ON THE REPRESENTATION OF LARGE EVEN INTEGER AS A SUM OF A PRODUCT OF AT MOST 3 PRIMES AND A PRODUCT OF AT MOST 4 PRIMES*

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0. Introduction

V. Brun [1] first proved in 1920 the following result:

Every large even integer is the sum of two integers each being a product of at most 9 primes. We denote this theorem by $(9,9)$, and we may define (a,b) similarly.

Brun's method and his result were improved by several mathematicians, namely:

$(7,7)$ (Rademacher, 1924) [2]
$(6,6)$ (Estermann, 1932) [3]
$(5,7)$, $(4,9)$, $(3,15)$, $(2,366)$ (Ricci, 1937) [11]
$(5,5)$ (Buchstab, 1938) [4]
$(4,4)$ (Buchstab, 1940) [5]

Professor Hua Loo Keng pointed out that $(4,4)$ may be possibly improved by the combination of the methods of Selberg [6], Brun and Buchstab. The purpose of this paper is to prove $(3,4)$, i.e. the following:

Theorem 1. *Every large even integer can be represented as a sum of a product of at most* 3 *primes and a product of at most* 4 *primes.*

Theorem 2. *There are infinitely many integers n such that n has at most* 3 *prime factors and $n+2$ has at most* 4 *prime factors.*

In this paper, we use $p, p', p'', \ldots, p_1, p_2, \ldots$ to denote prime numbers.

It seems possible to use the present method to prove $(3,3)$ but it needs some complicated numerical calculations.

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1. Some Computations

Lemma 1. *If $x \geq 1$ and $N \geq 1$, then*

$$\sum_{\substack{n \leq N \\ (n,x)=1}} \frac{|\mu(n)|2^{\Omega(n)}}{n} = \frac{1}{2}\prod_{p|x}\frac{p}{p+2}\prod_{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)\log^2 N$$

$$+ O(\log 2N \cdot \log\log 3xN) + O((\log\log 3x)^2),$$

where $\mu(n)$ denotes the Möbius function and $\Omega(n)$ the number of distinct prime factors of n.

We refer [7] for the proof.

Lemma 2. *Let $z \geq 1$, $g(1) = 1$, $g(2) = 1/2$, $g(p) = 2/p (p > 2)$ and $g(n) = \prod_{p|n} g(p)$ for square free number n. Then*

$$\sum_{\substack{n \leq z \\ 2\nmid n}} |\mu(n)|g(n)\prod_{p|n}(1-g(p))^{-1} = \frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\log^2 z + O(\log 2z \cdot \log\log 3z).$$

Proof. Set $\psi(q) = \prod_{p|q}(p-2)$. Then

$$\sum_{\substack{2 \leq z \\ 2\nmid n}} |\mu(n)|g(n)\prod_{p|n}(1-g(p))^{-1}$$

$$= \sum_{\substack{2\nmid n \\ n \leq z}} |\mu(n)|\frac{2^{\Omega(n)}}{n}\prod_{p|n}\frac{p}{p-2}$$

$$= \sum_{\substack{n \leq z \\ 2\nmid n}} |\mu(n)|\frac{2^{\Omega(n)}}{n}\prod_{p|n}\left(1+\frac{2}{p-2}\right)$$

$$= \sum_{\substack{n \leq z \\ 2\nmid n}} |\mu(n)|\frac{2^{\Omega(n)}}{n}\sum_{r|n}\frac{2^{\Omega(r)}}{\psi(r)}$$

$$= \sum_{\substack{r \leq z \\ 2\nmid r}} |\mu(r)|\frac{2^{2\Omega(r)}}{\psi(r)r}\sum_{\substack{s \leq z/r \\ (s,2r)=1}}\frac{|\mu(s)|2^{\Omega(s)}}{s}$$

$$= \sum_{\substack{r \leq z \\ 2\nmid r}} |\mu(r)|\frac{2^{2\Omega(r)}}{\psi(r)r}\left\{\frac{1}{2}\prod_{p}\frac{(p-1)^2(p+2)}{p^3}\prod_{p|2r}\frac{p}{p+2}\cdot\log^2\frac{z}{r} + O(\log 2z \cdot \log\log 3z)\right\}$$

$$= \frac{1}{4}\prod_{p}\frac{(p-1)^2(p+2)}{p^3}\log^2 z \cdot \sum_{\substack{r\le z\\ 2\nmid r}}\frac{4^{\Omega(r)}|\mu(r)|}{\prod_{p|r}(p^2-4)}$$

$$+O\left(\log 2z\cdot\sum_{\substack{r\le z\\ 2\nmid r}}\frac{4^{\Omega(r)}|\mu(r)|\log r}{\prod_{p|r}(p^2-4)}\right)+O(\log 2z\cdot\log\log 3z)$$

$$= \frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\cdot\log^2 z+O(\log 2z\cdot\log\log 3z).$$

The lemma is proved.

Lemma 3. *Let $z\ge 1$ and*

$$f(n)=\sum_{d|n}\frac{\mu(d)}{g\left(\frac{n}{d}\right)}=\frac{1}{g(n)}\prod_{p|n}(1-g(p)),$$

for square free number n. Then

$$\sum_{n\le z}\frac{|\mu(n)|}{f(n)}=\frac{1}{4}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\log^2 z+O(\log 2z\cdot\log\log 3z).$$

Proof.

$$\sum_{n\le z}\frac{|\mu(n)|}{f(n)}=\sum_{\substack{n\le z\\ 2\nmid n}}\frac{|\mu(n)|}{f(n)}+\sum_{\substack{n\le z\\ 2|n}}\frac{|\mu(n)|}{f(n)}$$

$$=\sum_{\substack{n\le z\\ 2\nmid n}}|\mu(n)|g(n)\prod_{p|n}(1-g(p))^{-1}+\frac{1}{f(2)}\sum_{\substack{n\le z/2\\ 2\nmid n}}|\mu(n)|g(n)\prod_{p|n}(1-g(p))^{-1}$$

$$=\frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\log^2 z+\frac{1}{8f(2)}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\log^2\frac{z}{2}+O(\log 2z\cdot\log\log 3z)$$

$$=\frac{1}{4}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\log^2 z+O(\log 2z\cdot\log\log 3z).$$

Lemma 4. *Let α,β be two numbers such that $2<\alpha<\beta$. Then*

$$\sum_{x^{1/\beta}<p\le x^{1/\alpha}}\frac{1}{p\log^2\frac{x}{p}}=\frac{1}{\log^2 x}\left\{\log\frac{\beta-1}{\alpha-1}+\frac{1}{\alpha-1}-\frac{1}{\beta-1}\right\}+O\left(\frac{1}{\log^3 x}\right).$$

See [4] and [8].

2. Theorem A

Let

$$(w) \quad a = 0 \text{ or } 1, \qquad 0 \le a_i, b_i < p_i, \qquad a_i \ne b_i \quad (1 \le i \le r)$$

be a set of numbers, where $3 = p_1 < p_2 < \cdots < p_r \le \xi$ are all the odd prime numbers $\le \xi$. Let $P_w(x, \xi)$ be the number of integers satisfying the following conditions:

$$n \le x, \quad n \equiv a \pmod 2, \quad n \not\equiv a_i \pmod{p_i}, \quad n \not\equiv b_i \pmod{p_i}\ (1 \le i \le r). \qquad (1)$$

It follows by Sun Zi theorem (Chinese Remainder Theorems) that the systems of congruences

$$\begin{cases} y \equiv 1 + a \pmod 2, \\ y \equiv a_i \quad \pmod{p_i} \quad (1 \le i \le r); \end{cases} \qquad \begin{cases} y \equiv 1 + a \pmod 2, \\ y \equiv b_i \quad \pmod{p_i} \quad (1 \le i \le r), \end{cases}$$

have unique solutions a^*, b^* respectively in the interval $0 \le y < 2p_1 \cdots p_r$.

Now we proceed to show that the number of integers n satisfying (1) is equal to the number satisfying the following:

$$n \le x, \quad (n - a^*)(n - b^*) \not\equiv 0 \pmod{p_i}\ (1 \le i \le r), \quad (n - a^*)(n - b^*) \not\equiv 0 \pmod 2. \qquad (2)$$

In fact, if n satisfies (1), then

$$(n - a^*)(n - b^*) \equiv (n - a_i)(n - b_i) \not\equiv 0 \pmod{p_i}\ (1 \le i \le r),$$
$$(n - a^*)(n - b^*) \equiv (a - 1 - a)^2 \equiv 1 \pmod 2.$$

On the other hand, if n satisfies (2), then

$$(n - a_i)(n - b_i) \equiv (n - a^*)(n - b^*) \not\equiv 0 \pmod{p_i}\ (1 \le i \le r),$$

i.e.

$$n \not\equiv a_i \pmod{p_i}, \quad n \not\equiv b_i \pmod{p_i}\ (1 \le i \le r).$$

We also have

$$(n - 1 - a)^2 \equiv (n - a^*)(n - b^*) \not\equiv 0 \pmod 2,$$

and thus $n \not\equiv 1 + a \pmod 2$, i.e. $n \equiv a \pmod 2$. Therefore n also satisfies (1).

Theorem A. *Let $c > 0$ and $P = \prod_{p \le \xi} p$. Then*

$$P_w(x,\xi) \le \frac{x}{\sum\limits_{\substack{1\le k\le \xi^c \\ k|p}} \frac{\mu^2(k)}{f(k)}} + O\left(\sum_{\substack{1\le k_1,\, k_2 \le \xi^c \\ k_1 | P \\ k_2 | P}} |\lambda_{k_1}\lambda_{k_2}| 2^{\Omega(k_1)} 2^{\Omega(k_2)} \right),$$

holds uniformly for any given (w), where $g(1) = 1,\ g(2) = 1/2,\ g(p) = 2/p (p > 2)$, $g(n) = \prod_{p|n} g(p)$ and $f(n) = \sum_{d|n} \mu(d) / g\left(\frac{n}{d}\right)$ for square free number n,

$$\lambda_n = \frac{\mu(n)}{g(n)f(n)} \sum_{\substack{1\le m\le \xi^c/n \\ (n,m)=1 \\ m|P}} \frac{\frac{\mu^2(m)}{f(m)}}{\sum\limits_{\substack{1\le l\le \xi^c \\ l|P}} \frac{\mu^2(l)}{f(l)}}.$$

Proof. Let $k|P$. Then

$$\begin{aligned} \sum_{\substack{k|(n-a^*)(n-b^*) \\ n\le x}} 1 &= 2^{\Omega(k)-\Omega(k,2)} \left[\frac{x}{k}\right] + O(2^{\Omega(k)}) \\ &= \frac{2^{\Omega(k)-\Omega((k,2))}}{k} x + O(2^{\Omega(k)}) \\ &= g(k)x + O(2^{\Omega(k)}). \end{aligned}$$

Since the number of integers satisfying (1) is equal to the number of integers satisfying (2), $\lambda_1 = 1$, and $\lambda_d = 0$ $(d > \xi^c)$, we have

$$\begin{aligned} P_w(x,\xi) &= \sum_{\substack{n\le x \\ ((n-a^*)(n-b^*),P)=1}} 1 = \sum_{n\le x} \sum_{d|((n-a^*)(n-b^*),P)} \mu(d) \\ &\le \sum_{n\le x} \left(\sum_{d|((n-a^*)(n-b^*),P)} \lambda_d \right)^2 = \sum_{\substack{d_1|P \\ d_1\le \xi^c}} \sum_{\substack{d_2|P \\ d_2\le\xi^c}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{\frac{d_1d_2}{(d_1,d_2)} \left| (n-a^*)(n-b^*) \right. \\ n\le x}} 1 \\ &= x \sum_{\substack{d_1|P \\ d_1\le \xi^c}} \sum_{\substack{d_2|P \\ d_2\le\xi^c}} \lambda_{d_1}\lambda_{d_2} g\left(\frac{d_1d_2}{(d_1,d_2)}\right) \\ &\quad + O\left(\sum_{\substack{d_1|P \\ d_1\le \xi^c}} \sum_{\substack{d_2|P \\ d_2\le\xi^c}} |\lambda_{d_1}\lambda_{d_2}| 2^{\Omega(d_1)+\Omega(d_2)} \right) = xQ + R. \end{aligned}$$

When n is square free, we have

$$\frac{1}{g(n)} = \sum_{\tau|n} \frac{1}{g(\tau)} \sum_{d|n/\tau} \mu(d) = \sum_{d\tau|n} \frac{\mu(d)}{g(\tau)} = \sum_{k|n}\sum_{d|k} \frac{\mu(d)}{g\left(\frac{k}{d}\right)} = \sum_{k|n} f(k)$$

and

$$\begin{aligned} Q &= \sum_{\substack{1\le d_1\le \xi^c \\ d_1|P}} \sum_{\substack{1\le d_2\le \xi^c \\ d_2|P}} \lambda_{d_1}\lambda_{d_2}\frac{g(d_1)g(d_2)}{g((d_1,d_2))} \\ &= \sum_{\substack{1\le d_1\le \xi^c \\ d_1|P}} \sum_{\substack{1\le d_2\le \xi^c \\ d_2|P}} \lambda_{d_1}\lambda_{d_2} g(d_1)g(d_2) \sum_{d|(d_1,d_2)} f(d) \\ &= \sum_{\substack{1\le d\le \xi^c \\ d|P}} f(d)\left(\sum_{\substack{1\le k\le \xi^c \\ d|k|P}} \lambda_k g(k)\right)^2 . \end{aligned}$$

Let

$$S = \sum_{\substack{1\le m\le \xi^c \\ m|p}} \frac{\mu^2(m)}{f(m)},$$

then

$$\lambda_k g(k) = \frac{1}{S}\sum_{\substack{1\le m\le \xi^c/k \\ (m,k)=1 \\ m|p}} \frac{\mu(k)\mu^2(m)}{f(k)f(m)} = \frac{1}{S}\sum_{\substack{1\le m\le \xi^c/k \\ (m,k)=1 \\ m|p}} \frac{\mu(mk)\mu(m)}{f(mk)},$$

$$\begin{aligned} \sum_{\substack{d|k|P \\ 1\le k\le \xi^c}} \lambda_k g(k) &= \frac{1}{S}\sum_{\substack{d|k|P \\ 1\le k\le \xi^c}} \sum_{\substack{1\le m\le \xi^c/k \\ m|P}} \frac{\mu(mk)\mu(m)}{f(mk)} \\ &= \frac{1}{S}\sum_{\substack{1\le r\le \xi^c \\ d|r|P}} \frac{\mu(r)}{f(r)} \sum_{d|k|r} \mu\left(\frac{r}{k}\right) \\ &= \frac{1}{S}\frac{\mu(d)}{f(d)}. \end{aligned}$$

Therefore

$$Q = \frac{1}{S}.$$

The theorem follows.

3. Applications of Theorem A

First, we estimate the second term of the inequality in Theorem A. Let $\xi > 3$ and $d(k) = \sum_{r|k} 1$. Then

$$2^{\Omega(k)} \le d(k) \quad \text{and} \quad \sum_{k\le \xi} d(k) = O(\xi \log \xi).$$

Hence

$$
\begin{aligned}
R &= O\left(\sum_{\substack{k_1\le \xi^c\\ k_1\mid P}}\ \sum_{\substack{k_2\le \xi^c\\ k_2\mid P}} |\lambda_{k_1}\lambda_{k_2}|2^{\Omega(k_1)}\cdot 2^{\Omega(k_2)}\right)\\
&= O\left(\left(\sum_{\substack{1\le k\le \xi^c\\ k\mid P}} |\lambda_k|2^{\Omega(k)}\right)^2\right) = O\left(\left(\sum_{\substack{1\le k\le \xi^c\\ k\mid P}} \frac{|\mu(k)|}{|f(k)g(k)|}2^{\Omega(k)}\right)^2\right)\\
&= O\left(\left(\sum_{1\le k\le \xi^c} \frac{|\mu(k)|2^{\Omega(k)}}{\prod\limits_{2<p\le \xi^c}\left(1-\frac{2}{p}\right)}\right)^2\right) = O\left(\log^4\xi\cdot\left(\sum_{1\le k\le \xi^c} d(k)\right)^2\right)\\
&= O(\xi^{2c}\log^6\xi). \qquad (3)
\end{aligned}
$$

If l is an integer ≥ 1 and $l < c \le l+1$, then

$$
\begin{aligned}
\sum_{\substack{1\le n\le \xi^c\\ n\nmid p}} \frac{|\mu(n)|}{f(n)} &= \sum_{1\le n\le \xi^c} \frac{|\mu(n)|}{f(n)} - \sum_{\xi<p\le \xi^c}\ \sum_{\substack{1\le n\le \xi^c\\ p\mid n}} \frac{|\mu(n)|}{f(n)}\\
&\quad + \sum_{\substack{\xi<p_1<p_2\\ p_1p_2\le \xi^c}}\ \sum_{\substack{1\le n\le \xi^c\\ p_1p_2\mid n}} \frac{|\mu(n)|}{f(n)} - \cdots + \cdots + (-)^l \sum_{\substack{\xi<p_1<\cdots<p_l\\ p_1p_2\cdots p_l\le \xi^c}}\ \sum_{\substack{1\le n\le \xi^c\\ p_1\cdots p_l\mid n}} \frac{|\mu(n)|}{f(n)}\\
&= \sum_{1\le n\le \xi^c} \frac{|\mu(n)|}{f(n)} - \sum_{\xi<p\le \xi^c} \frac{1}{f(p)} \sum_{\substack{1\le n\le \xi^c/p\\ (p,n)=1}} \frac{|\mu(n)|}{f(n)}\\
&\quad + \sum_{\substack{\xi<p_1<p_2\\ p_1p_2\le \xi^c}} \frac{1}{f(p_1)f(p_2)} \sum_{\substack{1\le n\le \xi^c/p_1p_2\\ (p_1p_2,n)=1}} \frac{|\mu(n)|}{f(n)}\\
&\quad - \cdots + \cdots + (-1)^l \sum_{\substack{\xi<p_1<\cdots<p_l\\ p_1\cdots p_l\le \xi^c}} \frac{1}{f(p_1)\cdots f(p_l)} \sum_{\substack{1\le n\le \xi^c/p_1\cdots p_l\\ (n,p_1\cdots p_l)=1}} \frac{|\mu(n)|}{f(n)}. \qquad (4)
\end{aligned}
$$

(i) Suppose $1 < c \le 2$. Since

$$
\sum_{\xi<p\le \xi^c} \frac{1}{f(p)} - \sum_{\xi<p\le \xi^c} \frac{2}{p} = \sum_{\xi<p\le \xi^c}\left(\frac{2}{p-2}-\frac{2}{p}\right) = O\left(\sum_{n>\xi}\frac{1}{n^2}\right) = O\left(\frac{1}{\xi}\right),
$$

it follows from (4) and Lemma 3 that

$$\begin{aligned}\sum_{\substack{1\le n\le \xi^c\\ n|P}} \frac{|\mu(n)|}{f(n)} &= \sum_{1\le n\le \xi^c} \frac{|\mu(n)|}{f(n)} - \sum_{\xi<p\le \xi^c} \frac{2}{p} \sum_{1\le n\le \xi^c/p} \frac{|\mu(n)|}{f(n)} + O\left(\frac{\log^2 \xi}{\xi}\right)\\
&= \frac{1}{4}\prod_{p>2} \frac{(p-1)^2}{p(p-2)} \log^2 \xi^c - \sum_{\xi<p\le\xi^c} \frac{2}{p}\cdot\frac{1}{4}\prod_{p>2}\frac{(p-1)^2}{p(p-2)} \log^2 \frac{\xi^c}{p}\\
&\quad + O(\log\xi \log\log\xi)\\
&= \frac{1}{4}\prod_{p>2} \frac{(p-1)^2}{p(p-2)} \{(2c-1)^2 - 2c^2\log c\}\log^2\xi + O(\log\xi\log\log\xi),\end{aligned}$$

where the following well-known formulas are used:

$$\begin{aligned}\sum_{p\le x} \frac{1}{p} &= \log\log x + c_1 + O\left(\frac{1}{\log x}\right), \quad c_1 \text{ is a constant},\\
\sum_{p\le x} \frac{\log p}{p} &= \log x + O(1),\\
\sum_{p\le x} \frac{\log^2 p}{p} &= \frac{1}{2}\log^2 x + O(\log x).\end{aligned}$$

Take $\xi = x^{1/2c}/\log^5 x$, and $2c = d$. Then it follows by Theorem A and (3) that

$$\begin{aligned}P_w(x, x^{1/d}) &\le P_w\left(x, \frac{x^{1/d}}{\log^5 x}\right)\\
&\le 2e^{-2\gamma}\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\cdot\Lambda(d)\frac{x}{\log^2 x} + O\left(\frac{x\log\log x}{\log^3 x}\right), \qquad (5)\end{aligned}$$

where γ is Euler constant and

$$\Lambda(d) = 2e^{2\gamma}\left[\frac{d^2}{(d-1)^2 - 2\left(\frac{d}{2}\right)^2\log\frac{d}{2}}\right], \quad 2\le d\le 4. \qquad (6)$$

(ii) Suppose $2 < c \le 3$. Since

$$\begin{aligned}&\sum_{\xi<p\le\xi^c}\frac{1}{f(p)}\sum_{1\le n\le\xi^c/p}\frac{|\mu(n)|}{f(n)} - \sum_{\xi<p\le\xi^c}\frac{1}{f(p)}\sum_{\substack{1\le n\le\xi^c/p\\ (p,n)=1}}\frac{|\mu(n)|}{f(n)}\\
&= O\left(\sum_{\xi<p\le\xi^c}\frac{1}{f(p)}\sum_{\substack{1\le n\le\xi^c/p\\ p|n}}\frac{|\mu(n)|}{f(n)}\right) = O\left(\sum_{\xi<p\le\xi^c}\frac{1}{f(p)^2}\log^2\xi\right) = O\left(\frac{\log^2\xi}{\xi}\right)\end{aligned}$$

and

$$\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\frac{1}{f(p)f(p')}-\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\frac{4}{pp'}=\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\left(\frac{2}{p-2}\cdot\frac{2}{p'-2}-\frac{2}{p-2}\cdot\frac{2}{p'}\right)$$
$$+\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\left(\frac{2}{p-2}\cdot\frac{2}{p'}-\frac{4}{pp'}\right)$$
$$=O\left(\sum_{\xi<p\le\xi^c}\frac{1}{p}\sum_{p'>\xi}\frac{1}{p'^2}\right)+O\left(\sum_{\xi<p'\le\xi^c}\frac{1}{p'}\sum_{p>\xi}\frac{1}{p^2}\right)$$
$$=O\left(\frac{1}{\xi}\right),$$

we have

$$\sum_{\substack{n\le\xi^c\\ n|P}}\frac{|\mu(n)|}{f(n)}=\sum_{n\le\xi^c}\frac{|\mu(n)|}{f(n)}-\sum_{\xi<p\le\xi^c}\frac{2}{p}\sum_{1\le n\le\xi^c/p}\frac{|\mu(n)|}{f(n)}$$
$$+\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\frac{4}{pp'}\sum_{1\le n\le\xi^c/pp'}\frac{|\mu(n)|}{f(n)}+O\left(\frac{\log^2\xi}{\xi}\right)$$
$$=\frac{1}{4}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\Bigg[(2c-1)^2-2c^2\log c$$
$$+\left(\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\frac{4}{pp'}\log^2\frac{\xi^c}{pp'}\right)\frac{1}{\log^2\xi}\Bigg]\log^2\xi+O(\log\xi\log\log\xi).$$

By (4) and Lemma 3. Take $\xi=x^{1/2c}/\log^5 x$ and $d=2c$. Then we have by Theorem A and (3) that (5) holds also for $4<d\le 6$, where

$$\Lambda(d)=2e^{2\gamma}\left(\frac{d^2}{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}+\delta\left(\frac{d}{2}\right)}\right),\quad 4\le d\le 6,\tag{7}$$

and

$$\delta(c)=4\sum_{\substack{\xi<p<p'\\ pp'\le\xi^c}}\frac{1}{pp'}\log^2\frac{\xi^c}{pp'}\Big/\log^2\xi,\quad c\ge 2.\tag{8}$$

Similarly we have

$$\Lambda(d)=2e^{2\gamma}\left(\frac{d^2}{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}+\delta\left(\frac{d}{2}\right)-\kappa\left(\frac{d}{2}\right)}\right),\quad 6\le d\le 8,\tag{9}$$

where

$$\kappa(c) = 8 \sum_{\substack{pp'p'' \le \xi^c \\ \xi < p < p' < p''}} \frac{1}{pp'p''} \log^2 \frac{\xi^c}{pp'p''} \Big/ \log^2 \xi, \quad c \ge 3. \tag{10}$$

The case of $d > 8$ may also be treated.

4. Two Iteration Theorems

Theorem B$_1$. *If $\Lambda_k(\alpha)$ and $\lambda_i(\alpha)$ $(2 \le \alpha \le 15)$ are two increasing functions with at most finite number of discontinuities of first kind such that*

$$P_w(x, x^{1/\alpha}) > \lambda_i(\alpha) \frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x} \log\log x\right), \quad 2 \le \alpha \le 15, \tag{11}$$

and

$$P_w(x, x^{1/\alpha}) < \Lambda_k(\alpha) \frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x} \log\log x\right), \quad 2 \le \alpha \le 15 \tag{12}$$

holds uniformly for (w), where $c = 2e^{-2\gamma} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)$ and γ denotes Euler constant, then

$$\psi(\alpha) = \begin{cases} 0, & 2 \le \alpha \le \tau, \\ \lambda_i(\beta) - 2\int_{\alpha-1}^{\beta-1} \Lambda_k(z) \frac{z+1}{z} dz, & 2 \le \tau \le \alpha \le \beta \le 15, \end{cases}$$

and

$$w(\alpha) = \Lambda_k(\beta) - 2\int_{\alpha-1}^{\beta-1} \lambda_i(z) \frac{z+1}{z^2} dz, \quad 2 \le \tau \le \alpha \le \beta \le 15$$

satisfies (11) and (12) respectively. We usually write $\psi(\alpha) = \lambda_{i+1}(\alpha)$ and $w(\alpha) = \Lambda_{k+1}(\alpha)$.

We refer to [4] for the proof.

Let

$$\begin{aligned} (w) \quad a = 0 \text{ or } 1, \quad 0 \le a_i < p_i \ (1 \le i \le r), \quad & 0 \le b_j < p_j \ (1 \le j \le s), \\ & a_v \ne b_v \quad (v = 1, 2, \ldots, \min(r, s)), \end{aligned}$$

be a set of integers, where $\xi = p_1 < \cdots < p_r \le y$ are all the odd prime numbers $\le z$. Let $P_w(x, y, z)$ be the number of integers satisfying

$$\begin{aligned} n \le x, \quad n \equiv a \,(\text{mod}\, 2), \quad & n \not\equiv a_i \,(\text{mod}\, p_i)\,(1 \le i \le r), \\ & n \not\equiv b_i \,(\text{mod}\, p_i)\,(1 \le i \le s). \end{aligned} \tag{13}$$

In particular, we have $P_w(x, y, z) = P_w(x, y)$, for $y = z$.

Theorem B_2. *If the inequality*

$$P_w(x, x^{1/\alpha}) < \Lambda(\alpha)\frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x}\log\log x\right), \quad 2 \le \alpha \le 15$$

holds uniformly for (w), where $\Lambda(d)$ is an increasing function which has at most finite number of discontinuities of the first kind, then for any given three positive numbers α, β, γ with $2 < \gamma \le \beta \le \alpha$, we have

$$P_w(x, x^{1/\gamma}, x^{1/\alpha}) > P_w(x, x^{1/\beta}, x^{1/\alpha}) - \Lambda\left(\frac{(\beta-1)\alpha}{\beta}\right) c \int_{\gamma-1}^{\beta-1} \frac{z+1}{z^2} dz \cdot \frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\log\log x\right).$$

Proof. If $p_m \ge p_r$, then the difference between $P_w(x, p_m, p_r)$ and $P_w(x, p_{m+1}, p_r)$ is equal to the number of integers satisfying

$$\begin{aligned} &n \le x, \quad n \equiv a \,(\mathrm{mod}\, 2), \quad n \not\equiv b_i \,(\mathrm{mod}\, p_i)\,(1 \le i \le r), \\ &\qquad n \not\equiv a_j \,(\mathrm{mod}\, p_j)\,(1 \le j \le m), \quad n \equiv a_{m+1} \,(\mathrm{mod}\, p_{m+1}). \end{aligned} \tag{14}$$

Denote by a_i^*, b_i^* and $\tilde{a}_m$ the respective solutions of the congruences

$$\begin{cases} p_{m+1}y + a_{m+1} \equiv a_i \,(\mathrm{mod}\, p_i) & (0 \le y < p_i), \\ p_{m+1}y + a_{m+1} \equiv b_i \,(\mathrm{mod}\, p_i) & (0 \le y \le p_i), \\ p_{m+1}y + a_{m+1} \equiv a \,(\mathrm{mod}\, 2) & (0 \le y \le 1). \end{cases}$$

Then $a_i^* \ne b_i^*$ $(1 \le i \le r)$ and therefore the number of integers satisfying (14) is equal to the number of integers satisfying

$$\begin{aligned} &n \le \frac{x - a_{m+1}}{p_{m+1}}, \quad n \equiv \tilde{a}_m \,(\mathrm{mod}\, 2), \quad n \not\equiv a_i^* \,(\mathrm{mod}\, p_i)\,(1 \le i \le m), \\ &\qquad\qquad n \not\equiv b_i^* \,(\mathrm{mod}\, p_i)\,(i \le r). \end{aligned} \tag{15}$$

Let

$$(w_m) \quad 0 \le \tilde{a}_m \le 1, \quad 0 \le a_i^* < p_i \ (i \le m), \quad 0 \le b_j^* < p_j \ (j \le r).$$

Then the number of integers satisfying (15) is $P_{w_m}(x - a_{m+1}/p_{m+1}, p_m, p_r)$. Hence

$$\begin{aligned} P_w(x, p_m, p_r) - P_w(x, p_{m+1}, p_r) &= P_{w_m}\left(\frac{x - a_{m+1}}{p_{m+1}}, p_m, p_r\right) \\ &\le P_{w_m}\left(\frac{x}{p_{m+1}}, p_m, p_r\right) \le P_{w_m}\left(\frac{x}{p_{m+1}}, p_r\right). \end{aligned} \tag{16}$$

Now let us arrange the prime numbers between $x^{1/\beta}$ and $x^{1/\gamma}$ as follows:

$$p_t \le x^{1/\beta} < p_{t+1} < \cdots < p_s \le x^{1/\gamma} < p_{s+1}.$$

We have

$$P_w(x, x^{1/\beta}, x^{1/\alpha}) = P_w(x, p_t, x^{1/\alpha}), \quad P_w(x, x^{1/\gamma}, x^{1/\alpha}) = P_w(x, p_s, x^{1/\alpha}).$$

Using (16) successively, we obtain

$$\begin{aligned}
P_w(x, x^{1/\beta}, x^{1/\alpha}) &\le P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \sum_{x^{1/\beta} < p_{i+1} \le x^{1/\gamma}} P_{w_i}\left(\frac{x}{p_{i+1}}, x^{1/\alpha}\right) \\
&= P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \sum_{x^{1/\beta} < p_{i+1} \le x^{1/\gamma}} P_{w_i}\left(\frac{x}{p_{i+1}}, \left(\frac{x}{p_{i+1}}\right)^{\frac{\log x^{1/\alpha}}{\log \frac{x}{p_{i+1}}}}\right) \\
&\le P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \sum_{x^{1/\beta} < p_{i+1} \le x^{1/\gamma}} P_{w_i}\left(\frac{x}{p_{i+1}}, \left(\frac{x}{p_{i+1}}\right)^{\frac{1}{\frac{(\beta-1)\alpha}{\beta}}}\right) \\
&\le P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \sum_{x^{1/\beta} < p_{i+1} \le x^{1/\gamma}} \Lambda\left(\frac{(\beta-1)\alpha}{\beta}\right) \frac{cx}{p_{i+1} \log^2 \frac{x}{p_{i+1}}} \\
&\quad + O\left(\sum_{x^{1/\beta} < p_{i+1} \le x^{1/\gamma}} \frac{x}{p_{i+1} \log^3 \frac{x}{p_{i+1}}} \log\log \frac{x}{p_{i+1}} \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
P_w(x, x^{1/\beta}, x^{1/\alpha}) &\le P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \Lambda\left(\frac{(\beta-1)\alpha}{\beta}\right) \left(\log \frac{\beta-1}{\gamma-1} + \frac{1}{\gamma-1} - \frac{1}{\beta-1}\right) \\
&\quad \times \frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x} \log\log x\right) \\
&= P_w(x, x^{1/\gamma}, x^{1/\alpha}) + \Lambda\left(\frac{(\beta-1)\alpha}{\beta}\right) \int_{\gamma-1}^{\beta-1} \frac{z+1}{z^2} dz \cdot \frac{cx}{\log^2 x} \\
&\quad + O\left(\frac{x}{\log^3 x} \log\log x\right),
\end{aligned}$$

by Lemma 4, the theorem is proved.

5. Proofs of Theorems

Let $\lambda(\alpha)$ and $\Lambda(\alpha)$ be two increasing functions with at most finite number of discontinuities such that

$$\begin{aligned}
&\lambda(\alpha) \frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x} \log\log x\right) \\
&\quad < P_w(x, x^{1/\alpha}) < \Lambda(\alpha) \frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x} \log\log x\right), \quad 2 \le \alpha \le 15,
\end{aligned}$$

holds uniformly on (w). We denote these functions by $\lambda_0(\alpha)$, $\Lambda_0(\alpha)$, $\lambda_1(\alpha)$, $\Lambda_1(\alpha), \ldots$.

By (6), (7), (9) and Theorem B_1, we have

α	10	9	8	7	6	5	4
$\lambda_0(\alpha)$	99.98181*	79.78469	60.88817	43.51554	26.70925	9.18109	0
$\Lambda_0(\alpha)$	100.02073*	82.7207	68.52511	54.39352	43.0082	34.89666	29.39023

Divide the interval $\alpha - 1 < x \le \beta - 1$ into n subintervals $u_i < x \le u_{i+1}$ ($i = 0, 1, \dots, n-1$), where $u_0 = \alpha - 1$ and $u_n = \beta - 1$. Since $\lambda(\alpha)$ and $\Lambda(\alpha)$ are increasing functions, we have

$$\int_{\alpha-1}^{\beta-1} \lambda(z)\frac{z+1}{z^2}dz \ge \sum_{s=0}^{n-1} \lambda(u_s) \int_{u_s}^{u_{s+1}} \frac{z+1}{z^2}dz,$$

and

$$\int_{\alpha-1}^{\beta-1} \Lambda(z)\frac{z+1}{z^2}dz \le \sum_{s=0}^{n-1} \Lambda(u_{s+1}) \int_{u_s}^{u_{s+1}} \frac{z+1}{z^2}dz.$$

Take $u_{s+1} - u_s = 0.02$. Starting from $\lambda_0(\alpha)$ and $\Lambda_0(\alpha)$, we obtain the following table by Theorem B_1 with several iterations

α	10	9	$\cdots$	6	5
$\lambda_i(\alpha)$	99.98181*	80.71187		29.28627	11.75811
$\Lambda_i(\alpha)$	100.02073*	81.36441		43.0082	34.89666

Therefore by Theorem B_2,

$$\begin{aligned}
P_w(x, x^{1/4}, x^{1/5}) &> P_w(x, x^{\frac{1}{4.2}}, x^{1/5}) - \Lambda\left(\frac{3.2\times 5}{4.2}\right)\int_3^{3.2} \frac{z+1}{z}dz \cdot \frac{cx}{\log^2 x} \\
&\quad + O\left(\frac{x}{\log^3 x}\log\log x\right) \\
&> P_w(x, x^{1/5}, x^{1/5}) - \left\{\Lambda\left(\frac{3.2\times 5}{4.2}\right)\int_3^{3.2}\frac{z+1}{z^2}dz\right. \\
&\quad + \Lambda\left(\frac{3.4\times 5}{4.4}\right)\int_{3.2}^{3.4}\frac{z+1}{z^2}dz + \Lambda\left(\frac{3.6\times 5}{4.6}\right)\int_{3.4}^{3.6}\frac{z+1}{z^2}dz \\
&\quad \left. + \Lambda\left(\frac{3.8\times 5}{4.8}\right)\int_{3.6}^{3.8}\frac{z+1}{z^2}dz + \Lambda(4)\int_{3.8}^{4}\frac{z+1}{z^2}dz\right\}\frac{cx}{\log^2 x} \\
&\quad + O\left(\frac{x}{\log^3 x}\log\log x\right) \\
&> (11.75811 - 10.75728)\frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x}\log\log x\right) \\
&= 1.00083\frac{cx}{\log^2 x} + O\left(\frac{x}{\log^3 x}\log\log x\right). \qquad (17)
\end{aligned}$$

*$\Lambda(10) = 100.02073$ and $\lambda(10) = 99.98181$ are taken from Buchstab [5].

(a) Suppose that x is an even number. Let $a = 1, a_i = 0, b_i = x \pmod{p_i}$ $(i = 1, 2, \ldots)$. Then it follows by (17) that there exists a constant x_0 such that

$$P_w(x, x^{1/4}, x^{1/5}) = \sum_{\substack{n \le x \\ p|n \Rightarrow p > x^{1/4} \\ p|(x-n) \Rightarrow p > x^{1/5}}} 1 > \frac{cx}{\log^2 x}, \quad x > x_0.$$

This shows that there exists an integer $n \le x$ such that the prime divisors of n and $x = n$ are greater than $x^{1/4}$ and $x^{1/5}$ respectively, i.e. the respective numbers of prime factors of n and $x - n$ are at most 3 and 4. Theorem 1 is proved.

(b) Take $a = 1, a_i = 0, b_i = p_i - 2\, (i = 1, 2, \ldots)$. Then (17) becomes

$$P_w(x, x^{1/4}, x^{1/5}) = \sum_{\substack{n \le x \\ p|n \Rightarrow p > x^{1/4} \\ p|(n+2) \Rightarrow p > x^{1/5}}} 1 > \frac{cx}{\log^2 x}, \quad x > x_0.$$

Theorem 2 follows.

A. Selberg announced that some results might be possibly obtained by his method, for example, (2, 3) in [9] and (3, 3) in [10]. However, the proofs of these results did not appeared in the literature till now.

The present method can also be used to prove the following results which will be published in other papers.

Let $F(x)$ be an irreducible polynomial of degree k without any fixed prime divisor and $\pi(N; F(x))$ be the number of integers x in the interval $1 \le x \le N$ such that $F(x)$ are primes. Then we have the following results:

(a) There exist infinitely many x such that $F(x)$ is a product of at most $[2.1k]$ primes.

(b)

$$\pi(N; F(x)) \le 2e^{\gamma}\mu_F \frac{N}{\log N} + o\left(\frac{N}{\log N}\right),$$

where γ is Euler constant, μ_F a constant depending on $F(x)$ and the constant implicit in "o" depending on $F(x)$ only.

Under the assumption of Generalised Riemann hypothesis (GRH), i.e. the assumption that the real parts of zeros of all Dirichlet's L-functions are $\le 1/2$, we have the following two results:

(c) Every large even integer is a sum of a prime and a product of at most 4 primes.

(d) $(p, p + 2)$ is called a pair of twin primes, if p and $p + 2$ are all prime numbers. Let $Z_2(N)$ be the number of primes such that $p \le N$ and $(p, p + 2)$ is twin primes. Then

$$Z_2(N) \le (8+\varepsilon) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^3 N}\right),$$

where ε is any pre-assigned positive number and the constant implicit in "O" depends on ε only.

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