Introduction

Geometry of classical mechanics and field theory is mainly differential geometry of finite-dimensional smooth manifolds, fibre bundles and Lie groups.

The key point why geometry plays a prominent role in classical field theory lies in the fact that it enables one to deal with invariantly defined objects. Gauge theory has shown clearly that this is a basic physical principle. At first, a pseudo-Riemannian metric has been identified to a gravitational field in the framework of Einstein's General Relativity. In 60-70th, one has observed that connections on a principal bundle provide the mathematical model of classical gauge potentials [120; 284; 442]. Furthermore, since the characteristic classes of principal bundles are expressed in terms of the gauge strengths, one can also describe the topological phenomena in classical gauge models [142]. Spontaneous symmetry breaking and Higgs fields have been explained in terms of reduced G-structures [341]. A gravitational field seen as a pseudo-Riemannian metric exemplifies such a Higgs field [230]. In a general setting, differential geometry of smooth fibre bundles gives the adequate mathematical formulation of classical field theory, where fields are represented by sections of fibre bundles and their dynamics is phrased in terms of jet manifolds [169].

Autonomous classical mechanics speaks the geometric language of symplectic and Poisson manifolds [1; 279; 426]. Non-relativistic time-dependent mechanics can be formulated as a particular field theory on fibre bundles over $\mathbb{R}$ [294].

At the same time, the standard mathematical language of quantum mechanics and perturbative field theory, except gravitation theory, has been long far from geometry. In the last twenty years, the incremental development of new physical ideas in quantum theory (including super- and BRST symmetries, geometric and deformation quantization, topological field theory, anomalies, non-commutativity, strings and branes) has called into play advanced geometric techniques, based on the deep interplay between algebra, geometry and topology.

Let us briefly survey some peculiarities of geometric and algebraic topological methods in quantum mechanics.

Let us recall that, in the framework of algebraic quantization, one associates to a classical system a certain (e.g., von Neumann, C^*-, canonical commutation or anticommutation relation) algebra whose different representations are studied. Quantization techniques under discussion introduce something new. Namely, they can provide non-equivalent quantizations of a classical system corresponding to different values of some topological and differential invariants. For instance, a symplectic manifold X admits a set of non-equivalent star-products indexed by elements of the cohomology group $H^2(X)[[\hbar]]$ [206; 340]. Thus, one may associate to a classical system different underlying quantum models. Of course, there is a question whether this ambiguity is of physical or only mathematical nature. From the mathematical viewpoint, one may propose that any quantization should be a functor between classical and quantum categories (e.g., some subcategory of Poisson manifolds on the classical side and a subcategory of C^*-algebras on the quantum side) [271]. From the physical point of view, dequantization becomes important.

There are several examples of *sui generis* dequantizations. For instance, Berezin's quantization [145] in fact is dequantization. One can also think of well-known Gelfand's map as being dequantization of a commutative C^*-algebra $\mathcal{A}$ by the algebra of continuous complex functions vanishing at infinity on the spectrum of $\mathcal{A}$. This dequantization has been generalized to non-commutative unital C^*-algebras [105; 239]. The concept of the strict C^*-algebraic deformation quantization implies an appropriate dequantization when $\hbar \to 0$ [269; 372]. In Connes' non-commutative geometry, dequantization of the spectral triple in the case of a commutative algebra $C^\infty(X)$ is performed in order to restart the original differential geometry of a spin manifold X [107; 368].

I.

Let us start with familiar differential geometry. There are the following reasons why this geometry contributes to quantum theory.

(i) Most of the quantum models comes from quantization of the original classical systems and, therefore, inherits their differential geometric properties. First of all, this is the case of canonical quantization which replaces the Poisson bracket $\{f, f'\}$ of smooth functions with the bracket $[\widehat{f}, \widehat{f}']$ of Hermitian operators in a Hilbert space such that Dirac's condition

$$[\widehat{f}, \widehat{f}'] = -i\hbar\widehat{\{f, f'\}}$$

holds. Let us mention Berezin–Toeplitz quantization [47; 145; 365] and geometric quantization [141; 401; 426; 438] of symplectic, Poisson and Kähler manifolds.

(ii) Many quantum systems are considered on a smooth manifold equipped with some background geometry. As a consequence, quantum operators are often represented by differential operators which act in a pre-Hilbert space of smooth functions. A familiar example is the Schrödinger equation. The Kontsevich deformation quantization is based on the quasi-isomorphism of the differential graded Lie algebra of multivector fields (endowed with the Schouten–Nijenhuis bracket and the zero differential) to that of polydifferential operators (provided with the Gerstenhaber bracket and the modified Hochschild differential) [219; 255].

(iii) In some quantum models, differential geometry is called into play as a technical tool. For instance, a suitable $U(1)$-principal connection is used in order to construct the operators $\widehat{f}$ in the framework of geometric quantization. Another example is Fedosov's deformation quantization where a symplectic connection plays a similar role [149]. Let us note that this application has stimulated the study of symplectic connections [165].

(iv) Geometric constructions in quantum models often generalize the classical ones, and they are build in a similar way. For example, connections on principal superbundles [21], graded principal bundles [405], and quantum principal bundles [293] are defined by means of the corresponding one-forms in the same manner as connections on smooth principal bundles with structure finite-dimensional Lie groups.

II.

In quantum models, one deals with infinite-dimensional smooth Banach and Hilbert manifolds and (locally trivial) Hilbert and C^*-algebra bundles. The definition of smooth Banach (and Hilbert) manifolds follows that of finite-dimensional smooth manifolds in general, but infinite-dimensional

Banach manifolds are not locally compact, and they need not be paracompact [273; 422]. In particular, a Banach manifold admits the differentiable partition of unity if and only if its model space does. It is essential that Hilbert manifolds (but not, e.g., nuclear manifolds) satisfy the inverse function theorem and, therefore, locally trivial Hilbert bundles are defined. However, they need not be bundles with a structure group.

(i) Infinite-dimensional Kähler manifolds provide an important example of Hilbert manifolds [327]. In particular, the projective Hilbert space of complex rays in a Hilbert space E is such a Kähler manifold. This is the space the pure states of a C^*-algebra A associated to the same irreducible representation π of A in a Hilbert space E [129]. Therefore, it plays a prominent role in many quantum models. For instance, it has been suggested to consider a loop in the projective Hilbert space, instead of a parameter space, in order to describe Berry's phase [7; 43]. We have already mentioned the dequantization procedure which represents a unital C^*-algebra by a Poisson algebra of complex smooth functions on a projective Hilbert space [105].

(ii) Sections of a Hilbert bundle over a smooth finite-dimensional manifold X make up a particular locally trivial continuous field of Hilbert spaces in [129]. Conversely, one can think of any locally trivial continuous field of Hilbert spaces or C^*-algebras as being the module of sections of a topological fibre bundle. Given a Hilbert space E, let $B \subset B(E)$ be some C^*-algebra of bounded operators in E. The following fact reflects the nonequivalence of Schrödinger and Heisenberg quantum pictures. There is the obstruction to the existence of associated (topological) Hilbert and C^*-algebra bundles $\mathcal{E} \to X$ and $\mathcal{B} \to X$ with the typical fibres E and B, respectively. Firstly, transition functions of $\mathcal{E}$ define those of $\mathcal{B}$, but the latter need not be continuous, unless B is the algebra of compact operators in E. Secondly, transition functions of $\mathcal{B}$ need not give rise to transition functions of $\mathcal{E}$. This obstruction is characterized by the Dixmier–Douady class of $\mathcal{B}$ in the Čech cohomology group $H^3(X, \mathbb{Z})$. There is the similar obstruction to the $U(1)$-extension of structure groups of principal bundles [73; 86]. One also meets the Dixmier–Douady class as the obstruction to a bundle gerbe being trivial [58; 87].

(iii) There is a problem of the definition of a connection on C^*-algebra bundles which comes from the fact that a C^*-algebra (e.g., any commutative C^*-algebra) need not admit non-zero bounded derivations. An unbounded derivation of a C^*-algebra A obeying certain conditions is an infinitesimal generator of a strongly (but not uniformly) continuous one-parameter group of automorphisms of A [62]. Therefore, one may introduce a connection on a C^*-algebra bundle in terms of parallel transport curves and operators, but not their infinitesimal generators [15]. Moreover, a representation of A does not imply necessarily a unitary representation of its strongly (not uniformly) continuous one-parameter group of automorphisms. In contrast, connections on a Hilbert bundle over a smooth manifold can be defined both as particular first order differential operators on the module of its sections [296] and a parallel displacement along paths lifted from the base [228].

(iv) Instantwise geometric quantization of time-dependent mechanics is phrased in terms of Hilbert bundles over $\mathbb{R}$ [174; 401]. Holonomy operators in a Hilbert bundle with a structure finite-dimensional Lie group are well known to describe the non-Abelian geometric phase phenomena [44]. At present, holonomy operators in Hilbert bundles attract special attention in connection with quantum computation and control theory [159; 181; 349].

III.

Geometry in quantum systems speaks mainly the algebraic language of rings, modules and sheaves due to the fact that the basic ingredients in the differential calculus and differential geometry on smooth manifolds (except non-linear differential operators) can be restarted in a pure algebraic way.

(i) Any smooth real manifold X is homeomorphic to the real spectrum of the $\mathbb{R}$-ring $C^\infty(X)$ of smooth real functions on X provided with the Gelfand topology [17; 233]. Furthermore, the sheaf C^∞_X of germs of $f \in C^\infty(X)$ on this topological space fixes a unique smooth manifold structure on X such that it is the sheaf of smooth functions on X. The pair (X, C^∞_X) exemplifies a local-ringed space. A sheaf $\mathfrak{R}$ on a topological space X is said to be a local-ringed space if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a local commutative ring [414]. One can associate to any commutative ring $\mathcal{A}$ the particular local-ringed space, called an affine scheme, on the spectrum $\operatorname{Spec}\mathcal{A}$ of $\mathcal{A}$ endowed with the Zariski topology [421].

Furthermore, one can assign the following algebraic variety to any commutative finitely generated $\mathcal{K}$-ring $\mathcal{A}$ over an algebraically closed field $\mathcal{K}$. Given a ring $\mathcal{K}[x]$ of polynomials with coefficients in $\mathcal{K}$, let us consider the epimorphism $\phi : \mathcal{K}[x] \to \mathcal{A}$ defined by the equalities $\phi(x_i) = a_i$, where a_i are generating elements of $\mathcal{A}$. Zeros of polynomials in $\operatorname{Ker}\phi$ make up an algebraic variety $\mathcal{V}$ whose coordinate ring $\mathcal{K}_\mathcal{V}$ is exactly $\mathcal{A}$. The subvarieties of $\mathcal{V}$ constitute the system of closed sets of the Zariski topology on $\mathcal{V}$ [397].

Every affine variety $\mathcal{V}$ in turn yields the affine scheme Spec$\mathcal{K}_{\mathcal{V}}$ such that there is one-to-one correspondence between the points of Spec$\mathcal{K}_{\mathcal{V}}$ and the irreducible subvarieties of $\mathcal{V}$. For instance, complex algebraic varieties have a structure of complex analytic manifolds.

(ii) Given a (connected) compact topological space X and the ring $\mathbb{C}^0(X)$ of continuous complex functions on X, the well-known Serre–Swan theorem [409] states that a $\mathbb{C}^0(X)$-module is finitely generated projective if and only if it is isomorphic to the module of sections of some (topological) vector bundle over X. Moreover, this isomorphism is a categorial equivalence [237], and its variant takes place if X is locally compact [369]. If X is a compact smooth manifold, there is the similar isomorphism of a finitely generated projective $C^\infty(X)$-modules on X to the modules of sections of some smooth vector bundle over X [430], and this is also true if X is not necessarily compact. A variant of the Serre–Swan theorem for Hilbert modules over non-commutative C^*-algebras holds [239].

(iii) Let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ a commutative $\mathcal{K}$-ring, and P, Q some $\mathcal{A}$-modules. The $\mathcal{K}$-linear Q-valued differential operators on P can be defined [202; 233; 261]. The representative objects of the functors $Q \to \mathrm{Diff}_s(P,Q)$ are the jet modules $\mathcal{J}^sP$ of P. Using the first order jet module $\mathcal{J}^1P$, one also restarts the notion of a connection on an $\mathcal{A}$-module P [260; 296]. Such a connection assigns to each derivation $\tau \in \mathfrak{d}\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ a first order P-valued differential operator ∇_τ on P obeying the Leibniz rule

$$\nabla_\tau(ap) = \tau(a)p + a\nabla_\tau(p).$$

For instance, if P is a $C^\infty(X)$-module of sections of a smooth vector bundle $Y \to X$, we come to the familiar notions of a linear differential operator on Y, the jets of sections of $Y \to X$ and a linear connection on $Y \to X$. Similarly, connections on local-ringed spaces are introduced [296]. In supergeometry, connections on graded modules over a graded commutative ring and graded local-ringed spaces are defined [21].

In non-commutative geometry, different definitions of a differential operator on modules over a non-commutative ring have been suggested [50; 136; 286]. Roughly speaking, the difficulty lies in the fact that, if ∂ is a derivation of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$, the product $a\partial$, $a \in \mathcal{A}$, need not be so. There are also different definitions of a connection on modules over a non-commutative ring [137; 267].

(iv) Let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ a (commutative or non-commutative) $\mathcal{K}$-ring, and $\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$. Derivations of $\mathcal{A}$ make up a Lie $\mathcal{K}$-algebra $\mathfrak{d}\mathcal{A}$. Let us consider the Chevalley–Eilenberg complex of $\mathcal{K}$-multilinear morphisms of $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$, seen as a $\mathfrak{d}\mathcal{A}$-module [160; 426]. Its subcomplex $\mathcal{O}^*(\mathfrak{d}\mathcal{A}, d)$ of $\mathcal{Z}(\mathcal{A})$-multilinear morphisms is a differential graded algebra, called the Chevalley–Eilenberg differential calculus over $\mathcal{A}$. It contains the minimal differential calculus $\mathcal{O}^*\mathcal{A}$ generated by elements da, $a \in \mathcal{A}$. If $\mathcal{A}$ is the $\mathbb{R}$-ring $C^\infty(X)$ of smooth real functions on a smooth manifold X, the module $\mathfrak{d}C^\infty(X)$ of its derivations is the Lie algebra of vector fields on X and the Chevalley–Eilenberg differential calculus over $C^\infty(X)$ is exactly the algebra of exterior forms on a manifold X where the Chevalley–Eilenberg coboundary operator d coincides with the exterior differential, i.e., $\mathcal{O}^*(\mathfrak{d}C^\infty(X), d)$ is the familiar de Rham complex. In a general setting, one therefore can think of elements of the Chevalley–Eilenberg differential calculus $\mathcal{O}^k(\mathfrak{d}\mathcal{A}, d)$ over an algebra $\mathcal{A}$ as being differential forms over $\mathcal{A}$. Similarly, the Chevalley–Eilenberg differential calculus over a graded commutative ring is constructed [160].

IV.

As was mentioned above, homology and cohomology of spaces and algebraic structures often play a role of *sui generis* hidden quantization parameters which can characterize non-equivalent quantizations.

(i) First of all, let us mention the abstract de Rham theorem [220] and, as its corollary, the homomorphism

$$H^*(X, \mathbb{Z}) \to H^*(X)$$

of the Čech cohomology of a smooth manifold X to the de Rham cohomology of exterior forms on X. For instance, the Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ of a $U(n)$-principal bundle $P \to X$ are represented by the de Rham cohomology classes of certain characteristic exterior forms $\mathcal{P}_{2i}(F_A)$ on X expressed into the strength two-form F_A of a principal connection A on $P \to X$ [142]. The Chern class c_2 of a complex line bundle plays a prominent role in many quantization schemes, e.g., geometric quantization.

The well-known index theorem establishes the equality of the index of an elliptic operator on a fibre bundle to its topological index expressed in terms of the characteristic forms of the Chern character, Todd and Euler classes. Let us note that the classical index theorem deals with linear elliptic operators on compact manifolds. They are Fredholm operators. In order to generalize the index theorem to non-compact manifolds, one either imposes conditions sufficient to force operators to be the Fredholm ones or considers the operators which are no longer Fredholm, but their index can be interpreted as a real number by some kind of averaging procedure [375].

(ii) Geometric quantization of a symplectic manifold (X, Ω) is affected by the following ambiguity. Firstly, the equivalence classes of admissible connections on a prequantization bundle (whose curvature obeys the prequantization condition $R = i\Omega$) are indexed by the set of homomorphisms of the homotopy group $\pi_1(X)$ of X to $U(1)$ [257; 312]. Secondly, there are non-equivalent bundles of half-forms over X in general and, consequently, the non-equivalent quantization bundles exist [141]. This ambiguity leads to non-equivalent quantizations.

(iii) The cohomology analysis gives a rather complete picture of deformation quantization of symplectic manifolds. Let $\mathcal{K}$ be a commutative ring and $\mathcal{K}[[h]]$ the ring of formal series in a real parameter h. Let us recall that, given an associative (resp. Lie) algebra A over a commutative ring $\mathcal{K}$, its Gerstenhaber deformation [166] is an associative (resp. Lie) $\mathcal{K}[[h]]$-algebra A_h such that $A_h/hA_h \cong A$. The multiplication in A_h reads

$$a * b = a \circ b + \sum_{r=1}^{\infty} h^r C_r(a, b)$$

where $\circ$ is the original associative (resp. Lie) product and C_r are 2-cochains of the Hochschild (resp. Chevalley–Eilenberg) complex of A. The obstruction to the existence of a deformation of A lies in the third Hochschild (resp. Chevalley–Eilenberg) cohomology group.

Let $A = \mathbb{C}^\infty(X)$ be the ring of complex smooth functions on a smooth manifold X. One considers its associative deformations A_h where the cochains C_r are bidifferential operators of finite order. The multidifferential cochains make up a subcomplex of the Hochschild complex of A, and its cohomology equals the space of multi-vector fields on X [433]. If $\mathbb{C}^\infty(X)$ is provided with the standard Fréchet topology of compact convergence for all derivatives, one can consider its continuous deformation. The corresponding subcomplex of the Hochschild complex of A is proved to have the same cohomology as the differential one [332].

Let now X be a symplectic manifold, and let $A = \mathbb{C}^\infty(X)$ be the Poisson algebra. Since the Poisson bracket is a bidifferential operator of order (1,1), one has studied the similar deformations of A where the cochains C_r are differential operators of order (1,1) with no constant term. The cohomology of the corresponding subcomplex of the Chevalley–Eilenberg complex of A equals the de Rham cohomology $H^*(X)$ of X [280]. The equivalence classes of Poisson deformations of the Poisson bracket on a symplectic manifold X are parameterized by $H^2(X)[[h]]$. A star-product on a Poisson manifold is defined as an associative deformation of $\mathbb{C}^\infty(X)$ such that $C_1(f,f') - C_1(f',f)$ is the Poisson bracket. The existence of a star-product on an arbitrary symplectic manifold has been proved in [125], and this is true for any regular Poisson manifold [150; 302]. Moreover, any star-product on a symplectic manifold is equivalent to Fedosov's one, and its equivalence classes are parameterized by $H^2(X)[[h]]$ [206; 340].

(iv) Let us also mention BRST cohomology, called into play in order to describe constrained symplectic systems [156; 217; 259]. Let (Z,Ω) be a symplectic manifold endowed with a Hamiltonian action of a Lie group G, $\widehat{J}$ the corresponding momentum mapping of Z to the Lie coalgebra $\mathfrak{g}^*$ of G, and $N = \widehat{J}^{-1}(0)$ a regular constraint surface. The classical BRST complex is defined as the bicomplex

$$B^{n,m} = \overset{n}{\wedge}\mathfrak{g}^* \otimes \overset{m}{\wedge}\mathfrak{g} \otimes C^\infty(Z),$$

where the n- and m-gradings are the ghost and antighost degrees, respectively. The differential

$$\delta : B^{*,*} \to B^{*+1,*}$$

is the coboundary operator of the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ of G with coefficients in the $\mathfrak{g}$-module $\mathfrak{g} \otimes C^\infty(Z)$, while

$$\partial : B^{*,*} \to B^{*,*-1}$$

is the Koszul boundary operator. The algebra B is provided with the graded Poisson bracket $[,]$, and there exists an element Θ of B, called the BRST charge, such that $[\Theta,\Theta] = 0$ and $D = [\Theta,.] = \delta + \partial$ up to extra terms of non-zero ghost number is the nilpotent classical BRST operator. The BRST cohomology is defined as the cohomology of this classical BRST operator. The BRST complex has been built for constrained Poisson systems [245] and time-dependent Hamiltonian systems with Lagrangian constraints [295] as an extension of the Koszul–Tate complex of constraints through introduction of ghosts. Quantum BRST cohomology has been studied in the framework of geometric [419] and deformation [49] quantization.

V.

Contemporary quantum models appeal to a number of new algebraic structures and the associated geometric techniques.

(i) For instance, SUSY models deal with graded manifolds and different types of supermanifolds, namely, H^∞-, G^∞-, GH^∞-, G-supermanifolds over (finite) Grassmann algebras, R^∞- and R-supermanifolds over Arens–Michael algebras of Grassmann origin and the corresponding types of DeWitt supermanifolds [21; 22; 69]. Their geometries are phrased in terms of graded local-ringed spaces. Let us note that one usually considers supervector bundles over G-supermanifolds. Firstly, the category of these supervector bundles is equivalent to the category of locally free sheaves of finite rank (in contrast, e.g., with GH^∞-supermanifolds). Secondly, derivations of the structure sheaf of a G-supermanifold constitute a locally free sheaf (this is not the case, e.g., of G^∞-supermanifolds). Moreover, this sheaf is again a structure sheaf of some G-superbundle (in contrast with graded manifolds). At the same time, most of the quantum models uses graded manifolds. They are not supermanifolds, though there is the correspondence between graded manifolds and DeWitt H^∞-supermanifolds. By virtue of the well-known Batchelor theorem, the structure ring of any graded manifold with a body manifold Z is isomorphic to the graded ring $\mathcal{A}_E$ of sections of some exterior bundle $\wedge E^* \to Z$. In physical models, this isomorphism holds fixed from the beginning as a rule and, in fact, by geometry of a graded manifold is meant the geometry of the graded ring $\mathcal{A}_E$. For instance, the familiar differential calculus in graded exterior forms is the graded Chevalley–Eilenberg differential calculus over such a ring.

(ii) Non-commutative geometry is mainly developed as a generalization of the calculus in commutative rings of smooth functions [107; 194; 267]. In a general setting, any non-commutative $\mathcal{K}$-ring $\mathcal{A}$ over a commutative ring $\mathcal{K}$ can be called into play. One can consider the above mentioned Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$, differential operators and connections on $\mathcal{A}$-modules (but not their jets). If the derivation $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ is a finite projective module with respect to the center of $\mathcal{A}$, one can treat the triple $(\mathcal{A}, \mathfrak{d}\mathcal{A}, \mathcal{O}^*\mathcal{A})$ as a non-commutative space. For instance, this is the case of the matrix geometry, where $\mathcal{A}$ is the algebra of finite matrices, and of the quantum phase space, where $\mathcal{A}$ is a finite-dimensional algebra of canonical commutation relations. Non-commutative field theory also can be treated in this manner [133; 359], though the bracket of space coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

in this theory is also restarted from Moyal's star-product $x^\mu \star x^\nu$ [99; 133].

A different linear coordinate product

$$[x^\mu, x^\nu] = ic^{\mu\nu}_\lambda x^\lambda$$

comes from Connes' non-commutative geometry [195].

In Connes' non-commutative geometry, the more deep analogy to the case of commutative smooth function rings leads to the notion of a spectral triple $(\mathcal{A}, E, \mathcal{D})$ [107; 109]. It is given by an involutive subalgebra $\mathcal{A} \subset B(E)$ of bounded operators on a Hilbert space E and an (unbounded) self-adjoint operator $\mathcal{D}$ in E such that the resolvent $(\mathcal{D} - \lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a compact operator and $[\mathcal{D}, \mathcal{A}] \subset B(E)$. Furthermore, one assigns to elements $\omega = a_0 da_1 \cdots da_k$ of the universal differential calculus $(\mathcal{O}^*\mathcal{A}, d)$ over $\mathcal{A}$ the operators

$$\pi(\omega) = a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]$$

in E. This however fails to be a representation of the differential algebra $\mathcal{O}^*\mathcal{A}$ because $\pi(\omega) = 0$ does not imply $\pi(d\omega) = 0$. The appropriate quotient

$$\mathcal{O}^*\mathcal{A} \ni \phi \to [\phi] \in \mathcal{O}^*_\mathcal{D}$$

together with the differential $d[\omega] := [d\omega]$ overcomes this difficulty, though $\pi([\omega])$ is not an operator in E. Let us note that other variants of spectral data, besides a spectral triple, are also discussed [158]. The algebra $C^\infty(X)$ and the Dirac operator $\mathcal{D}$ on a compact manifold X exemplifies Connes' commutative geometry [108; 368]. Spectral triples have been studied for non-commutative tori, the Moyal deformations of $\mathbb{R}^n$, non-commutative spheres 2-, 3- and 4-spheres [95; 110], and quantum Heisenberg manifolds [96].

(iii) Formalism of groupoids provides the above mentioned categorial C^*-algebraic deformation quantization of some class of Poisson manifolds [270; 271]. A groupoid is a small category whose morphisms are invertible [287; 367]. For instance, given an action of a group G on a set X on the right, the product $\mathfrak{G} = X \times G$ is brought into the action groupoid where:

- a pair $((x, g), (x', g'))$ is composable if and only if $x' = xg$,
- the inversion $(x, g)^{-1} = (xg, g^{-1})$,
- the partial multiplication $(x, g)(xg, g') = (x, gg')$,
- the range $r((x, g)) = (x, 1_G)$,
- the domain $l((x, g)) := (xg, 1_G)$.

The unit space $\mathfrak{G}^0 = r(\mathfrak{G}) = l(\mathfrak{G})$ of this groupoid is naturally identified to X. Any group bundle $Y \to X$ (e.g., a vector bundle) is a groupoid whose elements make up composable pairs if and only if they belong to the same fibre, and whose unit space is the set of unit elements of fibres of $Y \to X$. Let $\mathfrak{A} \to \mathfrak{G}^0$ be an Abelian group bundle over the unit space $\mathfrak{G}^0$ of a groupoid $\mathfrak{G}$. The pair $(\mathfrak{G}, \mathfrak{A})$ together with a homomorphism $\mathfrak{G} \to \operatorname{Iso}\mathfrak{A}$ is called the $\mathfrak{G}$-module bundle. One can associate to any $\mathfrak{G}$-module bundle a cochain complex $C^*(\mathfrak{G}, \mathfrak{A})$. Let $\mathfrak{A}$ be a $\mathfrak{G}$-module bundle in groups $U(1)$. The key point is that, similarly to the case of a locally compact group [129], one can associate a C^*-algebra $C^*(\mathfrak{G}, \sigma)$ to any locally compact groupoid $\mathfrak{G}$ provided with a Haar system by means of the choice of a two-cocycle $\sigma \in C^2(\mathfrak{G}, \mathfrak{A})$ [367]. The algebras $C^*(\mathfrak{G}, \sigma)$ and $C^*(\mathfrak{G}, \sigma')$ are isomorphic if σ and σ' are cohomology equivalent. If $\mathfrak{G}$ is an r-discrete groupoid, any measure λ of total mass 1 on its unit space $\mathfrak{G}^0$ induces a state of the C^*-algebra $C^*(\mathfrak{G}, \sigma)$.

A Lie groupoid is a groupoid for which $\mathfrak{G}$ and $\mathfrak{G}^0$ are smooth manifolds, the inversion and partial multiplication are smooth, while r and l are fibred manifolds. Since a Lie groupoid admits a Haar system, one can assign to it a C^*-algebra $C^*(\mathfrak{G})$. This assignment is functorial if certain classes of morphisms of Lie groupoids and C^*-algebras (isomorphism classes of regular bibundles and those of Hilbert bimodules, respectively) are considered [270].

A Lie groupoid is called symplectic if it is a symplectic manifold $(\mathfrak{G}, \Omega)$ such that the multiplication relation

$$(x, y) \to (xy, x, y)$$

is a Lagrangian submanifold of the symplectic manifold

$$(\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}, \Omega \ominus \Omega \ominus \Omega)$$

[27]. A Poisson manifold P is called integrable if there exists a symplectic groupoid $\mathfrak{G}(P)$ over P. It is unique up to an isomorphism. Integrable Poisson manifolds subject to a certain class morphisms (isomorphism classes of regular dual pairs) make up a suitable category **Poisson** [270]. Since the groupoid $\mathfrak{G}(P)$ is l- and l-simple connected, one considers the category $\mathbf{L\tilde{G}}$ of Lie groupoids possessing this property. Any Lie groupoid yields an associated Lie algebroid $A(\mathfrak{G})$ which is the restriction to $\mathfrak{G}^0$ of the vertical tangent bundle of the fibration $r : \mathfrak{G} \to \mathfrak{G}^0$ [287]. The key point is that, similarly to the dual of a Lie algebra, the dual $A^*(\mathfrak{G})$ of $A(\mathfrak{G})$ is a Poisson manifold. Then the assignment $\mathfrak{G} \mapsto A^*(\mathfrak{G})$ is a functor from $\mathbf{L\tilde{G}}$ to **Poisson** [271]. Let **LPoisson** denote its image. One can show that

$$\Upsilon : L^*(\mathfrak{G}) \mapsto C^*(\mathfrak{G})$$

is a functor from the category **LPoisson** to the above mentioned category of C^*-algebras [271]. It is a desired functorial quantization. This functor is equivariant under the Morita equivalence of Poisson manifolds in $\mathbf{L\tilde{G}}$ [444] and that of C^*-algebras [371]. Furthermore, the functorial quantization $A^*(\mathfrak{G}) \mapsto C^*(\mathfrak{G})$ is amplified into the above mentioned strict quantization of $C^*(\mathfrak{G})$ by an appropriate continuous field of C^*-algebras over $\mathbb{R}$ [271]. Connes' tangent groupoid provides an example of such strict quantization [269].

(iv) Hopf algebras and, in particular, quantum groups make a contribution to many quantum theories [97; 249; 292; 293]. At the same time, the development of differential calculus and differential geometry over these algebras has met difficulties. Given a (complex or real) Hopf algebra $H = (H, m, \Delta, \epsilon, S)$, one introduces the first order differential calculus (henceforth FODC) (Ω^1, d) over H just as for a non-commutative algebra. It is said to be left-covariant if Ω^1 possesses the structure of a left H-comodule

$$\Delta_l : \Omega^1 \to H \otimes \Omega^1$$

such that

$$\Delta_l(adb) = \Delta(a)(\mathrm{Id} \otimes d)\Delta(b), \qquad a, b \in H,$$

[249]. By virtue of Woronowicz's theorem [440], left-covariant FODCs are classified by right ideals

$$\mathcal{R} = \{x \in \operatorname{Ker} \epsilon \,:\, S(x_1)dx_2 = 0,\ x = \sum x_1 \otimes x_2)$$

of H contained in the kernel of its counit ϵ. The linear subspace

$$T = \{\chi \in H^* \,:\, \chi(1) = 0,\ \chi(\mathcal{R}) = 0\}$$

of the dual H^* is the quantum (enveloping) Lie algebra (quantum tangent space [214]) associated to the left-covariant FODC (Ω^1, d) (see [286] for a general construction of the enveloping algebra for a non-commutative FODC). A problem lies in the definition of vector fields as a sum $a^i u_i$ of invariant vector fields $u_i(a) = a_1\chi_i(a_2)$ [13] because they satisfy the deformed Leibniz rule deduced from the formula

$$\chi_i(ab) = \chi_i(a)\epsilon(b) + \sum_i f_{ij}(a)\chi_j(b),$$

where $\{\chi_i\}$ is a basis for T and f_{ij} are complex linear forms on H. One can model the vector fields obeying such a Leibniz rule by the so called Cartan pairs [50]. These are elements u of the right H-dual Ω^1 together with the morphisms

$$\widehat{u}: H \ni a \mapsto = u(da) \in H$$

which obey the relations

$$\widehat{(bu)}(a) = b\widehat{u}(a), \qquad \widehat{u}(ba) = \widehat{u}(b)a + \widehat{(ub)}(a).$$

Another problem of geometry of Hopf algebras is the notion of a quantum principal bundle [75; 82; 293]. In the case of Lie groups, there are two equivalent definitions of a smooth principal bundle, which is both a set of trivial bundles glued together by means of transition functions and a bundle provided with the canonical action of a structure group on the right. In the case of quantum groups, these two notions of a principal bundle are not matched, unless the base is a smooth manifold [139; 355].

- The first definition of a quantum principal bundle repeats the classical one and makes use of the notion of a trivial quantum bundle, a covering of a quantum space (e.g., by a family of non-intersecting closed ideals), and its reconstruction from local pieces [76] which however is not always possible [81].
- The second definition of a quantum principal bundle is algebraic [74; 293]. Let $\mathcal{H}$ be a Hopf algebra and $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra with respect to the coaction $\beta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$. Let

$$M = \{p \in \mathcal{P} \ : \ \beta(p) = p \otimes \mathbf{1}\}$$

be its invariant subalgebra. The triple $(\mathcal{P}, \mathcal{H}, \beta)$ is called a quantum principal bundle if the map

$$\mathrm{ver} : \mathcal{P} \underset{M}{\otimes} \mathcal{P} \ni (p \underset{M}{\otimes} q) \mapsto p\beta(q) \in \mathcal{P} \otimes \mathcal{H}$$

is a linear isomorphisms. This condition, called the Hopf–Galois condition, is a key point of this algebraic definition of a quantum principal bundle. By some reasons, one can think of it as being a *sui generis* local trivialization.

(v) Finally, one of the main point of Tamarkin's proof of the formality theorem in deformation quantization is that, for any algebra $\mathcal{A}$ over a field of characteristic zero, its Hochschild cochain complex and its Hochschild cohomology are algebras over the same operad [219; 411]. This observation has been the starting point of 'operad renaissance' [253; 297]. Monoidal categories provide numerous examples of algebras for operads. Furthermore, homotopy monoidal categories lead to the notion of a homotopy monoidal algebra for an operad. In a general setting, one considers homotopy algebras and weakened algebraic structures where, e.g., a product operation is associative up to homotopy [276]. Their well-known examples are A_∞-spaces and A_∞-algebras [403]. At the same time, the formality theorem is also applied to quantization of several algebraic geometric structures such as algebraic varieties [255; 450].