# Preface

This book started with the goal of explaining, to engineers and scientists, the advances made in the numerical computation of the isolated solutions of systems of nonlinear multivariate complex polynomials since the book of A. Morgan (Morgan, 1987). The writing of this book was delayed because of a number of surprising developments, which made possible numerically describing not just the isolated solutions, but also positive-dimensional solution sets of polynomial systems. The most recent advances allow one to work with individual solution components, which opens up new ways of solving a large system of polynomials by intersecting the solution sets of subsets of the equations. This collection of ideas, methods, and problems makes up the new area of *Numerical Algebraic Geometry.*

The heavy dependence of the new developments since (Morgan, 1987) on algebraic geometric ideas poses a serious challenge for an exposition aimed at engineers, scientists, and numerical analysts — most of whom have had little or no exposure to algebraic geometry. Furthermore most of the introductory books on algebraic geometry are oriented towards computational algebra, and give short shrift at best to the geometric results which underly the numerical analysis of polynomial systems. Even worse, from the standpoint of an engineer or scientist, such books typically aim to resolve algebraic questions and so do not directly address the numerical/geometric questions coming from applications.

Our approach throughout this book is to assume that we are trying to explain each topic to an engineer or scientist. We want to be accurate: we do not cut corners on giving precise definitions and statements. We give illustrative examples exhibiting all the phenomena involved, but we only give proofs to the extent that they further understanding.

The set of common zeros of a system of polynomials is not a manifold, but it is close to being one in the sense that exceptional points are rare. This vague statement can be made mathematically precise, and indeed, the theoretical underpinnings of our methods imply that we avoid such trouble spots "with probability one." The usual algebraic approaches to the subject do not show how familiar geometric notions from calculus relate to these solution sets. The geometric approach is harder, since to link concepts like prime ideals to algebraic sets with certain very nice geometric properties, you must use not only algebra, but topology, several complex variables, and partial differential equations. Doing this with full proofs would rule the book out for all but a very small audience. Yet the theory basically says that, *in any number of dimensions, solution sets are as nice as a few well chosen and simple examples would naively lead an engineer or scientist to expect.*

There remains a tension that we see no way to completely resolve. Dealing with polynomials and algebraic subsets of Euclidean space is basic, but this is not general enough to cover the applications common in engineering and science. For example, the use of products of projective spaces and multihomogeneous polynomials which live on them is extraordinarily useful, but these polynomials are not "functions" on the products of projective spaces. Working in an appropriate generality to cover everything needed would cast a pall over the whole book. Moreover, the early parts of the book need only advanced calculus and a few concepts from algebraic geometry. For this reason, we often restate results in different levels of generality in different parts of the book. We have also included an appendix with detailed statements of useful, more technical results from algebraic geometry.

Part One of the book is introductory.

Chapter 1 gives examples of polynomial systems as they arise in practice and gives an introduction to homotopy continuation, the numerical solution tool underlying our work.

Chapter 2 gives a more detailed discussion of homotopy continuation and what it means to be a complex or real solution of a system of polynomials.

Chapter 3 introduces some algebraic geometry and shows some of the ways it naturally presents itself, e.g., dealing with solutions at infinity and continuation paths going to infinity.

Chapter 4 gives a first discussion of generic points and probability-one algorithms. The powerful ability to choose "generic points" in Euclidean space increases the efficiency and stability of numerical algorithms and eliminates some problems that are endemic in exact symbolic procedures.

In Chapter 5, there is some detailed discussion of polynomials in just one variable. For example, we discuss the fundamental limitations that the number of digits available to us impose on our recognizing a zero of a polynomial.

Chapter 6 gives a brief discussion, with some pointers to the literature, of other approaches to solving systems of polynomials.

Part Two is devoted to the theory and practice of finding isolated solutions of polynomial systems. Here we consider the many special features of a polynomial system that make it amenable to efficient solution.

Chapter 7 explains the coefficient-parameter framework for systems arising in engineering and science. It is a compelling fact that almost all systems that arise in practice depend on parameters, and need to be solved many times for different values of the parameters. Thus it becomes worthwhile to spend extra computation solving such a system if that extra time, amortized over all the times we solve the system, leads to a more efficient and quicker average solution time. We include a case study of this approach applied to Stewart-Gough platform robots.

Polynomial systems arising in engineering and science tend to be sparse and highly structured. In Chapter 8, we give an extended discussion of such special structures. These features cause systems to have fewer solutions than would be naively expected. Taking advantage of this structure leads to more efficient homotopies and much faster solution times.

Chapter 9 gives case studies for systems arising from a number of different engineering and scientific applications. We have found that these systems present challenging problems and excellent trial grounds for improving our algorithms.

Chapter 10 covers endgame methods. These methods exploit continuation to improve the numerical accuracy of singular solutions, such as double or triple roots.

Chapter 11 deals with how to recognize and deal with problems that may occur. The probability-one methods we use are based on choosing generic points. If only we had computers with infinite precision, these methods would eliminate all manner of unpleasant difficulties, e.g., path crossing. Since real computers have only finite precision, the probability of "probability zero" events is very small, but positive. This chapter discusses how to detect the occurrence of such events, in the large problems occurring in engineering and science, and how to deal with them.

Part Three of the book shows how the ability to compute isolated solutions by homotopy continuation can be exploited to manipulate higher-dimensional solution sets of polynomial systems. To do so, we introduce "witness sets" to represent curves, surfaces and other algebraic-geometric sets as numerical objects. Witness sets and the underlying theory should be looked at as a new subject *Numerical Algebraic Geometry* whose relation to *Algebraic Geometry* is similar to the relation of *Numerical Linear Algebra* to *Linear Algebra*.

Chapter 12 introduces some needed material from algebraic geometry, such as the Zariski topology, its relation to the complex topology, the irreducible decomposition, constructible algebraic sets, and multiplicity.

Chapter 13 introduces the basic concepts of numerical algebraic geometry. Primary among these are *witness points*, which is the natural numerical data structure to encode irreducible algebraic sets. We also give an extensive discussion of the reduction to systems with the same number of equations as unknowns. Based on (Sommese & Wampler, 1996), the article where the *Numerical Algebraic Geometry* started, this chapter explains the numerical irreducible decomposition and how to compute "witness point supersets," a first approximation to the witness point sets occurring in the numerical irreducible decomposition.

Chapter 14 presents an alternative procedure to compute the "witness point supersets" of Chapter 13. We follow (Sommese & Verschelde, 2000), with some of the later improvements from (Sommese, Verschelde, & Wampler, 2004b). One novelty is the complete removal of slack variables.

Chapter 15 explains the algorithms to compute the numerical irreducible decomposition. This is primarily based on (Sommese, Verschelde, & Wampler, 2001a, 2001c, 2002b). The chapter ends with a section on singular path-tracking. We give some applications, mainly from the theory of mechanisms, which was a major motivation for our studying the numerical solution of polynomial systems.

Chapter 16 discusses briefly the recent algorithms of (Sommese et al., 2004b, 2004c) to find the numerical irreducible decomposition of the intersection of irreducible algebraic sets. This gives a new method which shows promise for solving large polynomial systems.

Appendix A collects in one place many useful results from algebraic geometry, including some structure theorems relating solutions sets of parameterized polynomial systems at generic points and particular points of the parameter space.

Appendix B lists some software packages available for solving polynomial systems by continuation.

Appendix C contains a users guide to HomLab, a suite of Matlab[1] routines provided by the authors for experimenting with polynomial continuation and working the numerous exercises in this book.

The bibliography is not meant to be exhaustive. At the present time, when a few keystrokes brings a deluge of references, the inclusion of everything of relevance on a topic as broad as polynomial systems would diminish the value of the bibliography as a tool for learning. Given this, we have followed the policy of only including references of such direct relevance to the topics we cover that they are referred to in the text.

Given the frequency with which web addresses change, we do not list explicit addresses of webpages in this book. We do mention numerous websites: it is easy to find their current coordinates by using a search engine.

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[1] "MATLAB" is a registered trademark of The Mathworks, Inc.

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