Preface

The present book focuses on the application of operator theory to oscillatory theory of differential equations and infinite systems of differential equations (the latter can be considered as one differential equation with operator-valued coefficients). Operator theory plays an exceptionally important role in modern mathematics and physics, especially in boundary value problems, quantum mechanics and theory of oscillations. Likewise, spectral analysis of differential operators is one of the most important areas of modern mathematical physics. Schrödinger operator alone (often called Sturm–Liouville operator in one-dimensional case) is the central topic of a series of well-known monographs[1].

Oscillatory theory of differential equations grew out of remarkable papers (called *Mémoires* at that time) [879], [880] and [881] (1829–1836) by Sturm and a short but significant paper [635] (1837) by Liouville and Sturm on second order linear ordinary differential equations with an eigenvalue parameter. With each passing year, the theory of Sturm–Liouville equations and their boundary value problems attracts even greater attention of mathematicians, physicists and engineers across the globe, owing to existing and newly emerging applications of Sturm–Liouville theory to diverse

[1] We will mention some monographs devoted to multi-dimensional Schrödinger operator: F. A. Berezin and M. A. Shubin [27]; K. Chadan and P. Sabatier [40]; H. Cycon, H. R. Froese, W. Kirch and B. Simon [50]; M. S. P. Eastham and H. Kalf [62]; K. Jörgens and J. Weidmann [85]; P. D. Lax and R. S. Phillips [108]; S. P. Merkur'ev and L. D. Faddeev [135]; Kh. Kh. Murtazin and V. A. Sadovnichiy [142]; R. G. Newton [146], [147]; A. G. Ramm [163]; M. Reed and B. Simon [168]; E. W. Schmid and H. Ziegelmann [177]; E. Ch. Titchmarsh [191, Vol. 2]; and J. Weidmann [197]. We will also mention some monographs devoted to Sturm–Liouville operator: Z. S. Agranovich and V. A. Marchenko [1], B. M. Levitan [112], B. M. Levitan and I. S. Sargasyan [113], V. A. Marchenko [121], and E. Ch. Titchmarsh [191, Vol. 1]. (Here we restrict ourselves to mathematical literature.)

areas of science, technology and mathematics, such as the theory of nonlinear evolution equations; quantum mechanics, including direct and inverse scattering problems (as already noted, Schrödinger equation coincides in many important cases with Sturm–Liouville equation); theory of oscillations; mathematical physics (classical and modern); theory of periodic and almost periodic systems and theory of stochastic (disordered) systems, including crystal lattices with impurities and quasi-crystals; geometry—in conjunction with the study of vector fields along a geodesic; symplectic geometry and topology. Alongside its numerous applications, the theory of Sturm–Liouville problems itself provides a beautiful arena for application of new mathematical theories and methods. At the same time, the theory of Sturm–Liouville problems stimulates the emergence and development of new mathematical methods (including the development of modern numerical methods and their computer applications [246]) and serves as the touchstone for their comparison and refinement.

In September 2003, on the occasion of the 200-th anniversary of the birth of Jack Charles François Sturm (1803–1855), an international colloquium with participation by numerous renowned mathematicians was organized in Geneva, Switzerland. Central theme of the colloquium was the development of Sturm–Liouville theory from its origin to the present. Highlights of the Sturm Bicentennial Colloquium include B. Simon's lecture on Sturm Oscillation and Comparison Theorems and some applications (see [862]) and W. N. Everitt's lecture [411] devoted to important landmarks in the development of Sturm–Liouville theory. For invited papers by participants of the Sturm Bicentennial Colloquium, see the book [8]. Additionally, W. N. Everitt created a catalogue of Sturm–Liouville differential equations; see [410].

It is well-known (see, for example, [165, Chapter VIII, Sec. 11]) that the investigation of different self-adjoint extensions of symmetric operators and the estimation of the location and multiplicity of their point spectra are among fundamental mathematical problems arising in any quantum-mechanical model. It turns out that the use of the interplay between oscillatory and spectral properties of differential operators, including the operators generated by finite or infinite systems of differential equations (in other words, differential equations with matrix or operator coefficients), provides effective means for determining the location and multiplicity of point spectra. The main part of the present book investigates the interplay between spectral and oscillatory properties for operator differential equations.

Sturm's oscillation theory and its various generalizations for ordinary differential equations and their finite systems, in connection with spectral theory, were considered in a series of well-known monographs[2].

Generalizations of alternation and comparison theorems to the case of finite systems are due to M. Morse [141], G. D. Birkhoff and M. R. Hestens [274], W. T. Reid [757], R. Bott [297] and H. M. Edwards [392]. An interesting topological treatment of Sturm's theorems and their connection with symplectic geometry is considered by V. I. Arnold [238]; however, the aim was not the novelty of mathematical results, for, as noted in [238], in an area as classical as Sturm's theory, it is difficult to keep an eye on all predecessors. In the present book, we compare the topological approach and operator-theoretical approach to Sturm's theorems. Various comparison theorems in Riemannian geometry are discussed in [76, Sec. 6].

In the present book, Sturm's oscillation theorem is transferred, in the corresponding form, to differential equations of arbitrary even order with operator-valued coefficients on finite and infinite intervals[3]. An important

[2]See A. G. Aslanyan and V. B. Lidskiy [12]; F. Atkinson [13]; R. Bellman [21]; E. A. Coddington and N. Levinson [46]; N. Dunford and J. T. Schwartz [58]; F. R. Gantmakher and M. G. Krein [69]; I. M. Glazman [70]; A. L. Gol'denveizer, V. B. Lidskiy and P. E. Tovstik [72]; P. Hartman [78]; I. T. Kiguradze and T. A. Chanturiya [89]; A. G. Kostyuchenko and I. S. Sargasyan [94]; K. Kreith [98]; B. M. Levitan and I. S. Sargasyan [113]; L. A. Pastur and A. L. Figotin [154]; G. Sansone [176]; V. N. Shevelo [178]; C. A. Swanson [187]; V. A. Yakubovich and V. M. Starzhinskiy [199] and [200]; J. Weidmann [196]. These monographs contain extensive bibliography on Sturm's oscillation theory and its connection to spectral theory. Note also that a generalization of Sturm's oscillation theorem to finite second-order systems was suggested by G. A. Bliss and I. J. Schoenberg [285]. However, this generalization characterizes oscillatory properties of those systems differently from the present book and connects the oscillatory properties to a sequence of numbers constructed in a special way. Except for the scalar case and related special cases, this sequence of numbers does not coincide with the spectrum.

[3]For scalar semi-bounded second-order problems, generalizations of the classical Sturm's Oscillation Theorem to the case of an infinite interval under various conditions are contained in the paper by H. Weyl [916] and in monographs by N. Dunford and J. T. Schwartz [58] and B. M. Levitan and I. S. Sargasyan [113]. For scalar differential equations of arbitrary even order, the interplay between oscillatory and spectral properties was investigated by M. G. Krein [585] and E. Heinz [506] (see also the monograph by I. M. Glazman [70]). The article [506] of E. Heinz, as noted in its preface by Van Der Waerden, is founded on the legacy of F. Rellich and represents the conclusion of investigations by F. Rellich [761]. For systems of fourth-order differential equations arising in the study of free oscillations of thin elastic shells, a generalization of Sturm's oscillation theorem is contained in the monographs by A. G. Aslanyan and V. B. Lidskiy [12] and A. L. Gol'denveizer, V. B. Lidskiy and P. E. Tovstik [72], and for finite fourth-order systems in the case of simple spectrum, in the article by A. Khalanai and Sh. Shandor [539] which also contains a generalization of results by G. A. Bliss and I. J. Schoenberg. The questions about oscillatory behavior of linear canonical systems were investigated in the

special case of this result is Morse Index Theorem [136], [159] and [160][4]. For differential equations of even order, we give a generalization of comparison and alternation theorems, as well as factorization theorems, generalizing to the considered case a theorem of Frobenius and M. G. Krein–Heinz–Rellich theorem. We note that "scalar" formulations and proofs of those theorems, based on the notion of determinant and the compactness of a finite-dimensional sphere, are not applicable in infinite-dimensional case. Our considerations encompass a broad class of canonical systems. As an application of factorization theorems, we obtain a generalization of Etgen–Pawlowski criterion [78] (second-order case) to equations of arbitrary even order with operator-valued coefficients.

We also establish an analogue of the oscillation theorem for discrete levels in the gap of continuous spectrum for equations of arbitrary even or odd order.

When applied to finite systems or scalar problems, the results of this book for infinite systems turn out to be at least as precise as the known results for the corresponding special cases. Furthermore, some results turn out to be new even in scalar and other special cases.

In conjunction with the investigation of oscillatory problems for infinite systems of differential equations, it was necessary to introduce a construction of the fundamental system of solutions subject to a self-adjoint boundary condition at a finite point or at infinity. In particular, we show that such fundamental solution defines the evolution of Lagrangian plane; more precisely, the evolution of its infinite-dimensional Hermitian analogue in a "doubled" (or, according to the order of the equation, the orthogonal sum of $2n$ identical Hilbert spaces) (separable) Hilbert space. The constructed fundamental solution is convenient in describing self-adjoint extensions of differential operators; in particular, in specifying boundary conditions at infinity. (For second-order scalar differential equation, the solution subject to a self-adjoint boundary condition at a singular end is constructed in [58].)

well-known works by V. B. Lidskiy [631] and V. A. Yakubovich [922] (the last two works consider the real case; certain results were transferred to the complex case by V. I. Khrabustovskiy [546]), in the work of V. I. Arnold [238], and, for second order differential equations with operator coefficients, in the works of G. J. Etgen and J. F. Pawlowski [403] and G. J. Etgen and R. T. Lewis [402]. Several criteria for finiteness of the negative spectrum, and, hence, for non-oscillatory behavior of a two-term differential equation of even order for $\lambda = 0$ with compact-operator coefficients were obtained by M. G. Gasymov, V. V. Zhikov and B. M. Levitan [444] and D. R. Yafaev [920].

[4]In [160] M. M. Postnikov noted that it is possible to generalize Morse Index Theorem to the general case of finite Sturm–Liouville systems.

In the present book, for differential equations of arbitrary order with operator coefficients, we use various methods for constructing the fundamental solution of a boundary-value problem and studying its properties.

The proof of oscillation theorems required a clarification of the dependence of eigenvalues and the greatest lower bound of the essential spectrum on the variable end-point of a finite or semi-infinite interval for semi-bounded differential operator of arbitrary even order with operator coefficients. The corresponding results of this book generalize a theorem of Courant and a theorem of M. G. Krein and affirmatively answer the question about strict monotonicity of eigenvalues under the extension of an unbounded domain, asked in [70, Chapter 4, Sec. 46].

In the absolutely indeterminate case, we describe all self-adjoint extensions of a symmetric differential operator of arbitrary (even as well as odd) order with operator coefficients on an infinite interval (real line or half-line). Additionally, we give a parametrization of the characteristic operator function and establish an explicit formula for finding this function. We also construct a solution of the Cauchy problem with the initial data at $(-\infty)$, and, thus, a fundamental solution[5].

Chapter 6 of the present book is devoted to investigation of perturbations of the spectrum and to a study of certain properties of solutions of periodic and almost periodic equations and finite systems. The methods of investigation here are connected to oscillation theory, and Chapter 6 can be considered as an application of those methods. Chapter 6 also contains detailed bibliographical comments.

The main part of the book uses the apparatus of the theory of Hermitian (in other words, self-adjoint) binary relations. For readers' convenience, the book is supplied by an Appendix containing the basics of the theory as well

[5] In the scalar case, a different form of the description of self-adjoint extensions of a second-order operator on an infinite interval was obtained by M. G. Krein [588] and C. Fulton [438]. For a scalar fourth-order equation and a two-term differential expression of an arbitrary even order, the same question was investigated by G. A. Mirzoev [680] and [681] simultaneously and independently of [541]–[543]. A description of all spectral functions of a scalar second-order differential operator was given, with the help of characteristic matrices, in the work of A. V. Shtraus [850] and V. M. Bruk [315], [316]. M. L. Gorbachuk [486] obtained a description of self-adjoint extensions of Sturm–Liouville operators with an operator potential in the absolutely indeterminate case. In the case when the values of deficiency indices coincide neither with the minimal possible ones (limit point case) nor with the maximal possible ones (the quasi-regular case), description of self-adjoint extensions of differential operators was investigated in the works by B. P. Allakhverdiev [221], I. M. Guseinov and R. T. Pashaev [503], F. G. Maksudov and B. P. Allakhverdiev [645], M. M. Malamud and V. I. Mogilevskiy [650], and V. I. Mogilevskiy [691]–[693].

as examples of the use of Hermitian relations. A Hermitian binary relation in a Hilbert space H is equivalent to specifying a Lagrangian plane or hypermaximal neutral lienal with respect to a skew-Hermitian inner product in a "doubled Hilbert space" $H \oplus H$. At the end of the Appendix, the reader can find some bibliographical comments.

Each chapter, and, thus the main part of the book, is based on the authors' results. In end-of-chapter bibliographical comments, we aimed not only to elucidate the history of a considered problem, but also to give a brief review of closely related areas, and, in some cases, to cite some important results of other authors or to review, for readers' convenience, some special facts used in the main text of the book.

In the English translation of the book, Bibliographical Comments are significantly expanded. Bibliographical Comments now contain separate subsections[6] discussing questions, such as a comparison of symplectic interpretation of Sturm's theorems and their operator-theoretical proofs; Marchenko's theorems on characteristic properties of Weyl solutions; a review of the history of Hilbert's 21-st problem and Bolibrukh's counterexample; Everitt–Zettl problem and Brusentsev's example; the results of Everitt, Kogan–Rofe-Beketov, Gilbert and Kalyabin in connection with the problem of possible values of deficiency indices of symmetric differential operators with complex-valued coefficients; Marchenko–Ostrovskiy and Pastur–Tkachenko theorems on the spectra of self-adjoint operators with periodic and limit-periodic coefficients; Batchenko–Gesztesy and Rofe-Beketov theorems on the spectra of non-self-adjoint differential operators (finite-zone type or with periodic coefficients); the results of V. I. Khrabustovskiy, who, at the request of the authors of this book, kindly wrote "The Inverse Sturm–Liouville Problem for the Spectral Matrix on the Whole Axis" (Sec. 6.6.8) and "Characteristic Operator and Boundary Value Problems with Separated Boundary Conditions" (Sec. A.4.3) containing his results; M. L. Gorbachuk theory of generalized boundary values; theory of abstract boundary conditions, connected with the names of J. W. Calkin, A. N. Kochubei and V. M. Bruk; Simon's and Rofe-Beketov's results on the strong resolvent convergence.

In comparison with the Russian edition of the book, the Bibliography now contains more than twice as many entries. The main text of the book

[6]The subsections are enumerated as follows: 4.5.1 denotes Subsection 1 of Section 5 of Chapter 4. The formulas are enumerated as follows: (3.5) denotes the formula 5 of Chapter 3. Theorems, Lemmas, Propositions, Corollaries, Remarks and Examples have separate enumerations containing two numbers, as the enumeration of formulas.

is also expanded, especially Appendix and Chapter 1, where new results are included.

The English translation of our book contains formulations of several open problems that can be used by young mathematicians as research topics.

Thus, the present translation can be considered in fact as a new book, with the title completed by the words "and Related Topics."

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