H)\,\mathrm{d}t \right\} \tag{3-20}$$

Thus all expectation values of time-ordered products are expressed in terms of moments of distributions of classical fields.

3-2-3 The Generating Functional

Usually, in quantum field theory, one is interested in the expectation values of time-ordered products of operators in the vacuum state. These suffice to give the S-matrix (Sec. 3-3-4). In the present case the vacuum is the ground state.

Using (3-20) we can write:

$$\langle 0 \mid T[Q(t_1) \ldots Q(t_n)] \mid 0 \rangle = \int \mathrm{d}q \, \mathrm{d}q' \Phi_0(q't') \Phi_0^*(qt) \times \int \frac{\mathscr{D}p \mathscr{D}q}{2\pi} q(t_1) \ldots q(t_n) \exp\left\{ \mathrm{i} \int_t^{t'} (p\dot{q} - H) \, \mathrm{d}t \right\} \tag{3-21}$$

where

$$\Phi_n(q, t) = \langle n \mid qt \rangle \tag{3-22}$$

$\mid n \rangle$ being energy eigenstates, namely,

$$H \mid n \rangle = E_n \mid n \rangle \tag{3-23}$$

The times t and t' in (3-21) are arbitrary, but they have to satisfy the condition $t < t_i < t'$.

The right-hand side of Eq. (3-21) can be generated by adding to H a term

$$H_{\text{ext}} = - \int_t^{t'} J(t) q(t) \, \mathrm{d}t \tag{3-24}$$

or to the operator Hamiltonian, a term $-JQ$. The steps of Sec. 3-2-1 can now be retracted to calculate $\langle q't' \mid qt \rangle^J$ which is the same function as (3-3), but calculated in the presence of the source. Then, the functional

$$Z\{J\} = \int \mathrm{d}q \, \mathrm{d}q' \, \Phi_0^*(q't') \langle q't' \mid qt \rangle^J \Phi_0(q, t) \tag{3-25}$$

will generate the quantities on the right-hand side of (3-21) via

$$(i)^n \langle 0 \mid T[Q(t_1) \ldots Q(t_n)] \mid 0 \rangle = \left. \frac{\delta^n Z\{J\}}{\delta J(t_1) \ldots \delta J(t_n)} \right|_{J=0} \tag{3-26}$$

We now use the fact that the source J vanishes for times outside the interval (t, t') to obtain an expression for Z in terms of the asymptotic behavior of transformation functions at large times. If $T' > t' > t > T$, then:

$$\langle Q'T' \mid QT \rangle^J = \int \mathrm{d}q \, \mathrm{d}q' \langle Q'T' \mid q't' \rangle \langle q't' \mid qt \rangle^J \langle qt \mid QT \rangle \tag{3-27}$$

where the first and last factors in the integrand are independent of J. In fact, we know from (3-7) that:

$$\langle qt | QT \rangle = \langle q | \exp[-iH(t-T)] | Q \rangle = \sum_n \Phi_n^*(q)\Phi_n(Q)\exp[-iE_n(t-T)] \tag{3-28}$$

$$\Phi_n(q) = \langle n | q \rangle \tag{3-29}$$

In order to project the ground state we continue T to the imaginary axis and then

$$\begin{aligned}\lim_{T\to i\infty} \exp(-iE_0T)\langle qt | QT \rangle &= \Phi_0(q)\Phi_0^*(Q)\exp(-iE_0t)\\ &= \Phi_0(q,t)\Phi_0^*(Q)\end{aligned} \tag{3-30}$$

which leads to

$$Z\{J\} = \lim_{\substack{T'\to -i\infty\\ T\to i\infty}} \frac{\langle Q'T' | QT \rangle^J}{\exp[-iE_0(T'-T)]\Phi_0(Q')\Phi_0^*(Q)} \tag{3-31}$$

Furthermore, one is usually interested in time-ordered products of operators divided by $\langle 0 | 0 \rangle$, and hence any factor in Eq. (3-31) which is independent of J can be ignored. $Z\{J\}$ is written as:

$$Z\{J\} = \lim_{\substack{T\to i\infty\\ T'\to -i\infty}} \int \mathcal{D}q \exp\left\{ i \int_T^{T'} [\mathcal{L}(q,\dot{q}) + Jq]\, dt \right\} \tag{3-32}$$

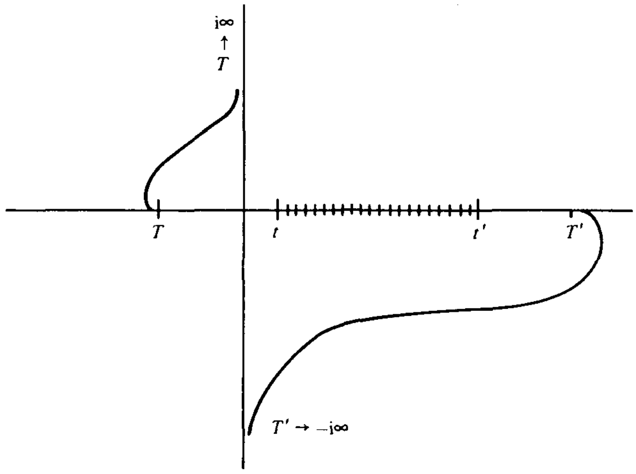

FIGURE 3-1
Time path used for the calculation of Z – the generating functional for real times – in the complex time plane. The source $J \neq 0$ only on an interval on the real axis.

One should keep in mind that it is only outside the interval (t, t') that we continue to the imaginary time axis. Because there $J = 0$, and hence we know the time-dependence explicitly (Eq. (3-28); see Fig. 3-1).

3-2-4 Analytic Continuation in Time – the Euclidean Theory

The apparent inconvenience, implied by the awkward path in the complex time plane, can be turned into an advantage by continuing all times to the imaginary axis – i.e., by performing a Wick rotation. All we have to do is to define time-ordering for imaginary times, and this we do by defining

$$t_i = -i\tau_i \tag{3-33}$$

so that the τ_i's can be ordered along the imaginary axis – to later t_i's correspond "later" τ_i. The only change that will occur in the previous discussion is that

$$\epsilon = \frac{t' - t}{n+1} \rightarrow \epsilon' = \frac{\tau' - \tau}{n+1} = i\epsilon \tag{3-34}$$

and we have to add a boundary condition insuring that the solutions of the Schrödinger equation for large imaginary times will remain finite. Effectively we are converting the Schrödinger equation into a diffusion equation.

The transformation function, Eq. (3-16), takes on the form

$$\begin{aligned} F &= \lim_{n\to\infty} \int \Pi \frac{\mathrm{d}q_i}{(2\pi\epsilon')^{1/2}} \exp\left\{-\epsilon' \sum_{n=1}^{n+1} \left[\frac{1}{2}\dot{q}^2 + V\left(\frac{q_i + q_{i-1}}{2}\right)\right]\right\} \\ &= \int \frac{\mathscr{D}q}{(2\pi\epsilon')^{1/2}} \exp\left\{-\int_\tau^{\tau'} \mathscr{L}_\mathrm{E}(\dot{q}, q)\,\mathrm{d}\tau\right\} \end{aligned} \tag{3-35}$$

where $\dot{q} = \mathrm{d}q/\mathrm{d}\tau$, and $\mathscr{L}_\mathrm{E}$ is the Euclidean version of the Lagrangian, which may remind some of a Hamiltonian, and which is analogous to the type of expressions we had in the statistical mechanical weights. If we remember that the field, in field theory, is the generalization of the coordinate, as a dynamical variable, then since the exponent in (3-35), as well as in (2-68), is expressed in terms of the coordinate and its derivatives, it should be properly called a Lagrangian.

Now the extraction of the ground state expectation value is obtained by simply letting $\tau \rightarrow -\infty$ and $\tau' \rightarrow +\infty$, and the Euclidean version of the generating functional becomes:

$$Z_\mathrm{E}\{J\} = \int \mathscr{D}q \exp\left\{-\int_{-\infty}^{\infty} [\mathscr{L}_\mathrm{E}(\dot{q}, q) - Jq]\,\mathrm{d}\tau\right\} \tag{3-36}$$

To regain our real time expectation values we use

$$\frac{1}{Z\{J\}} \frac{\delta^n Z}{\delta J(t_1)\ldots\delta J(t_n)}\bigg|_{J=0} = \frac{(\mathrm{i})^n}{Z_\mathrm{E}} \frac{\delta^n Z_\mathrm{E}}{\delta J(\tau_1)\ldots\delta J(\tau_n)}\bigg|_{\substack{J=0\\ \tau_i = it_i}} \tag{3-37}$$

Had we generated the Green functions in "momentum" space, which in the present case means frequency, then the analytic continuation to *the causal* functions would have been obtained by:

$$\frac{1}{Z\{J\}} \frac{\delta^n Z}{\delta J(k_1) \ldots \delta J(k_n)}\bigg|_{J=0} = \frac{(i)^n}{Z_E} \frac{\delta^n Z_E}{\delta J(K_1) \ldots \delta J(K_n)}\Bigg|_{\substack{J=0 \\ K_i = ik_i \\ \omega^2 = \omega^2 - i\epsilon}} \tag{3-38}$$

where ω^2 is the coefficient of q^2 in the Lagrangian.

Exercises 3-1 and 3-2 are illustrations of the mechanisms discussed above, in the case of the harmonic oscillator. There the original Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}\omega^2 q^2 \tag{3-39}$$

and its Euclidean version

$$\mathscr{L}_E = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\omega^2 q^2 \tag{3-40}$$

The Euclidean two-point Green functions in frequency and in time are, respectively (see Exercise 3-2)

$$\langle 0 \mid q(K)q(-K) \mid 0 \rangle \propto \frac{1}{K^2 + \omega^2} \tag{3-41}$$

$$\langle 0 \mid q(\tau_1)q(\tau_2) \mid 0 \rangle \propto -\frac{1}{2\omega} \exp(-\omega \mid \tau_1 - \tau_2 \mid) \tag{3-42}$$

The analytic continuation in time as given by (3-37) leads to:

$$\langle 0 \mid q(t_1)q(t_2) \mid 0 \rangle \propto$$

$$-\frac{1}{2\omega}\left\{\theta(t_1 - t_2)\exp[-i\omega(t_1 - t_2)] + \theta(t_2 - t_1)\exp[i\omega(t_1 - t_2)]\right\} \tag{3-43}$$

The continuation in Fourier space gives:

$$\langle 0 \mid q(k_1)q(-k_1) \mid 0 \rangle \propto \frac{1}{-k_1^2 + \omega^2 - i\epsilon} \tag{3-44}$$

which is the Fourier transform of (3-43). The boundary conditions imposed on the Euclidean theory determine those of the real time theory and fix the solution of the homogeneous equation.

3-3 FUNCTIONAL INTEGRALS FOR THE SCALAR BOSON FIELD THEORY

3-3-1 Introduction

In the present section we will consider, rather briefly and schematically, the procedure by which the quantities of main interest in quantum field theory are expressed in terms of functional integrals. These quantities are the Green functions, or time-ordered products of interacting fields, averaged in the vacuum state. The S-matrix, describing all possible scattering processes, is expressed in terms of these Green functions with their variables set on the mass shell and the part corresponding to stable incoming particles extracted.

On the one hand it is rather interesting to see the extent of the analogy between the formulation of a statistical mechanical problem given in Chapter 2, and the formulation of quantum field theory. On the other, in all situations of interest in the latter case, namely when infinities arise, the manipulations are of a purely formal character. The theory is then supplemented by additional prescriptions which render it finite in a systematic way, but most of the structure used in the derivation is lost.† The complications involved in resurrecting such things as field equations, interpolating fields, etc. have been discussed at length by Zimmermann (Brandeis Lecture Notes). Perhaps the most rational attitude is that of t'Hooft and Veltman (Diagrammar) in which the theory is postulated in terms of its regularized perturbation expansion. So there seems little point in giving a lengthy exposition.

Here we will follow the presentation of Fried[4] which in turn follows Symanzik. The logic is as follows:

(1) For an interacting bose field $\Phi(x)$ we can define the operator

$$T\{J\} = T \exp\left[\mathrm{i} \int_{-\infty}^{\infty} J(x)\Phi(x)\, \mathrm{d}^4x \right] \tag{3-45}$$

where $J(x)$ is a c-number source, and T is the time-ordering operator. The vacuum expectation values of time ordered products of Φ-fields are generated by

$$Z\{J\} = \langle 0 \mid T\{J\} \mid 0 \rangle \tag{3-46}$$

(2) If the dynamics of the field is described by a Lagrangian

$$\mathscr{L} = \mathscr{L}^{(0)}(\Phi) + \mathscr{L}_{\mathrm{Int}}(\Phi) \tag{3.47}$$

then

$$Z\{J\} = \mathscr{N}^{-1} \exp\left\{ \mathrm{i} \int \mathscr{L}_{\mathrm{Int}} \left[\mathrm{i} \frac{\delta}{\delta J(x)} \right] \mathrm{d}^4x \right\} Z^0\{J\} \tag{3-48}$$

with Z^0 the generating functional for the non-interacting theory. Z can be written also as:

$$Z\{J\} = \mathscr{N}^{-1} \int \mathscr{D}\phi \exp\left\{ \mathrm{i} \int [\mathscr{L}(\phi) + J\phi]\, \mathrm{d}^4x \right\} \tag{3-49}$$

with ϕ a c-number field.

† The reasons which make this strange logical procedure of Symanzik work are discussed by Lowenstein.[5]

(3) Finally one shows that, apart from a phase factor, the S-matrix is given by

$$S = :\exp\left[\int \Phi_{\mathrm{IN}}(x) K_x \frac{\delta}{\delta J(x)} \mathrm{d}^4x\right] : Z\{J\}\bigg|_{J=0} \tag{3-50}$$

where Φ_{IN} is the incoming field at $t \to -\mathrm{i}\infty$, K is the Klein–Gordon operator (it would have been the Dirac operator for fermions), and : : implies normal ordering – i.e., destruction operators to the right of the creation operators.

If the incoming fields are chosen to have the observed mass of the particles, then K_x plays the role of selecting out of any matrix element of S that part which corresponds to the poles associated with the incoming or outgoing real particles.

3-3-2 The Generating Functional for Green Functions

As implied in the previous section, we will consider the case of the scalar boson field. We denote the interacting field in the Heisenberg picture by $\Phi(x)$, where x is a space–time four-vector. If a term $\int J\Phi$ is added to the Lagrangian the additional time-dependence is generated by the operator

$$T_{t_1 t_2}\{J\} = T\exp\left[\mathrm{i}\int_{t_1}^{t_2} J(x)\Phi(x)\,\mathrm{d}^4x\right] \tag{3-51}$$

with T on the right-hand side indicating time-ordering. $J(x)$ is a c-number source, which has to be replaced by an anticommuting c-number in the case of fermions.

Taking functional derivatives with respect to $J(x)$ gives (see, e.g., Exercise 3-4)

$$\delta T_{t_1 t_2}\{J\}/\delta J(x) = 0$$

if x_0 is outside the interval (t_1, t_2) and

$$\delta T_{t_1 t_2}\{J\}/\delta J(x) = T[\mathrm{i}\Phi(x) T_{t_1 t_2}\{J\}] \tag{3-52}$$

Denoting by T the limit

$$T\{J\} = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} T_{t_1 t_2}\{J\}$$

we have

$$\delta^n T\{J\}/\delta J(x_1) \ldots \delta J(x_n) = T[(\mathrm{i})^n \Phi(x_1) \ldots \Phi(x_n) T\{J\}] \tag{3-53}$$

where it has been assumed that $T\{J\}$ will not be confused with the time-ordering operator.

It follows immediately that if $|0\rangle$ is the vacuum state, or, at this stage, any other state, we will have

$$(\mathrm{i})^n \langle 0 | T[\Phi(x_1) \ldots \Phi(x_n)] | 0 \rangle = \delta^n \langle 0 | T\{J\} | 0 \rangle / \delta J(x_1) \ldots \delta J(x_n) \Big|_{J=0} \tag{3-54}$$

which establishes that

$$Z\{J\} = \langle 0 | T\{J\} | 0 \rangle$$

is the generating functional for time-ordered vacuum expectation values.

3-3-3 The Generating Functional as a Functional Integral

If the Lagrangian density of our scalar boson field theory is split according to Eq. (3-47) with

$$\mathscr{L}_0(\Phi) = -\tfrac{1}{2}(\mu^2 \Phi^2 + \partial_\mu \Phi \partial^\mu \Phi) \tag{3-55}$$

then the equations of motion of the field, obtained by varying the action $\int \mathscr{L}$, take the form

$$K_x \Phi(x) = \frac{\partial \mathscr{L}_{\mathrm{Int}}(\Phi)}{\partial \Phi} \tag{3-56}$$

where

$$K_x = \mu^2 - (\nabla^2 - \partial_0^2) \equiv \mu^2 - \partial^2 \tag{3-57}$$

This equation is purely formal, since products of operators at the same point are problematic even for free fields. Nevertheless, we will not be discouraged by such trifles, and we proceed to calculate

$$K_x \langle 0 | T[\Phi(x) T\{J\}] | 0 \rangle = K_x \cdot \delta Z / \mathrm{i} \delta J(x)$$

Two terms appear in the derivative. One comes from the space–time variation of $\Phi(x)$, and the other from the time-ordering. The first is given by:

$$\begin{aligned} \langle 0 | T[K_x \Phi(x) T\{J\}] | 0 \rangle &= \langle 0 | [T \frac{\partial \mathscr{L}_{\mathrm{Int}}}{\partial \Phi} T\{J\}] | 0 \rangle \\ &= \mathscr{L}'_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)] Z\{J\} \end{aligned} \tag{3-58}$$

where we have used Eq. (3-56) and introduced the notation

$$\mathscr{L}'(A) = \frac{\partial \mathscr{L}}{\partial A}$$

The second term comes only from the time derivative present in K_x. It is left as an exercise (3-7) to show that this term can be written as:

$$-\mathrm{i}\int \mathrm{d}^3x' J(x', x_0) \langle 0 \mid T[\Phi(x', x_0), \dot{\Phi}(x, x_0)]\, T\{J\} \mid 0 \rangle$$

The equal time commutation relation between Φ and $\dot{\Phi}$ is like that between a coordinate and a momentum, namely:

$$[\Phi(x', x_0), \dot{\Phi}(x, x_0)] = \mathrm{i}\delta^3(x' - x) \qquad (3\text{-}59)$$

Thus we find

$$K_x \delta Z/\mathrm{i}\delta J(x) = J(x)Z + \mathscr{L}'_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\, Z \qquad (3\text{-}60)$$

If the theory were free, i.e. $\mathscr{L}_{\mathrm{Int}} = 0$, then with the same source we would have had

$$K_x \delta Z^0/\mathrm{i}\delta J(x) = J(x) Z^0 \qquad (3\text{-}61)$$

where Z^0 is the generating functional for the free theory. Using the identity

$$\left[\exp\left\{\mathrm{i}\int F[-\mathrm{i}\delta/\delta J(y)]\,\mathrm{d}^4y\right\}, J(x)\right] = F'[-\mathrm{i}\delta/\delta J(x)] \exp\left\{\mathrm{i}\int F[-\mathrm{i}\delta/\delta J]\right\} \qquad (3\text{-}62)$$

the proof of which is left as Exercise 3-8, one finds that the solution of Eq. (3-60) can be written in terms of $Z^0\{J\}$ as:

$$Z\{J\} = \exp\left\{\mathrm{i}\int \mathscr{L}_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\,\mathrm{d}^4x\right\} Z^0\{J\} \qquad (3\text{-}63)$$

Applying K_x to $\delta Z\{J\}/\mathrm{i}\delta J(x)$, and using (3-61) and (3-62), gives:

$$\begin{aligned}
&\exp\left\{\mathrm{i}\int \mathscr{L}_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\,\mathrm{d}^4x\right\} K_x \delta/\mathrm{i}\delta J(x) Z^0\{J\} \\
&\quad = \exp\left\{\mathrm{i}\int \mathscr{L}_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\,\mathrm{d}^4x\right\} J(x) Z^0\{J\} \\
&\quad = J(x)Z\{J\} + \left[\exp\left\{\mathrm{i}\int \mathscr{L}_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\,\mathrm{d}^4x\right\}, J(x)\right] Z^0\{J\} \\
&\quad = J(x)Z\{J\} + \mathscr{L}'_{\mathrm{Int}}[-\mathrm{i}\delta/\delta J(x)]\, Z\{J\}
\end{aligned}$$

Equation (3-60) determines Z up to a multiplicative constant which we choose by imposing the condition

$$Z\{J = 0\} = 1 \qquad (3\text{-}64)$$

To complete the calculation of Z we still have to solve (3-61) for $Z^0\{J\}$.

It is rather easy to guess the form of the solution of (3-61); it must be

$$Z^0\{J\} = \exp\left\{\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,d^4x\,d^4y\right\} \tag{3-65}$$

in which Δ is a solution of the inhomogeneous Klein–Gordon equation

$$K_x\Delta(x-y) = \delta(x-y) \tag{3-66}$$

But this equation admits many solutions depending on the choice of the solution of the homogeneous part, or, correspondingly, on the choice of boundary conditions. In our case the choice has been made by the requirement (3-54), i.e., that we generate time-ordered products.

We know on the one hand, from (3-65), that

$$\delta^2 Z^0/\delta J(x)\delta J(y)\,|_{J=0} = i\Delta(x-y) \tag{3-67}$$

On the other hand, using (3-54)

$$\delta^2 Z^0/\delta J(x)\delta J(y)|_{J=0} = -\langle 0\,|\,T[\Phi^0(x)\Phi^0(y)]\,|\,0\rangle \tag{3-68}$$

where Φ^0 denotes the free field. The vacuum expectation value of free fields is easy to calculate. All one has to do is to have the creation operators to the right, and if Φ^0 is written as

$$\Phi^0(x) = \Phi^+(x) + \Phi^-(x) \tag{3-69}$$

with the destruction operators included in Φ^+ and the creation operators in Φ^-. Then

$$\begin{aligned}\Delta(x-y) &= i\langle 0\,|\,\Phi^+(x)\Phi^-(y)\,|\,0\rangle\,\theta\,(x_0-y_0)\\ &\quad + i\langle 0\,|\,\Phi^+(y)\Phi^-(x)\,|\,0\rangle\,\theta(y_0-x_0)\\ &\equiv \Delta_c(x-y) = \int\frac{d^4k}{(2\pi)^4}\,\frac{\exp[ik(x-y)]}{k^2+\mu^2-i\epsilon}\end{aligned} \tag{3-70}$$

where $k^2 = \mathbf{k}^2 - k_0^2$. This is conventionally called the causal function. For more details the reader is referred to Bjorken and Drell[6] or Fried,[4] etc., and Exercises 3-9 to 3-13.

In Eq. (3-60) there are no operators and we can try to solve it without invoking any. Consider, for example, the functional

$$\bar{Z} = \int \mathscr{D}\phi\,\exp\left\{i\int[\mathscr{L}(\phi) + J\phi]\,d^4x\right\} \tag{3-71}$$

in which ϕ is, of course, a c-number function.

Then, just as for a usual integral, we have

$$\int \mathscr{D}\phi \frac{\delta}{\delta\phi(x)} \exp\left\{ i \int [\mathscr{L}(\phi) + J\phi]\, d^4x \right\} = 0 \qquad (3\text{-}72)$$

Performing the differentiation under the integral sign one finds

$$\int \mathscr{D}\phi\, i \frac{\partial \mathscr{L}(\phi)}{\partial\phi(x)} \exp\left\{ i \int [\mathscr{L}(\phi) + J\phi]\, d^4x \right\} + iJ(x)\bar{Z} = 0 \qquad (3\text{-}73)$$

But $\partial\mathscr{L}/\partial\phi$ is a polynomial in ϕ which can be generated by functional derivatives with respect to $J(x)$. In particular, the term $\partial\mathscr{L}_{\text{Int}}/\partial\phi(x)$ can be written as:

$$\mathscr{L}'_{\text{Int}}[-i\delta/\delta J(x)]$$

$\mathscr{L}^0$ is quadratic in ϕ, and thus $\partial\mathscr{L}^0/\partial\phi$ is linear. But because $\mathscr{L}^0$ includes derivatives of the field we have to exercise some caution in performing the functional derivative. The explicit calculation reads:

$$-\frac{i\delta}{\delta\phi(x)} \int d^4y \frac{1}{2} [m^2\phi^2(y) + \partial_\mu\phi\partial^\mu\phi] = -i \int d^4y [m^2\phi(y) + \partial_\mu\phi\partial^\mu]\, \delta(x-y)$$

$$= -i[m^2\phi(x) - \partial_\mu\partial^\mu\phi(x)] = -iK_x\phi(x)$$

Inserting it all back in (3-73) one finds that $\bar{Z}$ satisfies Eq. (3-60). If in addition we normalize $\bar{Z}$ in the same way in which we normalized Z, and write $m^2 - i\epsilon$ instead of m^2, then $\bar{Z}^0 = Z^0$ and $\bar{Z} = Z$.

What has been achieved, apart from showing that the generating functional in quantum field theory can be written in terms of a functional integral analogous to the ones appearing in statistical mechanics, is the derivation of Eqs. (3-63) and (3-65), which we recapitulate:

$$Z\{J\} = \exp\left\{ i \int \mathscr{L}_{\text{Int}}[-i\delta/\delta J(x)]\, dx \right\} \exp\left\{ \frac{i}{2} \int J(x)\Delta_c(x-y)J(y)\, dx\, dy \right\} \qquad (3\text{-}74)$$

This form applies to the Euclidean theory as well, and is very useful in generating the perturbation expansion, which is an expansion in the parameters of $\mathscr{L}_{\text{Int}}$. As we show in the next chapter.

3-3-4 The S-Matrix Expressed in Terms of the Generating Functional

For completeness we include a brief description, along traditional lines, of the procedure by which the S-matrix is expressed in terms of the generating functional Z.

As the Schrödinger states of the interacting system are extrapolated to times at $-\infty$ and at $+\infty$, they tend to free particle states called, respectively, the

IN and OUT states, which are denoted, respectively, $|\rangle_{\mathrm{IN}}$ and $|\rangle_{\mathrm{OUT}}$. These two complete sets of free states are constructed as a Fock space by the free field operators $\Phi_{\mathrm{IN}}(x)$ and $\Phi_{\mathrm{OUT}}(x)$. The S-matrix is defined as the unitary operator which connects the two complete sets. Namely,

$$\Phi_{\mathrm{OUT}}(x) = S^{+}\Phi_{\mathrm{IN}}(x)S \qquad (3\text{-}75)$$

and correspondingly

$$|\rangle_{\mathrm{OUT}} = S^{+}\,|\rangle_{\mathrm{IN}} \qquad (3\text{-}76)$$

In addition, asymptotic conditions are imposed. They read:

$$\lim_{x_0 \to \mp\infty} [\langle a\,|\Phi(x)|\,b\rangle - \langle a\,|\Phi_{\substack{\mathrm{IN}\\ \mathrm{OUT}}}|\,b\rangle] = 0 \qquad (3\text{-}77)$$

for any pair of normalizable states.

Two comments are in place here. Firstly, the imposition of strong asymptotic conditions, i.e., on the operators themselves, leads to the result that there is no scattering. Secondly, if one insists that the equal time commutation relation of the Φ's – the interacting field – be normalized to a δ function, as in (3-59), then another constant should be multiplying one of the matrix elements in (3-77). We leave this factor out. Furthermore, we will disregard the distinction between strong and weak asymptotic conditions in intermediate steps, and instead we will avoid using it where the weak condition does not suffice.

Since the field satisfies the field equation (3-56) we can write

$$\begin{aligned}\Phi(x) &= \Phi_{\mathrm{IN}}(x) + \int d^4y\,\Delta_{\mathrm{R}}(x-y)\frac{\partial \mathscr{L}_{\mathrm{Int}}(\Phi)}{\partial\Phi}\\ &= \Phi_{\mathrm{IN}}(x) + \int d^4y\,\Delta_{\mathrm{R}}(x-y)K_y\Phi(y) \qquad (3\text{-}78)\end{aligned}$$

where Φ_{IN} is a solution of the homogeneous equation and Δ_{R} is the Green function of the Klein–Gordon equation, which vanishes when its time argument is negative. Similarly, we can write

$$\Phi(x) = \Phi_{\mathrm{OUT}}(x) + \int d^4y\,\Delta_{\mathrm{A}}(x-y)K_y\Phi(y) \qquad (3\text{-}79)$$

Multiplying Eqs. (3-78) and (3-79) by $T\{J\}$ and by the operator of time-ordering, keeping in mind that Φ_{IN} is $\Phi(x)$ at $t=-\infty$ and Φ_{OUT} is $\Phi(x)$ at $t=+\infty$, we have, using (3-52),

$$\delta T/\delta J(x) = iT\{J\}\Phi_{\mathrm{IN}}(x) + \int d^4y\,\Delta_{\mathrm{R}}(x-y)K_y\,\delta T\{J\}/\delta J(y) \qquad (3\text{-}80a)$$

$$\delta T/\delta J(x) = i\Phi_{\mathrm{OUT}}(x)T\{J\} + \int d^4y\,\Delta_{\mathrm{A}}(x-y)K_y\,\delta T\{J\}/\delta J(y) \qquad (3\text{-}80b)$$

from which it follows that

$$(\Phi_{\mathrm{OUT}}T - T\Phi_{\mathrm{IN}}) = \mathrm{i} \int \mathrm{d}^4y \Delta(x-y) K_y \delta T\{J\}/\delta J(y) \tag{3-81}$$

where

$$\Delta(x) = \Delta_{\mathrm{A}}(x) - \Delta_{\mathrm{R}}(x) \tag{3-82}$$

The relations of the various Δ's are left as exercises (see, e.g., Exercise 3-13).

Next, multiplying both sides of (3-81) by S and using (3-75) one finds:

$$[\Phi_{\mathrm{IN}}(x), ST] = \mathrm{i} \int \mathrm{d}^4y \Delta(x-y) K_y \delta(ST)/\delta J(y) \tag{3-83}$$

since S is independent of J.

Using the identity (see Exercise 3-15)

$$[A, \mathrm{e}^B] = [A, B]\mathrm{e}^B \tag{3-84}$$

for operators A and B whose commutator is a c-number, we have

$$\begin{aligned}\Big[\Phi_{\mathrm{IN}}(x), \exp\Big\{\int \Phi^-_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}\exp\Big\{\int \Phi^+_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}\Big] \\ = \mathrm{i}\int \Delta(x-y) f(y)\,\mathrm{d}^4y \exp\Big\{\int \Phi^-_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}\exp\Big\{\int \Phi^+_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}\end{aligned} \tag{3-85}$$

in which $\Phi^-_{\mathrm{IN}}(x)$ is the part of Φ_{IN} which creates particles, while Φ^+ destroys them. Φ_{IN} is decomposed via

$$\Phi_{\mathrm{IN}}(x) = \Phi^+_{\mathrm{IN}}(x) + \Phi^-_{\mathrm{IN}}(x) \tag{3-86}$$

(see also Exercises 3-9 and 3-14), and using the definition of the normal product:

$$\exp\Big\{\int \Phi^-_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}\exp\Big\{\int \Phi^+_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\} \equiv\, :\exp\Big\{\int \Phi_{\mathrm{IN}}(x) f(x)\,\mathrm{d}x\Big\}:$$

the solution of Eq. (3-83) is now straightforward. It is given by

$$ST = C : \exp\Big\{\int \Phi_{\mathrm{IN}}(x) K_x \delta/\delta J(x)\,\mathrm{d}x\Big\} : Z\{J\} \tag{3-87}$$

as can be verified by calculating $[\Phi_{\mathrm{IN}}, ST]$ and using (3-85). C is an arbitrary constant.

Taking the vacuum expectation value of (3-87), and noting that

$$_{\mathrm{IN}}\langle 0 | : \mathrm{e}^R : | 0 \rangle_{\mathrm{IN}} = 1 \tag{3-88}$$

for any operator R, we find that

$$_{\rm IN}\langle 0 \,|ST\{J\}|\, 0 \rangle_{\rm IN} = CZ\{J\} \tag{3-89}$$

And since $T\{J\} \to 1$ as $J \to 0$, we have that

$$S = C : \exp\left\{ \int \Phi_{\rm IN}(x) K_x \delta/\delta J(x) \right\} : Z\{J\} \bigg|_{J=0} \tag{3.90}$$

If, furthermore, there are no external fields, $S^{+}|\,0\,\rangle_{\rm IN} = e^{i\Phi}|\,0\,\rangle_{\rm IN}$ and (3-89) implies that C is simply this phase factor.

EXERCISES

3-1 Calculate the Euclidean generating functional of the harmonic oscillator directly, in terms of a time-dependent as well as a frequency-dependent source.

3-2 Calculate the Euclidean two-point function.

(*a*) What equation does it satisfy?
(*b*) What boundary conditions have been assumed?

3-3 Show that the analytic continuation in time and frequency space are Fourier transforms of each other. Compare with the solution of Feynman and Hibbs.[7]

3-4 If $F\{J\}$ is a functional of J, then the functional derivative of F, $\delta F/\delta J(x)$ is defined by:

$$F\{J + \delta J\} - F\{J\} = \int dx \left(\frac{\delta F}{\delta J(x)} \right) \delta J(x) + 0(\delta J^2)$$

Using the fact that for bosons:

$$T(AB) = T(BA)$$

prove Eqs. (3-52) and (3-53).

3-5 Just as in usual integration we have

$$\int_{-\infty}^{\infty} dx \frac{d}{dx} f(x) = 0$$

if f is integrable, we have for functional integrals

$$\int \mathscr{D}\phi \frac{\delta}{\delta \phi(x)} F\{\phi\} = 0$$

By considering the expression

$$\int \mathscr{D}\phi\, \phi(x) \frac{\delta}{\delta\phi(y)} \exp\left\{ -\int \mathscr{L}[\phi]\, \mathrm{d}x \right\}$$

derive an equation for

$$\langle \phi(x)\, \phi(y) \rangle$$

assuming that

$$\mathscr{L}(\phi) = \tfrac{1}{2}(\partial\phi)^2 + P(\phi^2)$$

where P is a polynomial.

3-6 What would this equation read for the harmonic oscillator in Euclidean and real time?

What is the correspondence between the solutions?

3-7 Show that

$$\partial_0^2 \langle 0 \mid T[\Phi(x)T\{J\}] \mid 0 \rangle - \langle 0 \mid T[\partial_0^2\Phi(x)]T\{J\}] \mid 0 \rangle$$
$$= -\mathrm{i} \int \mathrm{d}^3x' J(x', x_0) \langle 0 \mid T[\Phi(x', x_0), \dot{\Phi}(x, x_0)]T\{J\} \mid 0 \rangle$$

3-8 Using the fact that

$$[\delta/\delta J(y), J(x)] = \delta(x-y)$$

prove identity (3-62).

3-9 The destructive and creative parts of the free field can be written as:

$$\Phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^4 k a(k) \exp(ikx)\theta(k_0)\delta(k_0^2 - k^2 - m^2)$$
$$\Phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^4 k a^+(k) \exp(-ikx)\theta(k_0)\delta(k_0^2 - k^2 - m^2)$$

with

$$[a(k), a(k')] = 0, \quad [a(k), a^+(k')] = \delta(k-k')$$

show that

$$[\Phi^+(x), \Phi^-(y)] = \mathrm{i}\Delta^+(x-y)$$
$$= -\frac{\mathrm{i}}{(2\pi)^3} \int \mathrm{d}^4 k\, \delta(k^2+m^2)\theta(k_0) \exp(\mathrm{i}kx)$$

and that Δ^+ satisfies the homogeneous Klein–Gordon equation.

3-10 Show that

$$\Delta_c(x-y) = -\theta(x_0 - y_0)\Delta^+(x-y) - \theta(y_0 - x_0)\Delta^+(y-x)$$

is the Green function for the Klein–Gordon equation.

3-11 By making a Fourier decomposition of the θ-functions show that

$$\Delta_c = \int \frac{d^4k}{(2\pi)^4} \frac{\exp[ik(x-y)]}{k^2 + m^2 - i\epsilon}$$

3-12 Show that in coordinate space Δ_c has the form

$$\Delta_c(x) = \frac{1}{4\pi}\delta(x^2) + \frac{im}{4\pi^2}\frac{\theta(x^2)}{\sqrt{x^2}}K_1(m\sqrt{x^2}) - \frac{m}{8\pi}\frac{\theta(-x^2)}{\sqrt{-x^2}}H_1^{(2)}(m\sqrt{-x^2})$$

where K and H are the conventional Bessel functions. Use this form to rationalize the name "causal" attached to this function.

3-13 Show that

$$\Delta_R(x) = -\theta(x_0)\Delta(x) \qquad \text{(retarded)}$$
$$\Delta_A(x) = \theta(x_0)\Delta(x) \qquad \text{(advanced)}$$

where $\Delta(x) = \Delta^+(x) - \Delta^+(-x)$, are Green functions of the Klein–Gordon equation.

3-14 Show, using results of Exercise 3-9, that

$$\langle 0 \mid [\Phi_{IN}(x), \Phi_{IN}(y)] \mid 0 \rangle = i\Delta(x-y)$$

3-15 Using the Baker–Housdorf formula

$$\exp(A+B) = \exp A \exp B \exp(-\tfrac{1}{2}[A,B])$$

where $[A,B]$ is a c-number, show that

$$[A, e^B] = [A,B]e^B$$

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