Chapter 1

Multinomial Lattices and Derivatives Pricing

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This article elaborates an n-order multinomial lattice approach to value derivative instruments on asset prices characterized by a lognormal distribution. Nonlinear optimization is employed, specified moments are matched, and n-order multinomial trees are developed. The proposed methodology represents an alternative specification to models of jump processes of order greater than three developed by other researchers. The main contribution of this work is pedagogical. Its strength is in its straightforward explanation of the underlying tree building procedure for which numerical efficiency is a motivation for actual implementation.

Keywords: Lattice; multinomial; derivatives; moment matching; numerical efficiency.

1. Introduction

Since the seminal article by Black and Scholes (BS, 1973), numerous methods for valuing derivative securities have been proposed. Merton (1973) extended the BS model to include valuing an option on a stock or index that pays continuous dividends. From this framework, the BS model was easily extended to currency options. In the case of exotic contracts where there is no closed form solution, various techniques have been elaborated including Monte-Carlo simulation, numerical integration, analytical and series approximation, jump

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The views expressed in this article are those of the authors and do not necessarily reflect the position of Fannie Mae, Santel or Wachovia Securities.

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processes, and finite difference methods. Parkinson (1977) applied a three-jump model via numerical integration to the valuation of American put options. Brennan and Schwartz (1978) demonstrated that the probabilities of a jump process approximation to the underlying diffusion process correspond to the coefficients of the difference equation approximation of the BS partial differential equation. Further, they demonstrated that the trinomial tree is equivalent to the explicit finite difference method and that a generalized multinomial jump process is equivalent to a complex implicit finite difference approximation. Courtadon (1982) suggested an alternative finite difference approximation.

Cox, Ross and Rubinstein (CRR, 1979) and Rendleman and Bartter (RB, 1979) introduced the two-state lattice approach, which proved to be a powerful tool that can be used to value a wide variety of contingent claims. Jabbour, Kramin and Young (2001) generalized the standard binomial approach and incorporate the main existing models as particular cases of an alternative approach to the specification of these lattices. Geske and Shastri (1985) compared a variety of approximation methods for contingent claims valuation, including the efficiency of the binomial lattice approach and finite difference method for option valuation. A number of alternative analytical approximations for continuous time valuation were suggested by Johnson (1983), Geske and Johnson (1984), Blomeyer (1986), Macmillan (1986), Whaley (1986), Barone-Adesi and Whaley (1987), and Omberg (1987). Boyle (1986) introduced a three-jump process as a modification of the CRR model in the case of a single state variable.

Boyle (1988) extended the lattice approach to option valuation in the case of two underlying state variables. Boyle's trinomial model was based on a moment matching methodology. The mean and variance of the discrete distribution were equated to those of the continuous lognormal distribution. By introducing a numerically optimized parameter, Boyle ensured non-negativity of the risk-neutral probabilities. Further, Boyle introduced a two-dimensional five-jump process for pricing options on two underlying assets that follow a bivariate lognormal distribution. This left the three or more state variable question unanswered. The difficulties associated with the practical implementation of Boyle's model to three or more state variables were connected with ensuring non-negative risk-neutral probabilities. Boyle, Evnine and Gibbs (1989) overcame this problem by equating the moment generating function of the approximating distribution to the true normal moment generating function. This technique can be easily generalized to k state variables. Kamrad and

Ritchken (1991) developed a multinomial lattice approximating method for valuing claims on several state variables that included many existing models as special cases. For example, the Kamrad and Ritchken (1991) model extended the model proposed by Boyle, Evnine and Gibbs (1989) offered some computational advantages by incorporating horizontal jump. Hull and White (1988) suggested a generalized version of the lattice approach to option pricing using a control variate technique and introduced a multivariate multinomial extension of the CRR model. Further, Hull and White (1994a, 1994b) proposed a robust two-stage procedure for one- and two-factor trinomial lattice models. Madan, Milne and Shefrin (1989) generalized the CRR model to the multinomial case to approximate a multi-dimensional lognormal process. They showed that the distribution of the discrete-time process converged to that of a one-dimensional lognormal process for a number of underlying assets, but they failed to specify the correlation structure among assets and establish convergence for general multivariate contingent claims prices. Hua (1990) solved this problem by using an alternative multinomial multivariate model.

Omberg (1988) derived a number of multinomial jump processes via pure Gauss-Hermite quadrature. The drawback to this method is that the nodes of the corresponding multinomial tree of order greater than three are not uniformly spaced (i.e., the tree is not recombining and the number of possible states increases geometrically with the number of time steps). To overcome this problem, Omberg (1988) suggested a modified Gauss-Hermite quadrature technique with uniform jumps and a lower degree of precision using Lagrangian polynomial interpolation to determine the value of the function at the Gaussian points.

Heston and Zhou (2000) investigated the rate of convergence of multinomial trees. They showed that the highest possible convergence rate for the lattice that ensures matching the first K central moments of the underlying stochastic process probability distribution, $\left(\frac{1}{(\sqrt{n})^{K-1}}\right)$, can be achieved with certainty only when the payoff of the derivative valued is continuously differentiable up to order $2K$ ($C^{(2K)}$). This condition is rarely satisfied. To overcome these difficulties, Heston and Zhou (2000) proposed a smoothing and adjustment approach and implemented them on trinomial and pentanomial lattices. Alford and Webber (2001) considered numerous techniques related to convergence and processing time improvement: smoothing, and Richardson extrapolation and truncation for multinomial lattices of order $(4m-1)$, where m is an integer, to achieve higher convergence rates for payoff functions with a finite set of critical points. This approach allowed one to match up to $(4m + 1)$ central moments of the underlying log-normal distribution. They concluded that the heptanomial lattice is the fastest and most accurate higher-order lattice. The focus of the last two papers was the application of multinomial trees to improve the rate of convergence of lattice methods. This was why Heston and Zhou (2000) and Alford and Webber (2001) considered numerous techniques for convergence improvement. Focusing on convergence, they considered only the lattices of orders 2, 3 and 5, and the latter — 3, 7, 11, 15, 19 etc. Working in a Black-Scholes world, they initially imposed a symmetry condition (the odd central moments are zero) on their systems and solved to match the even central moments consistent with a normal distribution. This is similar to the methodology specified in this paper.

In the Heston and Zhou (2000) and Alford and Webber (2001) papers, the methodology is not the focus and is therefore not as fully developed as in this paper. The purpose of this paper is pedagogical. It provides a step-by-step description of the moment matching technique, which is applied to develop n-order multinomial lattice parameterizations for a single-state option-pricing model. Thus, the underlying methodology is the focus. The remaining format of this paper is as follows. Section 2 provides a general description of n-order multinomial lattices. Section 3 defines the procedure when the underlying asset is described by a Geometric Brownian Motion process. Section 4 discusses practical implementation and provides numerical results. Section 5 gives conclusions.

2. A General Description of n-Order Multinomial Lattices

Consider a stochastic variable Q that follows an Ito process:

$$dQ = a(Q,t)\,dt + b(Q,t)\,dz, \tag{1}$$

where dt is an infinitely small increment of time, dz is a Wiener process, $a(Q,t)$, $b(Q,t)$ are some functions of Q and t is time. In a multinomial model of order n, for a short period of time Δt, the variable Q can move from Q_0 (the value at time zero) to $Q_0 + q_j$, with $j = \overline{1,n}$, where q_j is a change in the value of Q for time Δt and n is the number of possible jumps. The change of Q for time Δt has the following discrete distribution:

$$\{q_j \text{ with risk-neutral probability } p_j\}.$$

For a lattice approach, the first moment (M_1) of the distribution of the variable Q is given by the following:

$$M_1 = \sum_{j=1}^{n} p_j \cdot q_j. \tag{2}$$

To apply a moment matching technique and develop an n-order multinomial framework, one has to equate the first n central moments of the discrete lattice distribution to those of the specified continuous distribution. In order to match the first moment of the lattice approach for variable Q with the first moment consistent with the underlying process, one has to set:

$$M_1 = \sum_{j=1}^{n} p_j \cdot q_j = E(Q_{\Delta t}) = m.$$

The kth order central moment of the lattice approach for variable Q can be given as follows:

$$\tilde{m}_k = \sum_{j=1}^{n} p_j \cdot (q_j - m)^k = \sum_{j=1}^{n} p_j \cdot z_j^k,$$

where $z_j = q_j - m$. The first central moment $\tilde{m}_1$ is zero by construction, and the second central moment $\tilde{m}_2$ is set equal to the variance of the variable Q. To match the remaining central moments, it is necessary to specify the set of central moments of the variable Q determined by the moment generating function (MGF) $M(t)$ of the underlying distribution. The central moments of the distribution can be obtained by applying a Taylor series expansion to the MGF as follows:

$$M(t) = \sum_{j=0}^{\infty} M^{(j)}(0) \frac{t^j}{(j!)},$$

where $M^{(j)}(0)$ (the derivative of j order at time zero) represents the j order central moment. In order to set the lattice probability distribution consistent with a specified underlying distribution, one can apply a moment matching approach by solving the following nonlinear system with respect to the unknown parameters p_j and z_j, $j = \overline{1, n}$:

$$\tilde{m}_k = \sum_{j=1}^{n} p_j \cdot z_j^k = m_k^Q, \quad k = \overline{0, L}, \tag{3}$$

where m_k^Q is the central moment of order k of the continuous distribution and L is the number of moments matched. In order to specify the n-order multinomial lattice, it is sufficient to set $n+1$ equations, that is, $L=n$. The first equation is the condition that the probabilities sum to one. The remaining n equations match the first n central moments of the discrete distribution to those of the continuous underlying distribution. Solution vectors $[P]=\{p_j\}_{j=1}^n$ and $[Z]=\{z_j\}_{j=1}^n$ in this case are not unique because for $2n$ unknowns there are only $n+1$ nonlinear equations. In order to determine the unique solution $\{[P],[Z]\}$, one has to impose additional constraints. These constraints, if feasible, will affect only the convergence speed of the lattice model.

3. Multinomial Lattices and Lognormally Distributed Asset Prices

In a risk-neutral world, if one assumes that the stock price S follows a Geometric Brownian Motion process (GBM), then:

$$dS = rS\,dt + \sigma S\,dz, \tag{4}$$

where r is the instantaneous risk-free interest rate, and σ is the instantaneous volatility of the stock price. By using Ito's Lemma, one can show:

$$dX = \alpha\,dt + \sigma\,dz, \tag{5}$$

where $X=\ln(S)$ and $\alpha=\left(r-\frac{\sigma^2}{2}\right)$. As a result, $\ln(S)$ follows a generalized Wiener process for the time period $(0,t)$, where t is a point in time. The variable $\hat{X}=X_t-X_0=\ln\left(\frac{S_t}{S_0}\right)$ is distributed with a mean of $\alpha\cdot t$, a variance of $\sigma^2 t$ and S_0 and S_t represent the stock price at time 0 and t respectively. In a multinomial model of order n, the stock price can move from S_0 to $u_j\cdot S_0$, $j=\overline{1,n}$, where u_j is a proportional change in stock price for time t and n is the number of possible jumps. The variable $\hat{X}$ has the following discrete distribution:

$$\{q_j \text{ with risk-neutral probability } p_j\},$$

where $q_j=\ln(u_j)$. The first moment of the continuous underlying process for variable $\hat{X}$ is $E(\hat{X})=\alpha\cdot t=m$. The second moment $\tilde{m}_2$ is set equal to $\sigma^2\cdot\Delta t$. The moment generating function for a variable R that is normally distributed $R\sim N(\mu,\delta)$ is given by the following:

$$M(t)=e^{\mu\cdot t+\frac{1}{2}\cdot\delta^2\cdot t^2}.$$

Because the normal distribution is symmetrical, all odd central moments are zero. For the standard normal distribution W,

$$M(t) = e^{\frac{1}{2}\cdot t^2} = \sum_{j=0}^{\infty} \frac{t^{2j}}{(j!)\cdot 2^j}.$$

The (central) moment of order k represents the coefficient before t^k in the series above multiplied by $k!$ and can be given by the following formula:

$$m_k^W = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \prod_{i=1}^{k/2}(2\cdot i-1) & \text{if } k \text{ is even} \end{cases}.$$

For example,

$$\begin{cases} m_1^W = 0;\ m_3^W = 0;\ m_5^W = 0;\ m_7^W = 0;\ m_9^W = 0;\ \text{etc.} \\ m_2^W = 1;\ m_4^W = 3;\ m_6^W = 15;\ m_8^W = 105;\ m_{10}^W = 945;\ \text{etc.} \end{cases}$$

Analogously, for the variable R, it can easily be shown that the central moments are given by the following:

$$m_k^R = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \prod_{i=1}^{k/2}(2\cdot i-1)\delta^k & \text{if } k \text{ is even} \end{cases},$$

and:

$$\begin{cases} m_1^R = 0;\ m_3^R = 0;\ m_5^R = 0;\ m_7^R = 0;\ m_9^R = 0;\ \text{etc.} \\ m_2^R = \delta^2;\ m_4^R = 3\delta^4;\ m_6^R = 15\delta^6;\ m_8^R = 105\delta^8;\ m_{10}^R = 945\delta^{10};\ \text{etc.} \end{cases}$$

The lattice probability distribution consistent with a normal distribution can be obtained by solving the system (3), where $m_k^Q = m_k^W$, $z_j = \frac{q_j - m}{\delta}$, $\delta = \sigma\sqrt{t}$, $j = \overline{1,n}$, $k = \overline{0,L}$.

To illustrate the moment matching methodology, consider the binomial and trinomial models. In the first case, $n = 2$, thus the first two moments should be matched. In a binomial (two jump process) model, the stock price can either move up from S_0 to $u\cdot S_0$ or down to $d\cdot S_0$, where u and d are two parameters such that u is greater than one — to avoid arbitrage it is actually greater than e^{rt} — and d is less than one. Since the stock price follows a binomial process, the variable $\hat{X}$ has the following discrete distribution:

$$\begin{cases} \text{U with risk-neutral probability } p \\ \text{D with risk-neutral probability } (1-p) \end{cases},$$

where $U = \ln(u)$ and $D = \ln(d)$. For the binomial lattice, the system is given by the following:

$$p \cdot U + (1 - p) \cdot D = \alpha \cdot \Delta t,$$
$$p(1 - p)(U - D)^2 = \sigma^2 \Delta t.$$

This system of two equations and three unknowns U, D, and p can be further specified as follows:

$$\begin{aligned} p_1 + p_2 &= 1, \\ p_1 w_1 + p_2 w_2 &= 0, \\ p_1(w_1)^2 + p_2(w_2)^2 &= 1, \end{aligned} \tag{6}$$

where $p_1 = p$; $p_2 = 1 - p$; $w_1 = \frac{(U - \alpha\Delta t)}{\sigma\sqrt{\Delta t}}$; and $w_2 = \frac{(D - \alpha\Delta t)}{\sigma\sqrt{\Delta t}}$. It should be noted that a properly specified binomial lattice always results in a recombining tree. If one imposes the additional constraint that the third central moment is zero (this is consistent with normally distributed returns and may improve the convergence of the lattice approach but is not critical for the binomial model), then

$$p_1(w_1)^3 + p_2(w_2)^3 = 0. \tag{7}$$

The system (6) and (7) has four equations and four unknowns (p_1, w_1, p_2, w_2) and is complete. With constraint (7) the solution is trivial and unique: $p_1 = p_2 = \frac{1}{2}$, and $w_1 = -1,\ w_2 = 1$. This solution is equivalent to the specification of RB (1979) and Jarrow-Rudd (1983). As is well known, the standard binomial framework affords numerous specifications, which are fully discussed in Jabbour, Kramin and Young (2001).

For a trinomial (three-jump process) model, the system of the moment matching methodology is given by the following:

$$\begin{aligned} p_1 + p_2 + p_3 &= 1, \\ p_1 w_1 + p_2 w_2 + p_3 w_3 &= 0, \\ p_1(w_1)^2 + p_2(w_2)^2 + p_3(w_3)^2 &= 1, \\ p_1(w_1)^3 + p_2(w_2)^3 + p_3(w_3)^3 &= 0. \end{aligned} \tag{8}$$

The additional constraints can be imposed on the fourth and fifth moments as follows:

$$p_1(w_1)^4 + p_2(w_2)^4 + p_3(w_3)^4 = C, \tag{9}$$
$$p_1(w_1)^5 + p_2(w_2)^5 + p_3(w_3)^5 = 0. \tag{10}$$

In this case the complete system (8), (9) and (10) has a simple and unique analytical solution that can be obtained using pure Gauss-Hermite quadrature. The following is the parameterization of the system:

$$p_1 = p_3 = \frac{1}{2C}, \; p_2 = \frac{C-1}{C}, \; w_1 = -\sqrt{C}, \; w_2 = 0, \; w_3 = \sqrt{C}. \quad (11)$$

When one specifies the fourth moment of the lattice distribution corresponding to the fourth moment of the standard normal distribution (the kurtosis is equal to three, $C = 3$), the parameterization simplifies to the following as demonstrated by Omberg (1988):

$$p_1 = p_3 = \frac{1}{6}, \quad p_2 = \frac{2}{3}, \quad w_1 = -\sqrt{3}, \quad w_2 = 0, \quad w_3 = \sqrt{3}. \quad (12)$$

While there is no need to set the particular restrictions given by (9) and (10), which are consistent with the fourth and fifth moments of the normal distribution, these constraints should improve the convergence of the lattice approach in the case when payoff smoothness conditions (Heston and Zhou, 2000) are satisfied. For this case, the recombining condition is given by the following:

$$w_3 - w_2 = \Delta_2 = \Delta_1 = w_2 - w_1. \quad (13)$$

Therefore, the lattice consistent with the parameterization (11) recombines. Equation (10) can be considered a constraint that ensures a symmetrical lattice distribution. Parameter C represents a degree of freedom. The value of this parameter will not affect convergence to the correct value but rather the rate of convergence (Heston and Zhou, 2000). It is worth noting that multinomial trees of order higher than three obtained via pure Gauss-Hermite quadrature are not recombining. This does not diminish the theoretical importance of the technique but limits it as a practical method to applying n-order multinomial trees.

In general, the system for the moment matching approach can be mathematically represented as follows:

$$[P]^T[W^k] = m_k^W, \quad k = \overline{0, L}, \quad (14)$$

where $[W^k] = \{w_j^k\}_{j=1}^n$, $[W^0] = \{w_j^0\}_{j=1}^n = \{1\}_{j=1}^n = [J]$ and $[J]$ is a unit vector. Analogous to (13), in order to make the n-order multinomial tree recombine, one may impose the following constraints:

$$\Delta_{j+1} = \Delta_j, \quad j = \overline{1, n-2}, \quad (15)$$

where $\Delta_i \equiv w_{i+1} - w_i, i = \overline{1, n-1}$.[1] The nonlinear system (14) and (15) can be solved with respect to $[P]$ and $[W]$ numerically.[2]

Given a proper specification of an n-order multinomial lattice, the value of an option can be obtained through the usual backward recursion procedure:

$$f = e^{-r\Delta t} \sum_{k=1}^{n} p_k f_k = e^{-r\Delta t} \cdot [P]^T [F],$$

where Δt is the length of a time step, and $[F] = \{f_j\}_{j=1}^{n}$ represents the value of the option along a number of appropriate nodes of the n-order multinomial lattice.

4. Practical Implementation and Numerical Results

In this section, the practical implementation of n-order multinomial lattices is outlined and numerical results are provided. The first step of the approach is to determine the set of risk-neutral probabilities $[P]$ and jump parameters $[W]$. While a number of methods exist to implement this task one may minimize the following function:

$$\min_{[W],[P]} \left| [P]^T [W^K] - m_K^W \right|, \tag{16}$$

subject to constraints (14) and (15) where K is the minimum even number that is greater than n.This nonlinear optimization procedure ensures a minimum difference between the Kth central moment of the discrete distribution and that of the continuous distribution for the n-order multinomial model. While one does not have to specify this procedure to obtain the unknown tree parameters $[P]$ and $[W]$ (the satisfaction of constraints (14) and (15) and, perhaps, risk-neutral probability non-negativity constraints would be enough), the procedure (16) can accelerate convergence of the lattice approach via the output parameters. It

[1] While multinomial trees of order higher than two obtained via pure Gauss-Hermite quadrature do not recombine for the discrete-time GBM parameterization, a moment matching technique implemented through nonlinear optimization with constraints analogous to (15) does produce trees that recombine.

[2] Negative probabilities can easily be avoided by directly imposing the appropriate additional constraints: $0 \le p_i \le 1$.

is worth noting that the equality $[P]^T[W^K] = m_K^W$ cannot be always satisfied. Moreover, imposing an additional constraint:

$$[P]^T[W^I] = m_I^W, \tag{17}$$

where I is the minimum odd number that is greater than n, causes all remaining odd central moments to be equal to zero and thus ensures a symmetrical discrete distribution.

As discussed earlier, specifications of lattices with jump processes of order greater than three obtained using pure Gauss-Hermite quadrature do not recombine. Interestingly, a four-jump process lattice developed using the numerical procedure outlined above is degenerative and reduces to a trinomial tree. Thus the four-jump process lattice is redundant. All other n-order lattice parameterizations examined have a unique representation in terms of this algorithm. Below, in Table 1, are the risk-neutral probabilities, $[P]$, jump parameters, $[W]$, based on a lognormally distributed asset price, and thus normally distributed returns for lattices of order two through seven.[3]

Table 1. Risk-neutral probabilities $[P]$ and jump parameters $[W]$.

$[P]$	p1	p2	p3	p4	p5	p6	p7
n = 2	0.500000	0.500000					
n = 3	0.166667	0.666667	0.166667				
n = 4	0.000000	0.166667	0.666667	0.166667			
n = 5	0.013333	0.213334	0.546666	0.213334	0.013333		
n = 6	0.003316	0.081193	0.415492	0.415492	0.081193	0.003316	
n = 7	0.000802	0.026810	0.233813	0.477150	0.233813	0.026810	0.000802
$[W]$	w1	w2	w3	w4	w5	w6	w7
n = 2	−1.000000	1.000000					
n = 3	−1.732051	0.000000	1.732051				
n = 4	−3.464102	−1.732051	0.000000	1.732051			
n = 5	−2.738608	−1.369304	0.000000	1.369304	2.738608		
n = 6	−3.189031	−1.913419	−0.637806	0.637806	1.913419	3.189031	
n = 7	−3.594559	−2.396373	−1.198186	0.000000	1.198186	2.396373	3.594559

The above table presents the risk-neutral probabilities $[P]$ and jump parameters $[W]$ for jump processes of the order two through seven. These results are based on a lognormally distributed asset price with normally distributed returns.

[3]For the pentanomial and heptanomial trees, the probabilities $[P]$ and parameters $[W]$ are slightly different from those provided by Heston and Zhou (2000) and Alford and Webber (2001) respectively because the solution of underlying system is not unique.

Once the parameters of the discrete distributions $[P]$ and $[W]$ are specified, the tree building procedure for any n-order multinomial lattice is analogous to that of the binomial and trinomial trees. Option values are obtained through a recursive procedure.

In Table 2, n-order multinomial lattices are used to price European put options on a non-dividend paying stock. The underlying stock price distribution is assumed lognormal and thus the asset returns are normally distributed. The models considered are based upon two, three, five, six and seven-jump processes respectively. The stock price is set equal to 100. The three exercise prices considered are 90, 100 and 110. The time to expiration is one-year, the risk-free rate is 5% per annum and the volatility is 30%. The numbers of time steps considered include 25, 50 and 100. Lastly, the corresponding BS values and percentage errors — with respect to BS — are provided. As seen from the

Table 2. European put values for jump processes of order two, three, five, six and seven.

X	Time Steps (N)		2	3	5	6	7	Black-Scholes
90	25	Value	5.3943	5.2432	5.3280	5.2738	5.3309	5.3081
		Error	0.0162	−0.0122	0.0038	−0.0065	0.0043	
	50	Value	5.3378	5.3321	5.2948	5.2878	5.3043	
		Error	0.0056	0.0045	−0.0025	−0.0038	−0.0007	
	100	Value	5.3098	5.2994	5.3126	5.3032	5.3010	
		Error	0.0003	−0.0016	0.0009	−0.0009	−0.0013	
100	25	Value	9.4651	9.2700	9.3068	9.3838	9.3205	9.3542
		Error	0.0119	−0.0090	−0.0051	0.0032	−0.0036	
	50	Value	9.3211	9.3184	9.3352	9.3387	9.3413	
		Error	−0.0035	−0.0038	−0.0020	−0.0017	−0.0014	
	100	Value	9.3424	9.3404	9.3477	9.3492	9.3503	
		Error	−0.0013	−0.0015	−0.0007	−0.0005	−0.0004	
110	25	Value	14.7054	14.6176	14.6199	14.6756	14.6632	14.6553
		Error	0.0034	−0.0026	−0.0024	0.0014	0.0005	
	50	Value	14.6192	14.6583	14.6734	14.6662	14.6566	
		Error	−0.0025	0.0002	0.0012	0.0007	0.0001	
	100	Value	14.6829	14.6602	14.6544	14.6615	14.6625	
		Error	0.0019	0.0003	−0.0001	0.0004	0.0005	

The above table presents European put values for jump processes of order two, three, five, six and seven. The steps utilized include 25, 50 and 100. Option parameters are given by the following: $S = 100$; $X = 90$, 100, 110; $r = 5\%$; $v = 30\%$ p.a.; $T = 1$ year. BS values and percentage errors are reported.

results in Table 2, while there is no significant improvement in convergence with increase of the order of the multinomial lattice, the option values for all considered orders converge to the benchmark (BS) prices under decreasing step size.

While it was shown by Heston and Zhou (2000) that the convergence rate for the multinomial lattice is determined by the order of differentiability of the payoff function, and that, in general, numerical efficiency of the n-order multinomial lattice increases with n, it is the processing time that is the key measure, which defines computational efficiency among models of different orders or specifications. Numerical efficiency is not the focus of this work but may be considered using the techniques delineated by Kamrad and Ritchken (1991), Heston and Zhou (2000), and Alford and Webber (2001). Thus computational burden should be the subject of future efforts.

5. Conclusions

This article develops an n-order multinomial lattice approach to price options on assets that are characterized by a lognormal distribution with normally distributed returns. In order to determine an n-order multinomial lattice parameterization, a moment matching technique is implemented through nonlinear optimization. The focus of the paper is pedagogical and numerical results are provided for practical implementation purposes. While the numerical results are limited to asset with prices that are lognormally distributed, future research should focus on alternative moment generating functions. This is of crucial importance as alternative return distributions may provide a rich framework for reconciling theoretical option values with actual prices.

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