Chapter 1

Introduction

O brave new worlds, that have such people in them!
— E.A. Abbott in *Flatland*

Many of us have wondered some time or the other if one can have nontrivial science and technology in two space dimensions; but the usual feeling is that two space dimensions do not offer enough scope for it. This question, to the best of my knowledge, was first addressed in 1884 by E.A. Abbot in his satirical novel *Flatland* [1]. The first serious book on this topic appeared in 1907 entitled *An episode of Flatland* [2]. In this book C.H. Hinton offered glimpses of the possible science and technology in the flatland. A nice summary of these two books appeared as a chapter entitled *Flatland* in a book in 1969 edited by Martin Gardner [3]. Inspired by this summary, in 1979 A.K. Dewdney [4] published a book which contains several laws of physics, chemistry, astronomy and biology in the flatland. However, all these people missed one important case where physical laws are much more complex, nontrivial and hence interesting in the flatland than in our three dimensional world. I am referring here to the case of quantum statistics. In last two decades it has been realized that whereas in three and higher space dimensions all particles must either be bosons or fermions (i.e. they must have spin of $n\hbar$ or $(2n+1)\hbar/2$ with $n = 0, 1, 2, ...$ and must obey Bose-Einstein or Fermi-Dirac statistics respectively), in two space dimensions the particles can have any fractional spin and can satisfy *any* fractional statistics which is interpolating between the two. The particles obeying such statistics are generically called as *anyons*. In other words, if one takes one anyon slowly around the other then in general the phase acquired is $\exp(\pm i\theta)$. If $\theta = 0$ or π (modulo 2π) then the particles are bosons or fermions respectively while if $0 < \theta < \pi$ then the particles are termed as anyons.

In this book I plan to explore in detail the various facets of anyons. Before I go into the details, one might wonder if our discussion is merely of academic interest? The answer to the question is no. In fact it is a surprising fact that two, one and even zero dimensional experimental physics is possible in our three-dimensional world. A few lines of digression are called for here to explain how this is possible in our three dimensional world. The point is that because of the third law of thermodynamics, which states that all the degrees of freedom freeze out in the limit of zero temperature, it is possible to strictly confine the electrons to surfaces, or even to lines or points. Thus it may happen that in a strongly confining potential, or at sufficiently low temperatures, the excitation energy in one or more directions may be much higher than the average thermal energy of the particles, so that those dimensions are effectively frozen out. An illustration might be worth while here. Consider a two dimensional electron gas on which the first experiment was in fact done in 1966 [5]. The electrons are confined to the surface of a semiconductor by a strong electric field, and they move more or less freely along the surface. On the other hand, the energy E required to excite motion in the direction perpendicular to the surface is of the order of several milli-electron-Volt (meV). Now at a temperature of say $T = 1K$, the thermal energy is kT, where k is the Boltzmann constant. Thus if the transverse excitation energy is say 10 meV, then the fraction of electrons in the lowest excited transverse energy level is

$$e^{-\frac{E}{kT}} = e^{-100} \approx 10^{-44}. \tag{1.1}$$

which is zero for all practical purposes. Thus the electron gas is truly a two-dimensional gas. Few examples where planar experimental physics is possible are electron gas, surface layer studies and copper-oxide materials. Of course, even there, at the most basic level, the fundamental particles are certainly fermions or bosons. However, the most direct and appropriate discussion of the low energy behavior of a material is usually in terms of the quasi-particles. The hope is that at least in some of these cases the quasi-particles could be anyons. This hope has in fact been realized in the case of the fractionally quantized Hall effect where the quasi-particles are believed to be charged vortices i.e. charged anyons [6,7,8]. Three rather different experiments [9] seem to confirm the existence of fractionally charged excitations and hence indirectly of anyons.

There is another place where, at one stage many believed that anyons could play a major role. I have in mind here the high-T_c superconductors. To date the mechanism of superconductivity in these high-T_c materials is not known. Few years ago, several people were excited by the suggestion that anyons could provide the mechanism for superconductivity in these materials [10]. It soon turned out that these models provide a unique test of these ideas. In particular, they predicted the violation of the discrete symmetries of parity and time reversal invariance in these materials. Unfortunately the experiments performed in several laboratories [11,12] failed to observe the parity and the time reversal violation in the high-T_c superconductors. While these experiments have certainly dampened the interest of the physics community in anyons, they also showed that the anyon ideas are not merely esoteric and have experimental consequences which could be tested in the laboratories.

Another reason why I believe that the anyons would have relevance to the real world is because of the unwritten first law of physics which states that 'anything that is not forbidden is compulsory!'. Finally, anyons represent a challenge to all those people who think that they *know* quantum mechanics and statistical mechanics and that they could have contributed significantly to the development of these fields if only they had been born 60-70 years earlier!

At this stage it might be worthwhile to give a short historical review of the field. By its very nature, such a review is bound to be subjective and I apologize in advance to those authors who may feel that their contribution has not been given its due credit.

1.1 Historical Review

The concept of the indistinguishability of the identical particles has a deep meaning in quantum mechanics. Actually, this concept was introduced by John Willard Gibbs in classical statistical mechanics, much before the advent of quantum mechanics in order to resolve the famous Gibbs paradox. However, its ramifications are far deeper in quantum mechanics vis a vis the classical mechanics. For example, it was realized quite early that in quantum mechanics, the identical particles always interact simply because they are identical. As a result, the physical behavior of a collection of identical particles is influenced not only by the conventional interactions but also by the statistics they obey. In particular, it was realized that there are two kinds of quantum statistics in nature. It was shown that all particles have either half-integral or integral spin (in units of the Planck constant $\hbar$) and accordingly they satisfy Bose-Einstein [13,14] or Fermi-

Dirac [15,16] statistics respectively. It was also soon realized that there is an effective attraction between the bosons and an effective repulsion between the fermions [17,18], both of which are purely quantum mechanical in nature, and are refereed to as statistical interaction. It may be noted that it is this statistical attraction which gives rise to the accumulation of the Bose particles in the ground state which is at the heart of the phenomenon of Bose condensation. Similarly, it is the statistical repulsion between the fermions which gives rise to the famous Pauli exclusion principle. In fact, the stability of matter very much depends on the fermionic nature of the matter. Recall that according to our current picture, all matter in nature consists of quarks and leptons which are fermions.

The question that one wants to ask is whether Bose-Einstein and Fermi-Dirac are the only possible forms of quantum statistics in nature? For almost fifty years, it was believed that the answer to this question is yes. To understand why, let us go back in history a little bit. Immediately after quantum mechanics was formulated by Schrödinger and Heisenberg as it is known today, Heisenberg and Dirac extended the theory to systems of identical particles [16,19,20]. Their key observation was that all operators representing observables in such systems are necessarily symmetric under the interchange of the particle labels, if the particles are really indistinguishable. This statement has profound consequences since the symmetry operators preserve the symmetry of the wave functions. Clearly, if the operator O and the wave function ψ are both totally symmetric, then $O\psi$ is also totally symmetric while if O is symmetric but ψ is anti-symmetric, then $O\psi$ is totally anti-symmetric. This immediately explained as to why one has quantum theories of identical particles using only symmetric or only anti-symmetric wave functions. Since several consequences of both the Bose-Einstein and the Fermi-Dirac statistics were soon experimentally verified, hence no one really bothered to construct a more satisfactory theory.

The Heisenberg-Dirac theory, even though experimentally so successful, could however be questioned on philosophical ground. Consider for example the case when two particles are so far apart that they cannot be physically interchanged. Then, clearly, it does not matter if we symmetrize or anti-symmetrize the wave functions or do neither! So the question arises whether there is some postulate which is more fundamental than the symmetrization or anti-symmetrization postulate. Strictly speaking, the interchange of particle labels is a slightly misleading concept. If the particles are strictly identical, then an interchange of the identical particles is obviously an identity transformation. Now in quantum mechanics it is not uncommon that a physical identity transformation may be represented mathematically by a phase factor. As is well known, any permutation of bosons gives the trivial phase factor of $+1$ while even and odd permutation of fermions gives the phase factor of $+1$ or -1 respectively. An obvious question is, can we also have a more general complex phase factor instead of just $+1$ or -1?

This question was partly answered by Laidlaw and DeWitt [21] in 1971. They applied the Feynman path integral formalism to systems of identical particles. Note that in the path integral formalism, the interchange of identical particles has a clear physical meaning as a continuous process in which each particle moves along a continuous path. The path dependence of the interchange is all important since it relates the quantum mechanical concept of particle statistics to the topology of the configuration space. The phase factors associated with different interchange paths must define a representation of the first homotopy group of the configuration space. Unfortunately, Laidlaw and DeWitt confined their attention to only three dimensions and hence concluded that only bosons and fermions can exist thereby missing the more exotic possibilities in two space dimensions.

In 1977, using a more traditional approach to quantization, Leinaas and Myrheim [22] derived the same relationship between the particle statistics and topology but were bold enough to enquire about the possible quantum statistics in two dimensions. Their approach was based on the geometrical interpretation of wave functions which is the basis of the modern day gauge theories. They showed that in two dimensions, the space is multiply connected which results in the possibility of what they termed as the *intermediate statistics*. In particular, they showed that there exists a continuously variable parameter, which one can choose to be the phase angle θ (or $\alpha = \theta/\pi$), that characterizes different statistics: α equal to 0 or 1 corresponds to bosons or fermions respectively while $0 < \alpha < 1$ corresponds to more exotic possibilities. It may be noted that in principle, α can be a rational or an irrational number. As an illustration of their ideas, they explicitly solved the spectrum of two such exotic particles in a two-dimensional harmonic oscillator potential, and showed that as α goes from zero to 1, there is a continuous interpolation between the bosonic and the fermionic spectra. This simple calculation also showed that even the two-anyon spectrum is not related to the corresponding single particle energy levels thereby suggesting that even the simplest problem of a noninteracting anyon gas may not be easy to solve.

A clarification is in order at this stage. It should be noted here that the intermediate (or anyonic) statistics that we are talking about has nothing to do with para statistics. As we shall see in detail, while para statistics can exist in any dimension and correspond to higher dimensional representation of the permutation group, anyons can exist in only two space dimensions and the underlying group is the Braid group.

Few years later, Goldin, Menikoff and Sharp obtained the same results by an entirely different approach [23]. They studied the representations of the commutator algebra of of particle density and current operators. This algebra has commutation relations that are independent of the particle statistics, but the inequivalent representations correspond to different statistics.

It is fair to say that the idea of the intermediate statistics (in two space dimensions) did not receive enough attention in the physics community till the papers of Wilczek [24]. It is he who coined the name *anyons* for the two-dimensional identical particles obeying the intermediate statistics and since then it is being called as the anyonic or more generally as the *fractional statistics*. Apart from proposing the name, wilczek's main contribution was the flux-tube model for anyons in which anyons turn out to be point particles having both the electric charge and the magnetic flux i.e. they are point charged vortices. Wilczek also clearly spelled out the concept of statistical transmutation i.e. the fact that one can treat the noninteracting anyons as interacting bosons or interacting fermions. This idea has proved extremely useful in trying to work out the statistical mechanics of an ideal gas of anyons by treating it as an interacting Bose (or Fermi) gas. Now it is well known that the interacting Bose or Fermi gas problems are notoriously difficult and so this analogy is extremely useful in appreciating the difficulties involved in understanding even the seemingly simple problem of the noninteracting anyon gas. The flux-tube model also showed that the statistical interaction between the anyons is a vector and not a scalar long ranged interaction. This is a very important point because if it would have been a scalar long ranged interaction, going like $1/r$ at long distance, then in fact the virial expansion would not have existed for the noninteracting anyon gas!

Around the same time, Wilczek and Zee [25] constructed a model for neutral, extended anyons within the relativistic field theory formalism. In particular, they considered the $O(3)$ σ-model in 2+1 dimensions and showed that the solitons of this model acquire fractional spin and statistics in the presence of the Hopf-term which is an incarnation of the Chern-Simons term. This was an important development as the questions like the spin-statistics theorem can be handled rigorously, only within the formalism of the relativistic quantum field theory. Soon afterwords, Wu and Zee [26] showed that the same (extended) anyon solutions can also be constructed within the CP^1 model. The advantage over the σ model is that, in this case, the Hopf term can be written in a local form in terms of the CP^1 fields. It must be added here that the relativistic anyons are invariably extended objects and hence it is extremely difficult to do any calculations with them. Thus so far as the application of anyons to the real world is concerned, we shall mostly be using the non-relativistic flux tube model of anyons in which anyons are treated as point objects.

In 1983, the anyon ideas received a tremendous boost when it was realized that these ideas are not merely esoteric but can have applications in the real world. I have in mind the application in the context of the Fractional quantum Hall effect which was experimentally discovered in 1982 [27], soon after the discovery of the Integer quantum Hall effect [28]. It was Laughlin who proposed an explanation of the fractional quantum Hall effect [6]. According to him, the fractional quantization of the Hall resistance is the manifestation of a new state of matter, the incompressible quantum fluid, with elementary excitations which have fractional charge. Whereas Laughlin [6] and Haldane [7] suggested that these elementary excitations are fermions and bosons respectively, it was Halperin who correctly suggested that they are in fact anyons [8]. By using Berry phase calculations, soon it was proved by Arovas, Schrieffer and Wilczek [29] that these excitations indeed carry fractional charge and obey fractional statistics. Over the years, a number of improved theories of the fractional quantum Hall effect have been constructed [30], but all of them agree to the basic fact that the elementary excitations in the quantum Hall effect are indeed anyons. It might be added here that three rather different experiments seem to confirm the existence of the fractionally charged excitations [9] and hence indirectly of anyons.

In 1984, Wu [31] emphasized the fact that whereas the first homotopy group of the configuration space of identical particles in three and higher dimensions is the permutation group S_N, in two dimensions the corresponding group is the braid group B_N. It is amusing to note that the mathematicians arrived at exactly the same configuration space concept from the opposite direction, namely as a useful tool for studying the braid group [32]. Wu also made another important contribution [33]. He wrote down a class of exact solutions for three anyons in an external harmonic oscillator potential. In particular, he showed that an exact anyon state starting from the three-boson ground state *does not interpolate* to the three-fermion ground state but interpolates to an excited state so that there must be a crossing in the ground state of the three anyons. This was a remarkable result because it immediately showed that unlike the two-anyon case (where the ground state energy smoothly interpolated from the bosonic to the fermionic end), in the three-anyon case the three-body potential between the anyons must be playing a nontrivial role. This gave the first hint that the multi-anyon problems are going to be highly nontrivial.

The next obvious question to investigate was the behavior of an ideal gas of anyons. This is a kind of bench-mark study which is an absolute must before one can take into account the effect of interactions. Let us recall here that a similar study for an ideal Bose and Fermi gas was done right in the early days of quantum statistical mechanics. Of course that was easily done since the wave function for N-bosons or N-fermions is merely the product of the single particle wave functions, but with appropriate symmetry or anti-symmetry factors. Unfortunately, as was clear from the seminal paper of Leinaas and Myrheim [22], even the two anyon spectrum in an oscillator potential had no correlation with the single particle spectrum. The first step towards determining the equation of state of an anyon gas was taken by Arovas, Schrieffer, Wilczek and Zee [34,35] who calculated the second virial coefficient of a noninteracting anyon gas by treating it as an interacting Bose gas and showed that it has cusps at the bosonic values of $\theta = 2n\pi$. In particular, they considered two anyons in a circular box with hard walls. Few years later, Comtet, Georgelin and Ouvry [36] simplified the calculation by confining the particles in an external harmonic potential in the same way as Fermi had done for fermions [15], and essentially using the spectrum derived earlier by Leinaas and Myrheim. This calculation clearly showed the regularization independence of the second virial coefficient. In fact Arovas et al. [34] have also calculated the second virial coefficient by path integral methods without using any regulator (since there is no need to impose any finite area restriction) and of course all the methods give the same answer. The fact that the second virial coefficient of the noninteracting anyon gas turned out to be finite is a nontrivial statement since the noninteracting anyons experience a long ranged but *vector* interaction and it is not at all obvious that a cluster expansion should exist for the noninteracting anyon gas.. In this paper Arovas et al. also showed that one way to impart fractional statistics to the particles is to couple them with the Chern-Simmons term.

In 1986, Samir Paul and myself [37] considered an abelian Higgs model with the Chern-Simons term [38] in $2+1$ dimensions and showed that this model admits charged vortices of finite energy (in $2+1$ dimensions). As an extra bonus, we found that these vortices had non-zero angular momentum which in general could take any arbitrary value. This strongly suggested that these could in fact be extended charged anyons. This was rigorously proved by Fröhlich and Marchetti [39]. Thus this is the first field theory model for an extended charged anyon thereby generalizing the point charged vortex model of Wilczek. Sometimes later, Jatkar and I [40] showed that the abelian Higgs model with pure Chern-Simons term also admits charged vortex (i.e. extended anyon) solutions. This was important from condensed matter point of view since at long distance, the Chern-Simons term having only one derivative is expected to dominate over the Maxwell term which has two derivatives. Soon afterward, two groups simultaneously obtained the self-dual pure Chern-Simons vortex solutions [41]. The interesting point is that these are the noninteracting, though extended vortices. The other interesting point was that for the first time, one had found simultaneously the topological as well as the non-topological [42,43] self-dual solutions in a theory. Jackiw and Pi [44] considered the non-relativistic limit of this abelian Higgs model with pure Chern-Simons term and showed that this non-relativistic theory which can also be looked upon as an N-body problem with an attractive delta function interaction, admits self-dual charged vortex (i.e. anyon) solutions. The important point is that at the self dual point, the vortices are non-interacting though extended. Around the same time, Jackiw and Nair [45] obtained a wave equation for anyons.

Around 1988, Laughlin suggested that perhaps anyons could provide a mechanism for the high-T_c superconductors [10]. This really attracted the attention of the physics community to the ideas of fractional statistics. Two issues are involved here. Firstly, whether anyons indeed provide mechanism for superfluidity and superconductivity? Secondly, are the high-T_c superconductors anyonic in nature? It is the second question which attracted a lot of attention since we do not know the mechanism of high-T_c superconductivity even till today. Now as we have seen, anyons must necessarily violate the discrete symmetries of parity (P) and the time reversal invariance (T). It soon became clear that if anyons have anything to do with the high-T_c superconductors, then one must be able to see such a P and T violation in these materials. Unfortunately, precision measurements showed that these materials do not really show [11,12] such P and T violation. This dramatically reduced the enthusiasm of many people towards the fractional statistics. I must however add that in a way even this negative result is interesting because it shows that the anyon ideas are not esoteric but they could be tested in laboratories. After all physics is an experimental science and so a theory is useless unless it has some experimental consequences which can be tested in the near future. As far as the first issue is concerned, various calculations strongly suggest that anyons can certainly provide mechanism for superfluidity and hence superconductivity. What is not clear is if nature has made use of this mechanism in some superconductors.

In 1989, Comtet and Ouvry wrote an interesting paper [46] where they showed that the second virial coefficient of a noninteracting anyon gas is related to the axial anomaly in a 1+1 dimensional field theory. As soon as we saw this paper, it occured to us that perhaps the semi-classical approximation is also exact for the second virial coefficient. The point is, it is well known that the chiral anomaly gets contribution from only one loop (i.e. lowest order term in the semi-classical approximation) and since it is related to the second virial coefficient, hence that must also be one loop exact. This was subsequently proved by us [47]. The attractive point about the semi-classical approximation is that it does not require a knowledge of the quantum spectrum, which, in most cases including anyons, is very difficult to find. Besides, since there is no other scale in the problem apart from the thermal wave length, and further, since the semi-classical approximation for the virial expansion must be exact at very large temperatures, hence I believe that the semi-classical approximation must be exact for all the virial coefficients of the non-interacting anyon gas. Of course, it is not at all clear if one can really compute the higher virial coefficients within the semi-classical approximation.

Around 1991, a lot of work was done regarding the multi-anyon spectrum in a harmonic oscillator potential. Several people generalized the exact solutions of Wu [33] for three anyons and obtained a class of exact solutions for N-anyons in an oscillator potential [48] as well as in a uniform magnetic field [49] for all of which the energy varies linearly with the anyon parameter α. It soon became clear that these linear solutions form only a small subset among the full class of solutions, and that for the missing solutions, the energy must vary nonlinearly with the anyon parameter α. It is unfortunate that till today not even one of these nonlinear state is known analytically. It might be added here that recently a class of exact solutions for N-anyons have also been obtained in an N-body potential for which energy does not vary linearly with the anyon parameter α. It turns out that for all these levels, it is not the energy but $E^{-1/2}$ which is linear in the anyon parameter [50]. The perturbative [51], numerical [52] and semi-classical [53] studies have revealed several unusual and interesting features of the N-anyon spectrum in an oscillator potential as well as in a uniform magnetic field. For example, it is found that unlike the two-anyon case, the effective interaction around the three-fermion ground state is *repulsive* and not attractive as one had thought. As a result the ground state energy of three-anyons is not maximum at the fermionic end but at around $\alpha \equiv \theta/\pi = 0.71$. This has defied explanation so far. By now it is known that the same thing happens in the case of $6, 10, 15, ...$ anyons i.e. one finds that the effective interaction near the ground state of N-fermions very much depends on the value of N. In particular, so long as N is such that the fermions form closed shells (as happens for $N = 3, 6, 10...$), the effective interaction is *repulsive* while in all other cases it is *attractive.* Further, it has been shown that for large N, there must be at least $\sqrt{N}$ number of crossings in the ground state of N-anyons in a harmonic oscillator potential [54]. The fact that the N-anyon interaction around the fermionic ground state depends so sensitively on whether the shells are closed or not raises doubts about the validity of the approximate schemes like mean field theory which do not take into account this important fact.

At around the same time, there was also lot of activity regarding the third and higher virial coefficients of a noninteracting anyon gas. It was proved that unlike the second virial case, none of the higher virial coefficients contain a term which is linear in the anyon parameter α [55]. The semi-classical calculation of the third virial coefficient was also performed [47] which indicated that it is symmetric around $\alpha = 1/2$, i.e. $\theta = \pi/2$. The same symmetry was also apparent in the spectra of three anyons in an oscillator potential. Inspired by these observations, Sen [56] rigorously proved the symmetry of the third virial coefficient around $\alpha = 1/2$. Soon afterward, two groups [57] independently computed the third to sixth virial coefficients to second order in the anyon parameter. From the fourth virial coefficient onward, these expressions are quite complicated, having logarithms of algebraic numbers. It thus appears that the virial expansion for a noninteracting anyon gas may have a very complicated expansion in the power of the anyon parameter.

Even though so much work has been done in this field in the last two decades, I still feel that this area is still in its early developmental stage and it is not completely clear as to what direction this area will take in the future. This is because even the most basic problem of the statistical mechanics of a noninteracting anyon gas is still an unsolved problem. I strongly believe that unless one can solve this basic problem, no qualitative progress is possible in this field since in the absence of this bench-mark study, any calculation including interactions will always be unreliable.

As is clear from this (subjective) review of the field, several aspects of anyons have been explored in great detail in the last two decades and it is almost impossible to cover all these topics in this monograph. I have therefore, decided that instead of pretending to be objective, it is better to cover only those topics which I believe to be important. I have, however, included in Chapter 11 a brief description of the omitted topics and given a few references for each of these topics so that the interested reader can go back and trace other references and study these topics further. I have been fortunate in the sense that many of these topics have been adequately covered in the literature. I must also apologize to several authors whose work may not have been adequately quoted in the monograph in spite of my best attempts.

1.2 Plan of the Book

This book can be broadly divided into two parts. The first part has five chapters of which the first four are devoted to the various aspects of the non-relativistic model of anyons, while in the fifth I discuss the fractional exclusion statistics. The first part of the book can be understood by any one having a reasonable background in quantum and statistical mechanics. In particular no knowledge of many-body theory or quantum field theory is required in understanding this part. The second part also has five chapters and is at a slightly advanced level. A first course in quantum field theory and many-body theory is a prerequisite for really appreciating this part. In this part, I discuss the field theory models for anyons. In particular the role of the Chern-Simons term, which naturally provides a model for anyons, is discussed at length. Fractional quantum Hall effect and anyon superconductivity are also discussed in some detail. Finally, in Chapter 11, I give a brief description of the topics which have not been included in the book and give few references for each of these topics.

When one starts talking about anyons, the first obvious question is what are anyons and why are they possible only in two space dimensions and not in higher ones? This question is discussed at length in Chapter 2 where we shall see that whereas in two dimensions, the configuration space is multi-valued, it is only double-valued in three and higher space dimensions. A flux-tube model of anyon is also introduced here according to which an anyon can be looked upon as a point particle with a flux tube attached at its sight. This has an immediate consequence that one can freely transmute anyonic statistics with interaction. In particular, an ideal anyon gas is equivalent to an interacting Bose (or Fermi) gas. This also explains as to why even the problem of a non-interacting anyon gas is so difficult. It turns out that the group structure underlying the anyons is in fact the braid group. An elementary introduction to the braid group is also given in this chapter.

While talking about the new quantum statistics, the first obvious question is about the distribution function of an ideal gas of anyons. One would also like to know things like partition function and equation of state of the ideal gas. As a first step in that direction, the quantum mechanics of multi-anyons is discussed at length in Chapter 3. Exact solutions for two anyons in an oscillator as well as in several other potentials are discussed at length. The problem of three and multi-anyons in an oscillator potential and/or in a uniform magnetic field, as well as in an N-body potential, are also discussed and the difficulties in finding the full spectrum are pointed out. Using the second order perturbation theory, the ground state energy of three anyons in an oscillator potential is computed and using these results, a detailed discussion is made about the nature of the ground state of three and multi-anyons in an oscillator potential and/or in a uniform magnetic field. Finally, a brief discussion is also presented about the role anyons could play in quantum computations.

Chapters 3 and 4, in a sense are the heart of the book. After giving a brief introduction to the ideal Fermi and Bose gas in two dimensions, I discuss the various known results for an ideal gas of anyons. In particular, I discuss two different derivations for the computation of the second virial coefficient of an ideal anyon gas. Further, i show that the semi-classical techniques also reproduce the exact second virial coefficient of not only an ideal anyon gas but even of an interacting anyon gas so long as the interaction is scale invariant and hence does not introduce any extra dimensional parameter in the problem. It is worth emphasizing that to date, the full expressions for the third or higher virial coefficients are not known. However, some results for the higher virial coefficients are known which are discussed briefly and the difficulties in obtaining the exact results for the higher virial coefficients are pointed out.

Recently a new quantum statistics called Fractional Exclusion Statistics has been introduced which is valid in any dimension. This is discussed in some detail in Chapter 5. Various quantities like distribution function etc. are computed for an ideal gas and it is pointed out that the Calogero-Sutherland model represents an example of an ideal one dimensional gas with fractional statistics.

The phenomenon of fractional statistics is most naturally discussed within the framework of Chern-Simons theories and that will be done in the second part of the book. As a prelude to these discussions, I discuss the various properties of the Chern-Simons term in Chapter 6. In particular its role as a gauge field mass term and its behavior under the discrete transformations of parity (P) and time-reversal (T) is emphasized. It is worth noting that in the long wave length limit, this term having only one derivative (in two space and one time dimensions) is expected to dominate over the usual Maxwell kinetic energy term which has two derivatives.

In the presence of the Chern-Simons term, anyons can appear in two different ways and both approaches are elaborated in some detail. The charged vortex solutions in abelian Higgs model with Chern-Simons term are obtained in Chapter 7 and it is pointed out that these charged vortices represent the first relativistic model for (extended) charged anyons. I also construct the charged vortex solutions in pure Chern-Simons theory in both the relativistic and the non-relativistic settings. Finally, I also discuss an example of neutral relativistic anyons by considering the soliton solutions in the CP^1 model with the Hopf term which is one of the *avtars* of the Chern-Simons term.

In Chapter 8, I elaborate upon the other approach in which fundamental fields of theories with Chern-Simons term themselves carry fractional spin and obey fractional statistics. *A la* Dirac equation for the spin-1/2 fields, I also discuss here a relativistic wave equation for anyons.

Chapter 9 is devoted to a discussion about the many-anyon system within the mean field approach. Approximate schemes like RPA which go beyond the mean field approximation are also discussed. Qualitative arguments are also given here regarding the possibility of anyon superconductivity. It may be emphasized here that these arguments are quite robust and even though the high-T_c superconductors have perhaps nothing to do with the anyons, it is not unlikely that there may be other (yet to be discovered) superconductors in nature in which anyons could play an important role.

No discussion of anyons will be complete without an account of its most outstanding success–its application to the fractional quantum Hall effect where anyons appear as a quasi-particle excitations. After a brief discussion to the Landau level problem and to the quantum Hall effect, the trial wave-function approach of Laughlin [6] is discussed in some detail in Chapter 10. The Landau-Ginzberg-Chern-Simons approach is also briefly introduced.

Finally, in Chapter 11, a brief description of the omitted topics is given along with a few references for each of these topics so that the interested reader can trace back and study these topics further.

References

[1] E.A. Abbot, *Flatland* (Princeton Univ. Press, New Edition, 1991).
[2] C.H. Hinton, *An Episode Of Flatland* (1907).
[3] M. Gardner, in *The Unexpected Hanging And Other Mathematical Diversions*, ed. M. Gardner (Simon and Schuster 1969).
[4] A.K. Dewdney, *Two Dimensional Science and Technology, J. Recreate. Math.* **12** (1979) 16 ; For a short summary of the book see M. Gardner, *Sci. Ame., July Issue* (1980) 18.
[5] A.B. Fowler, F.F. Fang, W.E. Howard and P.J. Stiles, *Phys. Rev. Lett.* **16** (1966) 901.
[6] R.B. Laughlin, *Phys. Rev. Lett.* **50** (1983) 1395.
[7] F.D.M. Haldane, *Phys. Rev. Lett.* **51** (1983) 605.
[8] B.I. Halperin, *Phys. Rev. Lett.* **52** (1984) 1583; **52** (1984) E2390.
[9] R.G. Clark *et al.*, *Phys. Rev. Lett.* **60** (1988) 1747 ; J.A. Simmons *et al.*, **63** (1989) 1731 ; A.M. Chang and J.E. Cummingham, *Solid State Comm.* **72** (1989) 651 ; For a popular readable account see, A. Khurana, *Phys. Today* **43(1)** (1990) 19.
[10] R.B. Laughlin, *Phys. Rev. Lett.* **60** (1988) 2677 ; *Science* **242** (1988) 525.
[11] S. Spielman *et al.*, *Phys. Rev. Lett.* **65** (1990) 123.
[12] R.F. Kieff *et al.*, *Phys. Rev. Lett.* **64** (1990) 2082.
[13] S.N. Bose, *Zeits. f. Phys.* **26** (1924) 178.
[14] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss.* (1924) 26 ; (1925) 3,18.
[15] E. Fermi, *Zeits. f. Phys.* **36** (1926) 902.
[16] P.A.M. Dirac, *Proc. Roy. Soc. London* **A112** (1926) 661.
[17] R. K. Pathria, *Statistical Mechanics* (pergamon Press, Oxford, 1972).
[18] K. Huang, *Statistical mechanics* (John Wiley and Sons, Inc., 1963).
[19] W. Heisenberg, *Zeits. f. Phys.* **38** (1926) 411.
[20] P.A.M. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, Oxford, 1935).
[21] M.G.G. Laidlaw and C.M. DeWitt, *Phys. Rev.* **D3** (1971) 1375.
[22] J.M. Leinaas and J. Myrheim, *Nuovo Cim.* **B37** (1977) 1.
[23] G.A. Goldin, R. Menikoff and D.H. Sharp, *J. Math. Phys.* **21** (1980) 650 ; **22** (1981) 1664.
[24] F. Wilczek, *Phys. Rev. Lett.* **48** (1982) 1144 ; **49** (1982) 957.

[25] F. Wilczek and A. Zee, *Phys. Rev. Lett.* **51** (1983) 2250.
[26] Y.-S. Wu and A. Zee, *Phys. Lett.* **B147** (1984) 325.
[27] D.C. Tsui, H.L. Stormer and A.C. Gossard, *Phys. Rev. Lett.* **48** (1982) 1559.
[28] K. von Klitzing, G. Dorda and M. Pepper, *Phys. Rev. Lett.* **45** (1980) 494.
[29] D. Arovas, J.R. Schrieffer and F. Wilczek, *Phys. Rev. Lett.* **53** (1984) 722.
[30] J.K. Jain, *Phys. Rev.* **B40** (1989) 8079 ; **B41** (1991) 7653.
[31] Y.-S. Wu, *Phys. Rev. Lett.* **52** (1984) 2103.
[32] E. Fadell and L. Neuwirth, *Math. Scand.* **10** (1962) 111 ; R. Fox and L. Neuwirth, **10** (1962) 119.
[33] Y.-S. Wu, *Phys. Rev. Lett.* **53** (1984) 111; **53** (1984) E1028.
[34] D.P. Arovas, J.R. Schrieffer, F. Wilczek and A. Zee, *Nucl. Phys.* **B251** (1985) 117.
[35] J.S. Dowker, *J. Phys.* **A18** (1985) 3521.
[36] A. Comtet, Y. Georgelin and S. Ouvry, *J. Phys.* **A22** (1989) 3917.
[37] S.K. Paul and A. Khare, *Phys. Lett.* **B174** (1986) 420 ; *Phys. Lett.* **B177** (1986) E453.
[38] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975; *Ann. Phys.* **140** (1982) 372.
[39] J. Fröhlich and P.A. Marchetti, *Comm. Math. Phys.* **121** (1989) 177.
[40] D. Jatkar and A. Khare, *Phys. Lett.* **B236** (1990) 283.
[41] J. Hong, Y. Kim and P.Y. Pac, *Phys. Rev. Lett.* **64** (1990) 2230 ; R. Jackiw and E.J. Weinberg, **64** (1990) 2234.
[42] R. Jackiw, K. Lee and E.J. Weinberg, *Phys. Rev.* **D42** (1990) 3488.
[43] A. Khare, *Phys. Lett.* **B255** (1991) 393.
[44] R. Jackiw and S.-Y. Pi, *Phys. Rev. Lett.* **64** (1990) 2969; *Phys. Rev.* **D42** (1990) 3500.
[45] R. Jackiw and V.P. Nair, *Phys. Rev.* **D43** (1991) 1933.
[46] A. Comtet and S. Ouvry, *Phys. Lett.* **B225** (1989) 272.
[47] R.K. Bhaduri, R.S. Bhalerao, A. Khare, J. Law and M.V.N. Murthy, *Phys. Rev. Lett.* **66** (1991) 523.
[48] C. Chou, *Phys. Rev.* **D44** (1991) 2533 ; **D45** (1992) E1433 ; R. Basu, G. Date and M.V.N. Murthy, **B46** (1992) 3139 ; A.P. Polychronakos, *Phys. Lett.* **B264** (1991) 362.
[49] G.V. Dunne, A. Lerda, S. Sciuto and C.A. Trugenberger, *Nucl. Phys.* **B370** (1992) 601 ; K. Cho and C. Rim, *Ann. Phys.* **213** (1992) 295.
[50] A. Khare, *Phys. Lett.* **A221** (1996) 365.
[51] A. Khare and J. McCabe, *Phys. Lett.* **B269** (1991) 330 ; A. Khare, J. McCabe and S. Ouvry, *Phys. Rev.* **D46** (1992) 2714.
[52] M. Sporre, J.J.M. Verbaarschot, and I. Zahed, *Phys. Rev. Lett.* **67** (1991) 1813 ; M.V.N. Murthy, J. Law, M. Brack and R.K. Bhaduri, *Phys. Rev. Lett.* **67** (1991) 1817.
[53] F. Illuminati, F. Ravndal and J.A. Rudd, *Phys. Lett.* **A161** (1992) 323; M. Sporre, J.J.M. Verbaarschot, and I. Zahed, *Nucl. Phys.* **B389** (1993) 645.
[54] R. Chitra and D. Sen, *Phys. Rev.* **B 46** (1992) 10 923.
[55] J. McCabe and S. Ouvry, *Phys. Lett.* **B260** (1991) 113 ; A. Comtet, J. McCabe and S. Ouvry, *Phys. Lett.* **B260** (1991) 372 ; D. Sen, *Nucl. Phys.*

B360 (1991) 397.
[56] D. Sen, *Phys. Rev. Lett.* **68** (1992) 2977.
[57] A. Dasnières de Veigy and S. Ouvry, *Phys. Lett.* **B291** (1992) 130 ; *Nucl. Phys.* **B388** (1992) 715 ; R. Emparan and M.A. Valle Basagoiti, *Mod. Phys. Lett.* **A8** (1993) 3291.