

Introduction

The boundary-value problem

$$-y'' + q(x)y = \lambda y \tag{0.1}$$

$$y'(a) \cos \alpha - y(a) \sin \alpha = 0 \tag{0.2}$$

$$y'(b) \cos \beta + y(b) \sin \beta = 0 \tag{0.3}$$

is known in the literature as the Sturm–Liouville problem. (More generally, one can consider the Sturm–Liouville problem of the form (1.1)–(1.3) below.)

The boundary-value problem (0.1)–(0.3) is said to be regular if the interval $[a, b]$ is finite and the function $q(x)$ is integrable on $[a, b]$. If the interval $[a, b]$ is infinite or $q(x)$ is not integrable on $[a, b]$, the problem (0.1)–(0.3) is said to be singular.

For the regular Sturm–Liouville problem, classical Sturm Oscillation, Comparison and Alternation Theorems have been well known for a long time.

Theorem 0.1 (Oscillation Theorem) *There exists an unbounded increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ of the boundary-value problem (0.1)–(0.3). The eigenfunction corresponding to the eigenvalue λ_n has exactly $n - 1$ zeros in the interval (a, b) .*

Theorem 0.2 (Alternation Theorem) *The zeros of the n -th and $(n + 1)$ -st eigenfunctions of the boundary-value problem (0.1)–(0.3) alternate, i.e. between any two consecutive zeros of the n -th eigenfunction, there is a zero of the $(n + 1)$ -st eigenfunction.*

Theorem 0.2 is a particular case of the following fundamental Sturm Alternation Theorem.

Theorem 0.3 Assume that the following two equations are given:

$$-u'' + q_1(x)u = 0 \quad (0.4)$$

and

$$-v'' + q_2(x)v = 0. \quad (0.5)$$

If $q_1(x) > q_2(x)$ for all $x \in [a, b]$, then between every two zeros of any non-trivial solution of (0.4), there is at least one zero of every solution of (0.5).

Theorem 0.4 (Comparison Theorem) Assume that $u(x)$ is a solution of (0.4) satisfying the initial conditions

$$u(a) = \cos \alpha, \quad u'(a) = \sin \alpha,$$

and $v(x)$ is a solution of (0.5) with the same initial conditions. Additionally, assume that $q_1(x) > q_2(x)$ for all $x \in [a, b]$. Assume that $u(x)$ has n zeros in the interval $(a, b]$. Then $v(x)$ has at least n zeros in $(a, b]$, and the k -th zero of $v(x)$ is less than the k -th zero of $u(x)$.

Analogous results hold for semi-bounded problems on an infinite interval; see [58] and [113].

Below we formulate Morse Index Theorem mentioned in the Preface. This theorem pertains to Jacobi equation for a vector field along a geodesic γ on a smooth Riemannian manifold. The corresponding Jacobi equation reduces to the equation (0.1) with $\lambda = 0$ and $(m \times m)$ -matrix real symmetric coefficient $q(x)$ expressed, in a special way, through the Riemann curvature tensor and tangent vector to the geodesic—"velocity vector" (see [76, Sec. 4.2]⁷ and [136, Sec. 14]). The boundary conditions (0.2) and (0.3) satisfied by the vector solutions of the corresponding equation (0.1) now have the simplest possible form: $y(0) = y(1) = 0$.

Morse Index Theorem ([76, Sec. 4.6] or [136, Sec. 15])

Index of the Hessian for the action function

$$E_{**}: T\Omega_\gamma \times T\Omega_\gamma \rightarrow \mathbb{R}$$

*(i.e. maximal dimension of subspaces $T\Omega_\gamma$ on which the form E_{**} is negative definite) is finite and equal to the number of points of the geodesic $\gamma(t)$,*

⁷In [76] the Jacobi equation is reduced to a somewhat more general system. However, the corresponding system is equivalent to the above described equation through a unitary transformation in the L^2 -space of vector-valued functions.

$0 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ , where every conjugate point is counted according to its multiplicity. (In Morse terminology, this multiplicity is also called the index of the point t with respect to the endpoint $t = 0$ and the equation (0.1). The sum of indices of the point t along a given geodesic segment is called "Morse index of the given geodesic segment.")

There are many works devoted to further development of Morse theory. Here we note the works [311], [386], [392], [504], [509], [707], [711], [726], [840], [866], [884], [893], and the recent review by S. P. Novikov [710].

Morse index is a special case of Maslov index [128]–[131]; see also [10], [57], [137], [237] and [238]. Maslov index plays an important role in multidimensional global asymptotic methods of "quasi-classical" type (for instance, multidimensional generalization of the WKB method), in so-called "quantization conditions," etc., which are outside the scope of the present book.

In conclusion, we note that many linear partial differential equations can be successfully investigated when considered as Sturm–Liouville type equations in one unknown but with an unbounded operator coefficient. This approach requires the use of various methods, such as the theory of rigged spaces due to I. M. Gel'fand–A. G. Kostyuchenko [448] and Yu. M. Berezanskiy [24], [263], the theory of L. Schwartz–S. L. Sobolev distributions, and the theory of hyperfunctions and ultradistributions (see, for example, H. Komatsu [567], [568]). Starting with M. L. Gorbachuk [487], this approach was considered and further developed in the monographs [73], [74], [119], the review paper [496] and other works. For another approach to reduction of partial differential equations to differential equations with operator-valued coefficients, see, for example, the books [95] and [105].

The question about the extension of oscillatory theory to equations with unbounded operator-valued coefficients can still be considered as a problem open for investigations.