Chapter 1

Linear Quantum Mechanics: Its Successes and Problems

The quantum mechanics established by Bohr, de Broglie, Schrödinger, Heisenberg and Bohn in 1920s is often referred to as the linear quantum mechanics (LQM). In this chapter, the hypotheses of linear quantum mechanics, the successes of and problems encountered by the linear quantum mechanics are reviewed. The directions for further development of the quantum theory are also discussed.

1.1 The Fundamental Hypotheses of the Linear Quantum Mechanics

At the end of the 19th century, classical mechanics encountered major difficulties in describing motions of microscopic particles (MIPs) with extremely light masses ($\sim 10^{-23}$ – 10^{-26} g) and extremely high velocities, and the physical phenomena related to such motions. This forced scientists to rethink the applicability of classical mechanics and lead to fundamental changes in their traditional understanding of the nature of motions of microscopic objects. The wave-corpuscle duality of microscopic particles was boldly proposed by Bohr, de Broglie and others. On the basis of this revolutionary idea and some fundamental hypotheses, Schrödinger, Heisenberg, etc. established the linear quantum mechanics which provided a unique way of describing quantum systems. In this theory, the states of microscopic particles are described by a wave function which is interpreted based on statistics, and physical quantities are represented by operators and are given in terms of the possible expectation values (or eigenvalues) of these operators in the states (or eigenstates). The time evolution of quantum states are governed by the Schrödinger equation. The hypotheses of the linear quantum mechanics are summarized in the following.

(1) A state of a microscopic particle is represented by a vector in the Hilbert space, $|\psi\rangle$, or a wave function $\psi(\vec{r},t)$ in coordinate space. The wave function uniquely describes the motion of the microscopic particle and reflects the wave *nature of microscopic particles. Furthermore, if* β *is a constant, then both* $|\psi\rangle$ and $\beta|\psi\rangle$ describe the same state. Thus, the normalized wave function, which satisfies the condition $\langle\psi|\psi\rangle = 1$, is often used to describe the state of the particle.

(2) A physical quantity, such as the coordinate X, the momentum P and the energy E of a particle, is represented by a linear operator in the Hilbert space, and the eigenvectors of the operator form a basis of the Hilbert space. An observable mechanical quantity is represented by a Hermitian operator whose eigenvalues are real. Therefore, the values a physical quantity can have are the eigenvalues of the corresponding linear operator. The eigenvectors corresponding to different eigenvalues are orthogonal to each other. All eigenstates of a Hermitian operator span an orthogonal and complete set, $\{\psi_L\}$. Any vector of state, $\psi(\vec{r},t)$, can be expanded in terms of the eigenvectors:

$$\psi(\vec{r},t) = \sum_L C_L \psi_L(\vec{r},t), \qquad \text{or} \qquad |\psi(\vec{r},t)\rangle = \sum_L \langle\psi_L|\psi\rangle|\psi_L\rangle \tag{1.1}$$

where $C_L = \langle\psi_L|\psi\rangle$ is the wave function in representation L. If the spectrum of L is continuous, then the summation in (1.1) should be replaced by an integral: $\int dL \cdots$. Equation (1.1) can be regarded as a projection of the wave function $\psi(\vec{r},t)$ of a microscopic particle system on to those of its subsystems and it is the foundation of transformation between different representations in the linear quantum mechanics. In the quantum state described by $\psi(\vec{r},t)$, the probability of getting the value L' in a measurement of L is $|C_{L'}|^2 = |\langle\psi_{L'}|\psi\rangle|^2$ in the case of discrete spectrum, or $|\langle\psi_{L'}|\psi\rangle|^2 dL$ if the spectrum of the system is continuous. In a single measurement of any mechanical quantity, only one of the eigenvalues of the corresponding linear operator can be obtained, and the system is then said to be in the eigenstate belonging to this eigenvalue. This is a fundamental assumption of linear quantum mechanics concerning measurements of physical quantities.

(3) The average $\langle\hat{A}\rangle$ of a physical quantity A in an arbitrary state $|\psi\rangle$ is given by

$$\langle\hat{A}\rangle = \frac{\langle\psi|\hat{A}|\psi\rangle}{\langle\psi|\psi\rangle}, \tag{1.2}$$

or

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle,$$

if ψ is normalized. Possible values of A can be obtained through the determination of the above average. In order to obtain these possible values, we must find a wave function in which A has a precise value. In other words, we must find a state such that $\overline{(\Delta A)^2} = 0$, where $\overline{(\Delta A)^2} = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$. This leads to the following eigenvalue problem for the operator $\hat{A}$,

$$\hat{A}\psi_L = A\psi_L. \tag{1.3}$$

From the above equation we can determine the spectrum of eigenvalues of the operator $\hat{A}$ and the corresponding eigenfunctions ψ_L. The eigenvalues of $\hat{A}$ are possible values observed from a measurement of the physical quantity. All possible values of A in any other state are nothing but its eigenvalues in its own eigenstates. This hypothesis reflects the statistical nature in the description of motion of microscopic particles in the linear quantum mechanics.

(4) The Hilbert space in which the linear quantum mechanics is defined is a linear space. The operator of a mechanical quantity is a linear operator in this space. The eigenvectors of a linear operator satisfy the linear superposition principle. That is, if two states, $|\psi_1\rangle$ and $|\psi_2\rangle$ are both eigenfunctions of a given linear operator, then their linear combination

$$|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle, \tag{1.4}$$

where C_1 and C_2 are constants, also describes a state of the same particle. The linear superposition principle of quantum states is determined by the linear characteristics of the operators and this is why the quantum theory is referred to as linear quantum mechanics. It is noteworthy to point out that such a superposition is different from that of classical waves, it does not result in changes in probability and intensity.

(5) *The correspondence principle:* If two classical mechanical quantities, A and B, satisfy the Poisson brackets,

$$\{A, B\} = \sum_n \left(\frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q_n} \right)$$

where q_n and p_n are generalized coordinate and momentum in the classical system, respectively, then the corresponding operators $\hat{A}$ and $\hat{B}$ in quantum mechanics satisfy the following commutation relation:

$$[\hat{A}, \hat{B}] = (\hat{A}\hat{B} - \hat{B}\hat{A}) = -ih\{\hat{A}, \hat{B}\} \tag{1.5}$$

where $i = \sqrt{-1}$ and h is the Planck's constant. If A and B are substituted by q_n and p_n respectively, we have:

$$[\hat{p}_n, \hat{q}_m] = -ih\delta_{nm}, \quad [\hat{p}_n, \hat{p}_m] = 0, \quad \cdots .$$

This reflects the fact that values allowed for a physical quantity in a microscopic system are quantized, and thus the name "quantum mechanics". Based on this fundamental principle, the Heisenberg uncertainty relation can be obtained as follows,

$$\overline{(\Delta A)^2}\ \overline{(\Delta B)^2} \geq \frac{|C|^2}{4} \tag{1.6}$$

where $iC = [\hat{A}, \hat{B}]$ and $\Delta A = \langle \hat{A} - \langle \hat{A} \rangle \rangle$. For the coordinate and momentum operators, the Heisenberg uncertainty relation takes the usual form

$$|\Delta x||\Delta p \geq \frac{\hbar}{2}.$$

(6) The time dependence of a quantum state $|\psi\rangle$ of a microscopic particle is determined by the following Schrödinger equation:

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{1.7}$$

This is a fundamental dynamic equation for microscopic particle in space-time. $\hat{H}$ is the Hamiltonian operator of the system and is given by,

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V},$$

where $\hat{T}$ is the kinetic energy operator and $\hat{V}$ the potential energy operator. Thus, the state of a quantum system at any time is determined by the Hamiltonian of the system. As a fundamental equation of linear quantum mechanics, equation (1.7) is a linear equation of the wave function ψ which is another reason why the theory is referred as a linear quantum mechanics.

If the quantum state of a system at time t_0 is $|\psi(t_0)\rangle$, then the wave function and mechanical quantities at time t are associated with those at time t_0 by a unitary operator $\hat{U}(t, t_0)$, *i.e.*

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle, \tag{1.8}$$

where $\hat{U}(t_0, t_0) = 1$ and $\hat{U}^+\hat{U} = \hat{U}\hat{U}^+ = I$. If we let $\hat{U}(t, 0) = \hat{U}(t)$, then the equation of motion becomes

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}\hat{U}(t) \tag{1.9}$$

when $\hat{H}$ does not depend explicitly on time t and $\hat{U}(t) = e^{-i(\hat{H}/\hbar)t}$. If $\hat{H}$ is an explicit function of time t, we then have

$$\hat{U}(t) = 1 + \frac{1}{ih}\int_0^t dt_1\hat{H}(t_1) + \frac{1}{(ih)^2}\int_0^t dt_1\hat{H}(t_1)\int_0^{t_1} dt_2\hat{H}(t_2) + \cdots . \tag{1.10}$$

Obviously, there is an important assumption here: the Hamiltonian operator of the system is independent of its state, or its wave function. This is a fundamental assumption in the linear quantum mechanics.

(7) *Identical particles:* No new physical state should occur when a pair of identical particles is exchanged in a system. In other words, the wave function satisfies $\hat{P}_{kj}|\psi\rangle = \lambda|\psi\rangle$, where $\hat{P}_{kj}$ is an exchange operator and $\lambda = \pm 1$. Therefore, the wave function of a system consisting of identical particles must be either symmetric, ψ_s, $(\lambda = +1)$, or antisymmetric, ψ_a, $(\lambda = -1)$, and this property remains invariant with time and is determined only by the nature of the particle. The wave function of a boson particle is symmetric and that of a fermion is antisymmetric.

(8) *Measurements of physical quantities:* There was no assumption made about measurements of physical quantities at the beginning of the linear quantum mechanics. It was introduced later to make the linear quantum mechanics complete. However, this is a nontrivial and contraversal topic which has been a focus of scientific debate. This problem will not be discussed here. Interested reader can refer to texts and references given at the end of this chapter.

1.2 Successes and Problems of the Linear Quantum Mechanics

On the basis of the fundamental hypotheses mentioned above, Heisenberg, Schrödinger, Bohn, Dirac, and others established the theory of linear quantum mechanics which describes the properties and motions of microscopic particle systems. This theory states that once the externally applied potential fields and initial states of the particles are given, the states of the particles at any time later and any position can be determined by the linear Schrödinger equation, equations (1.7) and (1.8) in the case of nonrelativistic motion, or equivalently, the Dirac equation and the Klein-Gordon equation in the case of relativistic motion. The quantum states and their occupations of electronic systems, atoms, molecules, and the band structure of solid state matter, and any given atomic configuration are completely determined by the above equations. Macroscopic behaviors of systems such as mechanical, electrical and optical properties may also be determined by these equations. This theory also describes the properties of microscopic particle systems in the presence of external electromagnetic field, optical and acoustic waves, and thermal radiation. Therefore, to a certain degree, the linear quantum mechanics describes the law of motion of microscopic particles of which all physical systems are composed. It is the foundation and pillar of modern physics.

The linear quantum mechanics had great successes in descriptions of motions of microscopic particles, such as electron, phonon, photon, exciton, atom, molecule, atomic nucleus and elementary particles, and in predictions of properties of matter based on the motions of these quasi-particles. For example, energy spectra of atoms (such as hydrogen atom, helium atom), molecules (such as hydrogen molecule) and compounds, electrical, optical and magnetic properties of atoms and condensed matters can be calculated based on linear quantum mechanics and the calculated results are in good agreement with experimental measurements. Being the foundation of modern science, the establishment of the theory of quantum mechanics has revolutionized not only physics, but many other science branches such as chemistry, astronomy, biology, *etc.*, and at the same time created many new branches of science, for example, quantum statistics, quantum field theory, quantum electronics, quantum chemistry, quantum biology, quantum optics, *etc.* One of the great successes of the linear quantum mechanics is the explanation of the fine energy spectra of hydrogen atom, helium atom and hydrogen molecule. The energy spectra predicted by linear quantum mechanics for these atoms and molecules are completely in agreement with experimental data. Furthermore, modern experiments have demonstrated that the results of the Lamb shift and superfine structure of hydrogen atom and the anomalous magnetic moment of the electron predicted by the theory of quantum electrodynamics are in agreement with experimental data within an order of magnitude of 10^{-5}. It is therefore believed that the quantum electrodynamics is one of most successful theories in modern physics.

Despite the great successes of linear quantum mechanics, it nevertheless encountered some problems and difficulties. In order to overcome these difficulties, Einstein had disputed with Bohr and others for the whole of his life and the difficulties still remained up to now. Some of the difficulties will be discussed in the next section. These difficulties of the linear quantum mechanics are well known and have been reviewed by many scientists. When one of the founders of the linear quantum mechanics, Dirac, visited Australia in 1975, he gave a speech on the development of quantum mechanics in New South Wales University. During his talk, Dirac mentioned that at the time, great difficulties existed in the quantum mechanical theory. One of the difficulties referred to by Dirac was about an accurate theory for interaction between charged particles and an electromagnetic field. If the charge of a particle is considered as concentrated at one point, we shall find that the energy of the point charge is infinite. This problem had puzzled physicists for more than 40 years. Even after the establishment of the renormalization theory, no actual progress had been made. Such a situation was similar to the unified field theory for which Einstein had struggled for his whole life. Therefore, Dirac concluded his talk by making the following statements: It is because of these difficulties, I believe that the foundation for the quantum mechanics has not been correctly laid down. As part of the current research based on the existing theory, a great deal of work has been done in the applications of the theory. In this respect, some rules for getting around the infinity were established. Even though results obtained based on such rules agree with experimental measurements, they are artificial rules after all. Therefore, I cannot accept that the present foundation of the quantum mechanics is completely correct.

However, what are the roots of the difficulties of the linear quantum mechanics that evoked these contentions and raised doubts about the theory among physicists? Actually, if we take a closer look at the history of physics, one would know that not so many fundamental assumptions were required for all physical theories but the linear quantum mechanics. Obviously, these assumptions of linear quantum mechanics caused its incompleteness and limited its applicability.

It was generally accepted that the fundamentals of the linear quantum mechanics consist of the Heisenberg matrix mechanics, the Schrödinger wave mechanics, Born's statistical interpretation of the wave function and the Heisenberg uncertainty principle, *etc.* These were also the focal points of debate and controversy. In other words, the debate was about how to interpret quantum mechanics. Some of the questions being debated concern the interpretation of the wave-particle duality, probability explanation of the wave function, the difficulty in controlling interaction between measuring instruments and objects being measured, the Heisenberg uncertainty principle, Bohr's complementary (corresponding) principle, single particle versus many particle systems, the problems of microscopic causality and probability, process of measuring quantum states, *etc.* Meanwhile, the linear quantum mechanics in principle can describe physical systems with many particles, but it is not easy to solve such a system and approximations must be used to obtain approximate solutions. In doing this, certain features of the system which could be important have to be neglected. Therefore, while many enjoyed the successes of the linear quantum mechanics, others were wondering whether the linear quantum mechanics is the right theory of the real microscopic physical world, because of the problems and difficulties it encountered. Modern quantum mechanics was born in 1920s, but these problems were always the topics of heated debates among different views till now. It was quite exceptional in the history of physics that so many prominent physicists from different institutions were involved and the scope of the debate was so wide. The group in Copenhagen School headed by Bohr represented the view of the main stream in these discussions. In as early as 1920s, heated disputes on the statistical explanation and completeness of wave function arose between Bohr and other physicists, including Einstein, de Broglie, Schrödinger, Lorentz, *etc.*

The following is a brief summary of issues being debated and problems encountered by the linear quantum mechanics.

(1) First, the correctness and completeness of the linear quantum mechanics were challenged. Is linear quantum mechanics correct? Is it complete and self-consistent? Can the properties of microscopic particle systems be completely described by the linear quantum mechanics? Do the fundamental hypotheses contradict each other?

(2) Is the linear quantum mechanics a dynamic or a statistical theory? Does it describe the motion of a single particle or a system of particles? The dynamic equation seems an equation for a single particle, but its mechanical quantities are determined based on the concepts of probability and statistical average. This caused confusion about the nature of the theory itself.

(3) How to describe the wave-particle duality of microscopic particles? What is the nature of a particle defined based on the hypotheses of the linear quantum mechanics? The wave-particle duality is established by the de Broglie relations. Can the statistical interpretation of wave function correctly describe such a property? There are also difficulties in using wave package to represent the particle nature of microscopic particles. Thus describing the wave-corpuscle duality was a major challenge to the linear quantum mechanics.

(4) Was the uncertainty principle due to the intrinsic properties of microscopic particles or a result of uncontrollable interaction between the measuring instruments and the system being measured?

(5) A particle appears in space in the form of a wave, and it has certain probability to be at a certain location. However, it is always a whole particle, rather than a fraction of it, being detected in a measurement. How can this be interpreted? Is the explanation of this problem based on wave package contraction in the measurement correct?

Since these are important issues concerning the fundamental hypotheses of the linear quantum mechanics, many scientists were involved in the debate. Unfortunately, after being debated for almost a century, there are still no definite answers to most of these questions. We will introduce and survey some main views of this debate in the following.

As far as the completeness of the linear quantum mechanics was concerned, Von Neumann provided a proof in 1932. According to Von Neumann, if O is a set of observable quantities in the Hilbert space Q of dimension greater than one, then the self-adjoint of any operator in this set represents an observable quantity in the same set, and its state can be determined by the average $\langle \hat{A} \rangle$ for the operator $\hat{A}$. If this average value satisfies $\langle 1 \rangle = 1$, we have $\langle r\hat{A} \rangle = r\langle \hat{A} \rangle$ for any real constant r. If A is non-negative, then $\langle \hat{A} \rangle \geq 0$. If $A, B, C, \cdots$ are arbitrary observable quantities, then, there always exists an observable $A + B + C + \cdots$ such that $\langle \hat{A} + \hat{B} + \hat{C} + \cdots \rangle = \langle \hat{A} \rangle + \langle \hat{B} \rangle + \langle \hat{C} \rangle + \cdots$. Von Neumann proved that there exists a self-adjoint operator $\hat{A}$ in Q such that $\langle A^a \rangle \neq \langle A \rangle^a$. This implies that there always exists an observable quantity A which is indefinite or does not have an accurate value. In other words, the states as defined by the average value are dispersive and cannot be determined accurately, which further implies that states in which all observable quantities have accurate values simultaneously do not exist. To be more concrete, not all properties of a physical system can possess accurate values. At this stage, this was the best the theory can do. Whether it can be accepted as a complete theory is subjective. It seemed that any further discussion would lead to nowhere.

It was realized later that Von Neumann's theorem was mathematically flawless but ambiguous and vague in physics. In 1957, Gleason made two modifications to Von Neumann's assumptions: Q should be the Hilbert space of more than two dimensions rather than one; and A, B, C, $\cdots$ should be limited to commutable self-adjoint operators in Q. He verified that Von Neumann's theorem is still valid with these assumptions. Because the operators are commutable, the linear superposition property of average values is, in general, independent of the order in which experiments are performed. Hence, these assumptions seem to be physically acceptable. Furthermore, Von Neumann's conclusion ruled out some nontrivial hidden variable theories in the Hilbert space with dimensions of more than two.

However, in 1966, Bell indicated that Gleason's theorem can essentially only remove the hidden variable theories which are independent of environment and arrangements before and after a measurement. It would be possible to establish hidden variable theories which are dependent on environment and arrangements before and after a measurement. At the same time, Bell argued that since there are more input hidden variables in the hidden variable theory than in quantum mechanics, there should be new results that may be compared with experiments, thus to verify whether the quantum mechanics is complete.

Starting from an ideal experiment based on the localized hidden variables theory and the average value $q(a, b) = \int A_a(\lambda) B_b(\lambda) d\lambda$, Bohm believed that some features of a particle could be obtained once those of another particle which is remotely separated from the first are measured. This indicates that correlation between particles exists which could be described in terms of "hidden parameters". Based on this idea, Bell proposed an inequality which is applicable to any "localized" hidden variables theory. Thus, the natures of correlation in a system of particles predicted by the Bell's inequality and quantum mechanics would differ appreciably which can be used to verify which of the two is correct.

To this end, we discuss a system of spin correlation. We shall first discuss spin correlation from the point of view of quantum mechanics. Assume that there exists a system which consists of two particles A and B, both of spin 1/2, but the total spin of the system is zero. Let A_a be the spin component measured along a direction specified by a unit vector $\hat{a}$, and similarly B_b the spin component measured along a direction specified by a unit vector $\hat{b}$. According to linear quantum mechanics, it is easy to write down the components of the spin operators along directions $\hat{a}$ and $\hat{b}$. They are $(\hat{\sigma}_A \cdot \hat{a})/2$ and $(\hat{\sigma}_B \cdot \hat{b})/2$, respectively, where $\hat{\sigma}_A/2$ and $\hat{\sigma}_B/2$ are the spin operators of particles A and B in terms of the Pauli matrices, respectively. $(\hat{\sigma}_A \cdot \hat{a})/2$ and $(\hat{\sigma}_B \cdot \hat{b})/2$ can be regarded as projections of the spin operators on the unit vectors $\hat{a}$ and $\hat{b}$, respectively. The spin correlation function, $q(a, b)$, may be defined as the average of the product of A_a and B_b, *i.e.* $q(a, b) = 4\overline{A_a \cdot B_b}$, where the factor of 4 is due to "normalization", the horizontal line above $A_a \cdot B_b$ denotes the statistical average of the product of A_a and B_b over all possible results of measurements. According to linear quantum mechanics, we have

$$\overline{A_a \cdot B_b} = \frac{1}{4}\langle 0^+|(\hat{\sigma}_A \cdot \hat{a})(\hat{\sigma}_B \cdot \hat{b})|0^+\rangle$$

where $|0^+\rangle$ represents the spin wave function with zero total spin, of the system consisting of particles A and B of spin 1/2, and can be expressed as

$$|0^+\rangle = \frac{1}{\sqrt{2}}\left[\psi_{+\frac{1}{2}}(A)\psi_{-\frac{1}{2}}(B) - \psi_{-\frac{1}{2}}(A)\psi_{+\frac{1}{2}}(B)\right].$$

$\langle 0^+|$ in the above equations is the Hermitian conjugate of $|0^+\rangle$. Using the above expression and the rules of Pauli matrix, we can obtain

$$q(a, b) = 4\overline{A_a \cdot B_b} = -\hat{a} \cdot \hat{b}.$$

According to this equation, $q(a, b) = -1$ if $\hat{a} = \hat{b}$, which results in "negative" correlation for spin projections measured in the same direction.

On the other hand, if we start from Bell's localized hidden variable theory, we obtain the following Bell's inequality:

$$|q(\hat{a}, \hat{b}) - q(\hat{a}, \hat{c})| \leq 1 + q(\hat{a}, \hat{c}).$$

This involves measurements of the spin components in three directions, specified by unit vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$, respectively, in contrast to the previous case which involves only two directions. If we let $\hat{a} = \hat{b} = \hat{c} = \hat{n}$, then Bell's inequality becomes $q(\hat{n} \cdot \hat{n}) \geq -1$, which is the same as that given by quantum mechanics. Different results can be expected if three directions are really involved in the measurements. For example, if the angles between $\hat{a}$ and $\hat{b}$ and between $\hat{b}$ and $\hat{c}$ are 60° and that between $\hat{a}$ and $\hat{c}$ is 120°, then we have $q(\hat{a},\hat{b}) = q(\hat{b},\hat{c}) = 1/2$, and $q(\hat{a},\hat{c}) = -1/2$ according to quantum mechanics. Substituting these into the Bell's inequality, it is evident that

$$\left|\frac{1}{2}+\frac{1}{2}\right| \leq 1-\frac{1}{2}$$

which results in $1 \leq 1/2$ that does not make any sense.

It is clearly seen that spin correlation described in linear quantum mechanics contradicts the Bell's inequality. That is to say that all statistical predictions of linear quantum mechanics cannot be obtained from the localized hidden variable theory. In some special cases, if statistical predictions based on linear quantum mechanics are correct, then the localized hidden variable theory does not hold, and vice versa. However, whether the Bell's inequality is correct remained a question.

Since then many physicists, for example Wigner in 1970, had also derived the Bell's inequality using analytical methods which were quite different from Bell's approach. Unfortunately, only single state of particles with zero spin was discussed in an ideal experiment setting. This is equivalent to assume that two particles of spin 1/2 always reach the instrument and therefore the instrument always measures a definite spin along a given axis. Such a measurement is very hard to realize in actual experiments.

This prompted Clayser *et al.* to generalize Bell's inequality by removing the restrictions of single state and spin 1/2, in 1969. The Clayser's generalized inequality

$$|q(\hat{a},\hat{b}) - q(\hat{a},\hat{b}')| \leq 2 \pm [q(\hat{a}',\hat{b}') + q(\hat{a}',\hat{b})]$$

is based on some more common and realistic experimental conditions. If $q(\hat{a}',\hat{b}) = -1$, the Clayser's inequality reduces to the Bell's inequality. Bell himself also obtained the same result in 1971. Since 1972, many experiments, as shown in Table 1.1, have been carried out and results have been reported to verify which theory, the Bell's inequality of localized hidden variable or the linear quantum mechanics, correctly describes the motion of the microscopic particle.

Among the nine experiments listed in Table 1.1, seven of them gave supports to linear quantum mechanics and only two experimental findings are in agreement with the Bell's inequality. It seems that the experimental results are in favor of the linear quantum mechanics than Bell's localized hidden variable theory. This shows that linear quantum mechanics does not satisfy the requirement of localization. The results, however, cannot exclusively confirm its validity either.

1.3 Dispute between Bohr and Einstein

While the view on linear quantum mechanics and its interpretation by Bohr and others in the Copenhagen school dominated the debate, many prominent physicists respected Einstein as the authority who had doubted and continuously criticized

Table 1.1 List of experiments to verify Bell's inequality.

No.	Author(s)	Date	Experiment	Results
1	S. T. Freedman J. F. Clauser	1972	Low-energy photon radiation in transitional process of a calcium atom	Supports linear quantum mechanic
2	R. A. Holt F. M. Pipkin	1973	Low-energy photon radiation in transitional process of mercury-198 atoms	Supports Bell's inequality
3	J. F. Clauser	1976	Low-energy photon radiation in transitional process of mercury-202 atoms	Supports linear quantum mechanic
4	E. S. Firg R. C. Thomson	1976	Low-energy photon radiation in transitional process of mercury-202 atom	Supports linear quantum mechanic
5	G. Fioraci S. Gutkowski S. Natarrigo R. Pennisi	1975	High-energy photon annihilation of electron – positron pair (γ ray)	Supports Bell's inequality
6	J. Kasday J. Ulman Wu Jianxiong	1975	High-energy photon annihilation of electron – positron pair (γ ray)	Supports linear quantum mechanic Supports linear quantum mechanic
7	M. Lamchi-Rachti W. Mitting	1976	Atomic pair in single state	Supports linear quantum mechanic
8	Aspect P. Grangier G. Roger	1981	Cascade photon radiation in transitional process of atoms	Supports linear quantum mechanic
9	P. Grangier P. Grangier G. Roger	1982	Cascade photon radiation in transitional process of ^{46}Ca	Supports linear quantum mechanic

Bohr's interpretation. This resulted in a life-long dispute between Bohr and Einstein, which was unprecedented and went through three stages.

The first stage was during the period from 1924 to 1927 when the theory of quantum mechanics had just been established. Einstein proceeded from his own philosophical belief and his scientific goal for an exact description of causality in the physical world, and expressed his extreme unhappiness with the probability interpretation of linear quantum mechanics. In a letter to Born on December 4, 1926, Einstein said that "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not the real thing (*der Wahre Jakob*). The theory says a lot, but it does not bring us any closer to the secret of the "Old One." I, at any rate, am convinced that *He* is not playing at dice."

The second stage was from 1927 to 1930. After Bohr had put forward his complementary principle and had established his interpretation as the main stream interpretation, Einstein was extremely unhappy. His main criticism was directed at the uncertainty relation on which Bohr's complementary principle was based. At the 5th (1927) and the 6th (1930) International Meetings of Physics at Solway, Einstein proposed two ideal experiments (double slit diffraction and photon box) to prove that the uncertainty relation and formalism of the quantum mechanics contradict each other, and thus to disprove Bohr's complementary principle. But Einstein's idea was demolished each time by Bohr through resourceful analysis. Since then, Einstein had to accept the logical consistency of quantum mechanics and turned his criticism to the completeness of the linear quantum mechanics theory.

The third stage was from 1930 until the death of Einstein. The dispute during this period is reflected in the debate between Einstein and Bohr over the EPR paradox proposed by Einstein together with Podolsky and Rosen. This paradox concerned the fundamental problem of the linear quantum mechanics, *i.e.*, whether it satisfied the deterministic localized theory and the microscopic causality. Since some of the subsequent experiments seem to support the linear quantum mechanics, instead of the Bell inequality, it is necessary to understand the nature of the EPR paradox and results it brought about.

The EPR paradox will be briefly introduced below.

Consider a system consisting of two particles which move in opposite directions. For simplicity but without losing its generality, we assume that the initial relativistic momentum of the pair of particles is $p = 0$. Then there must be $p_1 = -p_2$ after the two particles interact and depart. However, the magnitude and direction of the momentum of each particle are not known. Assume that the momentum of particle 1 is measured, by a detector, and the value $p_1 = +a$ is obtained, then the momentum of the particle 2 is determined and it can only be $p_2 = -a$ according to conservation of momentum in the linear quantum mechanics. However, in the light of the hypothesis of contraction of wave packet in the measuring process, the plane wave with momentum $p_1 = a_1$ is "selected" out by the detector from the wave packet $\psi_1(X_1)$ describing particle 1. In accordance with the traditional linear quantum mechanics, this process of "spectrum resolution" is due to some kind of "uncontrollable interaction" between the instrument and the wave packet. Under the influence of such an "uncontrollable interaction", the momentum of particle 1 could be $p_1 = a$, or $p_1 = b$, $\cdots$. However, what is surprising is that there is always $p_2 = -a$ as long as $p_1 = a$ is measured by the detector. This means that this value should be obtained regardless of the measurement on the wave packet $\psi_2(X_2)$ is made or not. In other words, when the wave packet $\psi_1(X_1)$ is measured and contracted, the wave packet $\psi_2(X_2)$ for particle 2 will also be automatically contracted. A series of questions then arise. For example, what mechanism makes this possible? Does this occur instantaneously, or is it propagating at speed of light according to the special theory of relativity? How can the wave packet contraction caused by measurement automatically guarantee the conservation of momentum? It is very difficult to answer these questions. Only after careful studies by Einstein and others, the following conclusions were obtained: either the description of the linear quantum mechanics was incomplete, or the linear quantum mechanics didn't satisfy the criterion of "localization". Einstein tended to believe that physical phenomena must satisfy the criterion of "localization", *i.e.* physical quantities cannot propagate with speed greater than the speed of light. Thus, he thought that the linear quantum mechanics is an incomplete theory. Due to this remarkable analysis by Einstein, many physicists began to explore the theory of "hidden parameters" of the linear quantum mechanics.

The "queries" to the linear quantum mechanics by Einstein and others had indeed created quite a stir. Bohr had to respond in his own capacity to these queries. In 1935, Bohr published a short essay in Physical Review in which he argued that if a system consists of two local particles 1 and 2, then this system should be described by a wave function $\psi(1,2)$. In such a case, the local particles 1 and 2 are no longer mutually independent entities. Even though they are spatially separated at the instant the system is probed, they cannot be considered as independent entities. Thus, there is no basis for statements such as measurement of subsystem 1 could not influence subsystem 2 within the framework of the linear quantum mechanics, and the idea of Einstein *et al.* cannot be accepted. Essentially, Bohr was not really against the "paradox" proposed by Einstein and others, but only confirmed that linear quantum mechanics might not satisfy the principle of localization. Bohr further commented that in the final decisive steps of measurement in Einstein's ideal experiment, even though there was no mechanical interference to the system being probed, influence on experimental conditions did exist. Thus, Einstein's arguments could not verify their conclusion that the description of quantum mechanics is incomplete.

Many scientists who followed closely the thought of localization and incompleteness of the linear quantum mechanics by Einstein and others believed that there could exist a hidden variables theory behind linear quantum mechanics which might be able to interpret the probability behavior of microscopic particle. The concept of "hidden variables" was proposed soon after linear quantum mechanics was born. However, it was disapproved by Von Neumann in 1932. For a long time since then, no one had mentioned this problem. After the second World War, Einstein repeatedly criticized the linear quantum mechanics and suggested that any actual state should be completely described.

Motivated by this thought, Bohm put forward the first systematic "hidden variable theory" in 1952. He believed that the statistical characteristics of linear quantum mechanics is due to some "background" fluctuations hidden behind the quantum theory. If we can find the hidden function for a microscopic particle, then a deterministic description could be made for a single particle. But how can the existence of such hidden variables be proved? Bohm proposed two experiments, to measure the spin correlation of a single proton and the polarization correlation in annihilating radiation of photons, respectively. It was realized later that in Bohm's theory the single state ψ is essentially a slowly varying state which describes states of a fluid with random fluctuations. Since the wave function itself cannot have such random fluctuation, a hidden variable could not be introduced. Bohm's theory mentioned above was referred to as a random hidden-variables theory.

However, if the motion of particles can also be considered as a stable Markov process. A steady state solution of the Schrödinger equation can then be given from a steady distribution of the Markov chain, and if the Fock-Planck equation was taken as the dynamic equation of microscopic particle, a new "hidden variables theories" of linear quantum mechanics can be set up. After Bell established his inequality on the basis of Bohm's deterministic "localized variables theory" in 1966, various attempts were made to experimentally verify which theory is the right theory and to settle the dispute once and for all. As mentioned earlier, majority of the experiments supported the linear quantum mechanics at that time, and it was clear that not all the predictions by the linear quantum mechanics can be obtained from the localized hidden variables theory. Thus the "hidden variable theory" was abandoned.

To summarize, the long dispute between Bohr and Einstein was focused on three issues. (1) Einstein upheld to the belief that the microscopic world is no different from the macroscopic world, particles in the microscopic world are matters and they exist regardless of the methods of measurements, any theoretical description to it should in principle be deterministic. (2) Einstein always considered that the theory of the linear quantum mechanics was not an ultimate and complete theory. He believed that quantum mechanics is similar to classical optics. Both of them are correct theories based on statistical laws, *i.e.*, when the probability $|\psi(\vec{r},t)|^2$ of a particle at a moment t and location r is known, the average value of an observable quantity can be obtained using statistical method and then compared with experimental results. However, the understanding to processes involving single particle was not satisfactory. Hence, $\psi(\vec{r},t)$ cannot give everything about a microscopic particle system, and the statistical interpretation cannot be ultimate and complete. (3) The third issue concerns the physical interpretation of the linear quantum mechanics. Einstein was not impressed with the attempt to completely describing some single processes using linear quantum mechanics, which he made very clear in a speech at the fifth Selway International Meeting of physics. In an article, "Physics and Reality", published in 1936 in the Journal of the Franklin Institute, Einstein again mentioned that what the wave function ψ describes can only be a many-particle system, or an assemble in terms of statistical mechanics, and under no circumstances, the wave function can describe the state of a single particle. Einstein also believed that the uncertainty relation was a result of incompleteness of the description of a particle by $\psi(\vec{r},t)$, because a complete theory should give precise values for all observable quantities. Einstein also did not accept the statistical interpretation, because he did not believe that an electron possess free will. Thus, Einstein's criticism against the linear quantum mechanics was not directed towards the mathematical formalism of the linear quantum mechanics, but to its fundamental hypotheses and its physical interpretation. He considered that this is due to the incomplete understanding of the microscopic objects. Moreover, the contradiction between the theory of relativity and the fundamental of the linear quantum mechanics was also a central point of dispute. Einstein made effort to unite the theory of relativity and linear quantum mechanics, and attempted to interpret the atomic structure using field theory. The disagreements on several fundamental issues of the linear quantum mechanics by Einstein and Bohr and their followers were deep rooted and worth further study. This brief review on the disputes between the two great physicists given above should be useful to our understanding on the nature and problems of the linear quantum mechanics. It should set the stage for the introduction of nonlinear quantum mechanics.

1.4 Analysis on the Roots of Problems of Linear Quantum Mechanics and Review on Recent Developments

The discussion in the previous section shows that the disputes and disagreement on several fundamental issues of the linear quantum mechanics are deep rooted. Almost all prominent physicists were involved to a certain degree in this dispute which lasted half of a century, which is extraordinary in the history of science. What is even more surprising is that after such a long dispute, there have been no conclusions on these important issues till now. Besides what have been mentioned above, there was another fact which also puzzled physicists. As it is know, the concept of "orbit" has no meaning in quantum mechanics. The state of a particle is described by the wave function ψ which spreads out over a large region in space. Even though this suggests that a particle does not have a precise location, in physical experiments, however, particles are always captured by a detector placed at an exact position. Furthermore, it is always one whole particle, rather than a fraction of it, being detected. How can this be interpreted by the linear quantum mechanics? Given this situation, can we consider that the linear quantum mechanics is complete? Even though the linear quantum mechanics is correct, then it can only be considered as a set of rules describing some experimental results, rather than an ultimate complete theory. In the meantime, the indeterministic nature of the linear quantum mechanics seems against intuition. All these show that it is necessary to improve and further develop the linear quantum mechanics. Attempt to solve these problems within the framework of the linear quantum mechanics seem impossible. Therefore, alternatives that go beyond the linear quantum mechanics must be considered to further develop the quantum mechanics. To do this, one must thoroughly understand the fundamentals and nature of the linear quantum mechanics and seriously consider de Broglie's idea of a nonlinear wave theory.

Looking back to the development and applications of the linear quantum mechanics for almost a century, we notice that the splendidness of the quantum mechanics is the introduction of a wave function to describe the state of particles and the expression of physical quantities by linear Hermitian operators. Such an approach is drastically different from the traditional methods of classical physics and took the development of physics to a completely new stage. This new approach has been successfully applied to some simple atoms and molecules, such as hydrogen atom, helium atom and hydrogen molecule, and the results obtained are in agreement with experimental data. Correctness of this theory is thus established. However, besides being correct, a good theory should also be complete. Successful applications to a subset of problems does not mean perfection of the theory and applicability to any physics system. Physical systems in the world are manifold and every theory has its own applicable scope or domain. No theory is universal.

From the above discussion, we see that the most fundamental features of the linear quantum mechanics are its linearity and the independence of the Hamiltonian of a system on its wave function. These ensure the linearity of the fundamental dynamic equations, *i.e.*, the Schrödinger equation is a linear equation of the wave function, all operators in the linear quantum mechanics are linear Hermitian operators, and the solutions of the dynamic equation satisfy the linear superposition principle. The linearity results in the following limitations of the linear quantum mechanics.

(1) The linear quantum mechanics is a wave theory and it depicts only the wave feature, not their corpuscle feature, of microscopic particles. As a matter of fact, the Schrödinger equation (1.7) is a wave equation and its solution represents a probability wave. To see this clearly, we consider the wave function $\psi = f \cdot \exp(-iEt/\hbar)$ and substitute it into (1.7). If we let $n^2 = (E-U)/(E-C) = k^2/k_0^2$, where C is a constant, and $k_0^2 = 2m(E-C)/\hbar^2$, then (1.7) becomes

$$\frac{\partial^2 f}{\partial t^2} + k^2 n^2 f = 0.$$

This equation is nothing but that of a light wave propagating in a homogeneous medium. Thus, the linear Schrödinger equation (1.7) is only able to describe the wave feature of the microscopic particle. In other words, when a particle moves continuously in the space-time, it follows the law of linear variation and disperses over the space-time in the form of a wave. This wave feature of a microscopic particle is mainly determined by the kinetic energy operator, $\hat{T} = -(\hbar^2/2m)\nabla^2$, in the dynamic equation (1.7). The applied potential field, $V(x,t)$ is imposed on the system by external environment and it can only change the wave form and amplitude, but not the nature of the wave. Such a dispersion feature of the microscopic particle ensures that the microscopic particle can only appear with a definite probability at a given point in the space-time. Therefore, the momentum and coordinate of the microscopic particle cannot be accurately measured simultaneously, which lead to the uncertainty relation in the linear quantum mechanics. Therefore, the uncertainty relation occurs in linear quantum mechanics is an inevitable outcome of the linear quantum mechanics.

(2) Due to this linearity and dispersivity, it is impossible to describe the corpuscle feature of microscopic particles by means of this theory. In other words, the wave-corpuscle duality of microscopic particle cannot be completely described by the dynamic equation in the linear quantum mechanics, because external applied potential fields cannot make a dispersive particle an undispersive, localized particle, and there is no other interaction that can suppress the dispersion effect of the

kinetic energy in the equation. Thus a microscopic particle always exhibits features of a dispersed wave and its corpuscle property can only be described by means of Born's statistical interpretation of the wave function. This not only exposes the incompleteness of the hypotheses of linear quantum mechanics, but also brings out an unsolvable difficulty, namely, whether the linear quantum mechanics describes the state of a single particle or that of an assemble of many particles.

As it is known, in linear quantum mechanics, the corpuscle behavior of a particle is often represented by a wave packet which can be a superposition of plane waves. However, the wave packet always disperses and attenuates with time during the course of propagation. For example, a Gaussian wave packet given by

$$\psi(x,t=0) = e^{-\alpha_0^2 x^2/2}, \quad |\psi|^2 = e^{-\alpha_0^2 x^2} \tag{1.11}$$

at $t = 0$ becomes

$$\begin{aligned} \psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t/2m)} dk \\ &= \frac{1}{\alpha_0 \sqrt{1/\alpha_0^2 + i\hbar t/m}} e^{-(x^2/2)(1/\alpha_0^2 + i\hbar t/m)}, \\ |\psi|^2 &= \frac{1}{\sqrt{1 + (\alpha_0^2 \hbar t/m)^2}}\, e^{-x^2/\alpha_t^2} \end{aligned} \tag{1.12}$$

after propagating through a time t, where

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t=0) e^{ikx} dk, \quad \alpha_t = \frac{1}{\alpha_0}\sqrt{1 + i\frac{\alpha_0^2 \hbar}{m} t}.$$

This indicates clearly that the wave packet is dispersed as time goes by. The uncertainty in its position also increases with time. The corresponding uncertainty relation is

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1 + \frac{\alpha_0^4 \hbar^2 t^2}{m^2}},$$

where

$$\Delta p = \frac{\hbar \alpha_0}{\sqrt{2}}, \quad \Delta x = \frac{1}{\sqrt{2}\alpha_0}\sqrt{1 + \frac{\alpha_0^4 \hbar^2 t^2}{m^2}}.$$

Hence, the wave packet cannot be used to describe the corpuscle property of a microscopic particle. How to describe the corpuscle property of microscopic particles has been an unsolved problem in the linear quantum mechanics. This is just an example of intrinsic difficulties of the linear quantum mechanics.

(3) Because of the linearity, the linear quantum mechanics can only be used in the case of linear field and medium. This means that the linear quantum mechanics is suitable for few-body systems, such as the hydrogen atom and the helium atom, *etc.* For many-body systems and condensed matter, it is impossible to solve the

wave equation exactly and only approximate solutions can be obtained in the linear quantum mechanics. However, doing so loses the nonlinear effects due to intrinsic and self-interactions among the particles in these matters. Therefore, the scope of application of the linear quantum mechanics is limited. Moreover, when this theory is applied to deal with features of elementary particles in quantum field theory, the difficulty of infinity cannot always be avoided and this shows another limitation of this theory. Therefore, it is necessary to develop a new quantum theory that can deal with these complex systems.

From the discussion above, we learned that linearity on which linear quantum mechanics is based is the root of all the problems encountered by the linear quantum mechanics. The linearity is closely related to the assumption that the Hamiltonian operator of a system is independent of its wave function, which is true only in simple and uniform physical systems. Thus the linearity greatly limited the applicable scope and domain of the linear quantum mechanics. It cannot be used to study the properties of many-body, many-particle, nonlinear and complex systems in which there exist complicated interaction, the self-interaction, and nonlinear interactions among the particles and between the particles and the environment.

Since the wave feature of microscopic particle can be well described by the wave function, one important issue to be looked into in further development of quantum mechanics is the description of corpuscle feature of microscopic particles, so that the new quantum theory should completely describe the wave-corpuscle duality of microscopic particles. However, this is easily said than done. To this respect, it is useful to review what has already been done by the pioneers in this field, as we can learn from them and get some inspiration from their work.

One can learn from the history of development of the theory of superconductivity. It is known that the mechanism of superconductivity based on electron-phonon interaction was proposed by Fröhlich as early as in 1951. But Fröhlich failed to establish a complete theory of superconductivity because he confined his work to the perturbation theory in the linear quantum mechanics, and superconductivity is a nonlinear phenomenon which cannot be described by the linear quantum mechanics. Of course, this problem was finally solved and the nonlinear BCS theory was established in 1975. This again clearly demonstrated the limitation of the linear quantum mechanics. This problem will be discussed in the next chapter in more details.

In view of this, in order to overcome the difficulties of the linear quantum mechanics and further develop the theory of quantum mechanics, two of the hypotheses of the linear quantum mechanics, *i.e.* linearity of the theory and independence of the Hamiltonian of a system on its wave function must be reconsidered. Further development must be directed toward a nonlinear quantum theory. In other words, nonlinear interaction should be included into the theory and the Hamiltonian of a system should be related to the wave function of the system.

The first attempt of establishing a nonlinear quantum theory was made by de

Broglie, which was described in his book: "the nonlinear wave theory". Through a long period of research, de Broglie concluded that the theory of wave motion cannot interpret the relation between particle and wave because the theory was limited to a linear framework from the start. In 1926, he further emphasized that if $\psi(\vec{r}, t)$ is a real field in the physical space, then the particle should always have a definite momentum and position. de Broglie assumed that $\psi(\vec{r}, t)$ describes an essential coupling between the particle and the field, and used this concept to explain the phenomena of interference and diffraction.

In 1927, de Broglie put forward a "dual solution theory" in a paper published in J. de Physique. de Broglie proposed that two types of solutions are permitted in the dynamic equation in the linear quantum mechanics. One is a continuous solution, $\psi = Re^{i\theta}$, with only statistical meaning, and this is the Schrödinger wave. This wave can only have statistical interpretation and can be normalized. It does not represent any physical wave. The other type, referred as a u wave, has singularities and is associated with spatial localization of the particle. The corpuscle feature of a microscopic particle is described by the u wave and the position of a particle is determined by a singularity of the u wave. de Broglie generalized the formula of the monochromatic plane wave and stipulated a rule of associating the particle with the propagation of the wave. The particle would move inside its wave according to de Broglie's dual solution theory. This suggests that the motion of the particle inside its wave is influenced by a force which can be derived from a "quantum potential". This quantum potential is proportional to the square of the Planck constant and is dependent on the second derivative of the amplitude of the wave. It can also be given in terms of the change in the rest mass of the particle. In the case of a monochromatic plane wave, the quantum potential is zero. In 1950s, de Broglie further improved his "dual solution theory". He proposed that the u wave satisfies an undetermined nonlinear equation, and this led to his own "nonlinear wave theory". However, de Broglie did not give the exact nonlinear equation that the u wave should satisfy. This theory has serious difficulties in describing multiparticle systems and the s state of a single-particle. The theory also lacked experimental verification. Thus, even though it was supported by Einstein, the theory was not taken seriously by the majority and was gradually forgotten.

Although de Broglie's nonlinear wave theory was incorrect, some of his ideas, such as the quantum potential, the u wave of nonlinear equation which is capable of describing a physical particle, provided inspiration for further development of quantum mechanics.

As mentioned above, de Broglie stated that the quantum potential is related to the second derivative of the amplitude of the wave function $\psi = Re^{i\theta}$. Bohm, who proposed the theory of localized hidden variables in 1952, derived this quantum potential. It is independent of the phase, θ, of the wave function, and is represented in the form of $V = h^2\nabla^2 R/2mR$, where R is the amplitude of the wave, m is the mass of the particle, and h the Planck constant. With such a quantum potential, Bohm believed that the motions of microscopic particles should follow the Newton's equation, and it is because of the "instantaneous" action of this quantum potential, a measurement process is always disturbed. The latter, however, was less convincing.

Quantum potential and nonlinear equations were again introduced in the Bohm-Bohr theory proposed in 1966. They assumed that, in the dual Hilbert space, there exists a dual vector, $|\psi_1\rangle$ and $|\sigma\rangle$, which satisfy

$$|\psi_1\rangle = \sum_n a_n|A_n\rangle, \qquad |\sigma\rangle = \sum_k \sigma_k|A_k\rangle,$$

where σ is a hidden variable and satisfies the Gaussian distribution in an equilibrium state. They introduced a nonlinear term in the Schrödinger equation, to represent the effect arising from the quantum measurement, and determined the equation containing the nonlinear term based on the relation among the particles, the environment and the hidden variables. Attempts were made to solve the problem concerning the influence of the measuring instruments on the properties of particles being probed. de Broglie pointed out that the quantum potential can be expressed in terms of the change in the rest mass of the particle and tried to interpret Bohm's quantum potential based on the counteraction of the u wave and domain of singularity. Thus, the quantum potential arises from the interaction between particles. It is associated with nonlinear interaction and is able to change the properties of particles. These were encouraging. It seemed promising to make microscopic particles measurable and deterministic by adding a quantum potential with nonlinear effect to the Schrödinger equation, and eventually to have deterministic quantum theory. A delighted Dirac commented that the results will ultimately prove that Einstein's deterministic or physical view is correct.

In summary, we started the chapter with a review on the hypotheses on which the linear quantum mechanics was built, and the successes and problems of theory. We have seen that the linear quantum mechanics is successful and correct, but on the other hand, it is incomplete. Some of its hypotheses are vague and non-intuitive. Moreover, it is a wave theory and cannot completely describe the wave-corpuscle duality of microscopic particles. Therefore, improvement and further development on the linear quantum mechanics are required. The dispute between Einstein and Bohr, the recent work done by de Broglie, Bohm, and Bohr provided positive inspiration for further development of the quantum theory. From the above discussion, the direction for a complete theory seems clear: it should be a nonlinear theory. Two of its fundamental hypotheses of linear quantum mechanics, linearity and independence of the Hamiltonian of a systems on its wave function, must be reconsidered.

However, at what level of theory will the problems of the linear quantum mechanics be solved? What would be a good physical system to start with? What would be the foundation of a new theory? These and many other important questions can only be answered through further research. It is clear that the new theory should not be confined to the scope and framework of the linear quantum mechanics. The work of de Broglie and Bohm gave us some good motivations, but their ideas cannot be indiscriminately borrowed. One must go beyond the framework of the linear quantum mechanics and look into the nonlinear scheme. To establish a new and correct theory, one must start from the phenomena and experiments which the linear quantum mechanics failed, or had difficulty, to explain, and uses completely new concepts and new approaches to study these unique quantum systems. This is the only way to clearly understand the problems in the linear quantum mechanics. For this purpose, we will review some macroscopic quantum effects in the following chapter because these experiments form the foundation of the nonlinear quantum mechanics.

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