Chapter 1

Introduction to Steady State Bifurcation Theory

In this chapter, we present some classical theorems on steady state bifurcations, including the Lyapunov-Schmidt procedure, bifurcation theorems from eigenvalues of odd multiplicity, and bifurcation theorems for potential operators. The presentation we adopt here is to use the Lyapunov-Schmidt procedure to link all these theorems in a natural way.

The version of the Lyapunov-Schmidt procedure presented here differs slightly from the one given in classical textbooks. The later is done by decomposing the space into the direct sum of the eigenspace and its complement. While the Lyapunov-Schmidt procedure presented here is based on the decomposition of the space into the direct sum of the generalized eigenspace and its complement. This Lyapunov-Schmidt procedure is more nature, and much more convenient to study steady state bifurcations. In fact, it is this difference, together with other ingredients, including in particular the spectral theorem in Chapter 3, that made many problems more accessible.

Another important ingredient for the Lyapunov-Schmidt procedure presented here is the introduction of the normalization of the bifurcation equation (1.19), which is crucial for the new steady state bifurcation theory from higher order terms regardless of the multiplicity of the linearized eigenproblems given in Chapter 4.

1.1 Implicit Function Theorem

Let X, Y, Z be Banach spaces, $U \subset X \times Y$ an open set, and $F : U \to Z$ a mapping. Let $(x_0, y_0) \in U$ satisfy

$$F(x_0, y_0) = 0. \tag{1.1}$$

We consider the solvability near (x_0, y_0) of the equation

$$F(x, y) = 0. \tag{1.2}$$

The following well known implicit function theorem is of fundamental importance in the bifurcation theory.

Theorem 1.1 (Implicit Function Theorem). *Let $F \in C^k(U, Z)$ be k-th order differentiable ($k \geq 1$), and satisfy (1.1). Assume that the derivative operator $D_x F(x_0, y_0) : X \to Z$ has a bounded inverse. Then, the following assertions hold true.*

(1) There exist a neigborhood $V \subset X$ of x_0 and a neighborhood $W \subset Y$ of y_0 such that as $y \in W$, (1.2) has a unique solution $x = \Phi(y) \in V$.
(2) The mapping $\Phi : W \to V$ is k-th differentiable, especially if F is analytic, then Φ is also analytic.
(3) If $D_y F(x_0, y_0) = 0$, then we have

$$\Phi'(y_0) = 0, \quad i.e. \ \Phi(y) = o(\|y - y_0\|). \tag{1.3}$$

Remark 1.1 Actually, if the derivations of F with respect to y vanish up to m-th order at (x_0, y_0):

$$F_y^{(n)}(x_0, y_0) = 0, \qquad \forall\, 1 \leq n \leq m,$$

then the implicit function satisfies that

$$\frac{d^j \Phi(y_0)}{dy^j} = 0, \quad j = 1, \cdots, m;$$

namely, $\Phi(y) = o(\|y - y_0\|^m)$.

1.2 Basics of Topological Degree Theory

1.2.1 *Brouwer degree*

Let $\bar{\Omega} \subset \mathbb{R}^n$ be a bounded open set, $F : \bar{\Omega} \to \mathbb{R}^n$ a continuous mapping. $p \in \mathbb{R}^n$ with $p \notin F(\partial\Omega)$. We begin with the definition of Brouwer degree, which assigns each triple (F, Ω, p) to an integer, denoted by $\deg(F, \Omega, p)$, which is called the Brouwer degree of F in Ω with respect to p.

A point $x_0 \in \Omega$ is called a regular point of F if the Jacobian matrix $DF(x_0)$ is invertible; otherwise x_0 is called a critical point of F. A point $p \in \mathbb{R}^n$ is called a regular value of F if either $F^{-1}(p) = \emptyset$ or each point

$x \in F^{-1}(p)$ is a regular point. A point $p \in \mathbb{R}^n$ is called a critical value of F if it is not a regular value. By the Sard theorem [Milnor, 1965], for $F \in C^1(\Omega, \mathbb{R}^n)$ the set of critical values of F has measure zero in $\mathbb{R}^n$.

Definition 1.1 We define for each triple (F, Ω, p) with $p \notin F(\partial\Omega)$ the Brouwer degree as follows,

(1) if $F \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $p \in \mathbb{R}^n$ is a regular value of F, then

$$\deg(F, \Omega, p) = \sum_{y \in F^{-1}(p)} \operatorname{sign} \det DF(y),$$

where $\det DF(y)$ is the determinant of $DF(y)$;

(2) if $F \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $p \in \mathbb{R}^n$ is a critical value of F, then we take a regular value p_1 of F with $|p_1 - p| < \operatorname{dist}(p, F(\partial\Omega))$ and define that

$$\deg(F, \Omega, p) = \deg(F, \Omega, p_1);$$

(3) if $F \in C(\bar{\Omega}, \mathbb{R}^n)$ is a continuous function, we take a C^1 function $F_1 \in C^1(\bar{\Omega}, \mathbb{R}^n)$ such that

$$\sup_{x \in \bar{\Omega}} |F(x) - F_1(x)| < \operatorname{dist}(p, F(\partial\Omega)),$$

and define the Brouwer degree by

$$\deg(F, \Omega, p) = \deg(F_1, \Omega, p).$$

It is routine, but lengthy, to justify that the Brouwer degree is well-defined, and enjoys the following basic properties; see among others [Nirenberg, 2001; Chen, 1981]:

1. NORMALIZATION:

$$\deg(\mathrm{id}, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ 0 & \text{if } p \notin \Omega, \end{cases}$$

where $\mathrm{id} : \Omega \to \mathbb{R}^n$ is the identity.

2. DOMAIN ADDITIVITY: If $\Omega_1, \Omega_2 \subset \Omega$ with $\Omega_1 \cap \Omega_2 = \emptyset$, $p \notin F(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$\deg(F, \Omega, p) = \deg(F, \Omega_1, p) + \deg(F, \Omega_2, p).$$

3. HOMOTOPY INVARIANCE: Let $H : \bar{\Omega} \times [0, 1] \to \mathbb{R}^n$ be continuous, and for each $t \in [0, 1]$, $p \notin H(\partial\Omega, t)$. Then

$$\deg(H_0, \Omega, p) = \deg(H_1, \Omega, p),$$

where $H_0(x) = H(x,0)$, and $H_1(x) = H(x,1)$.

4. KRONECKER'S EXISTENCE THEOREM: If $\deg(F,\Omega,p) \neq 0$, then the equation $F(x) = p$ has at least one solution in Ω.
5. EXCISION PROPERTY: If $D \subset \bar{\Omega}$ is a closed set, and $p \notin F(D)$, then
$$\deg(F,\Omega,p) = \deg(F,\Omega \setminus D,p).$$
6. CONNECTED DOMAIN PROPERTY: Let $p : \mathbb{R}^1 \to \bar{\Omega} \in C^0$. If $p(t) \notin F(\partial\Omega)$ for any $t \in \mathbb{R}^1$, then
$$\deg(F,\Omega,p(t)) = \text{const}.$$
7. BOUNDARY PROPERTY: If $F, G \in C(\bar{\Omega},\mathbb{R}^n)$ with $F|_{\partial\Omega} = G|_{\partial\Omega}$, then
$$\deg(F,\Omega,p) = \deg(G,\Omega,p).$$
8. POINCARÉ–BOHL THEOREM: Let $F, G \in C(\bar{\Omega},\mathbb{R}^n)$. If
$$tF(x) + (1-t)G(x) \neq p, \quad \forall\, t \in [0,1], \forall x \in \partial\Omega,$$
then
$$\deg(F,\Omega,p) = \deg(G,\Omega,p).$$

1.2.2 *Basic theorems of Brouwer degree*

We introduce here four basic theorems for the Brouwer degree, which will be useful in the later discussion. They are the Reduction Theorem, Multiplication Formula, Borsuk Theorem, and Even Mapping Theorem.

Theorem 1.2 (Reduction Theorem). *Let $\mathbb{R}^m \subset \mathbb{R}^n$ $(n > m)$, and $\Omega \subset \mathbb{R}^n$ a bounded open set, $f \in C(\bar{\Omega},\mathbb{R}^m)$. If $p \in \mathbb{R}^m$ and $p \notin F(\partial\Omega)$, where $F(x) = x - f(x)$, then*
$$\deg(F,\Omega,p) = \deg(G,\Omega \cap \mathbb{R}^m,p)$$
where $G = F|_{\Omega\cap\mathbb{R}^m}$.

Example 1.1 Consider the following two mappings
$$F(x) = (g_1(x),\cdots,g_m(x),x_{m+1},\cdots,x_n),$$
$$G(x_1,\cdots,x_m) = (g_1(x_1,\cdots,x_m,0),\cdots,g_m(x_1,\cdots x_m,0)),$$
for any $x \in \Omega \subset \mathbb{R}^n$, $0 \in \Omega$. Based on Theorem 1.2, we have
$$\deg(F,\Omega,0) = \deg(G,\Omega \cap \mathbb{R}^m,0).$$

Theorem 1.3 (Multiplication Formula). *Let $\Omega_1, \Omega_2, \subset \mathbb{R}^n$ be bounded open sets, $F \in C(\bar{\Omega}_1, \mathbb{R}^n)$, $G \in C(\bar{\Omega}_2, \mathbb{R}^n)$, $F(\bar{\Omega}_1) \subset \Omega_2$. Let Ω_α be the connected domains of $\Omega_2 \setminus F(\partial\Omega_1)$. If $p \notin G \circ F(\partial\Omega_1) \cup G(\partial\Omega_2)$, then*

$$\deg(G \circ F, \Omega_1, p) = \sum_\alpha \deg(G, \Omega_\alpha, p) \deg(F, \Omega_1, p_\alpha),$$

where $p_\alpha \in \Omega_\alpha$. By the connected domain property, the degree $\deg(G \circ F, \Omega_1, p)$ *is independent of $p_\alpha \in \Omega_\alpha$.*

Theorem 1.4 (Borsuk Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $0 \in \Omega$, and $F \in C(\bar{\Omega}, \mathbb{R}^n)$. If Ω is symmetric on $x = 0$, i.e. $x \in \Omega \Rightarrow -x \in \Omega$, and*

$$F(x) = -F(-x) \neq 0, \quad \forall\, x \in \partial\Omega,$$

then $\deg(F, \Omega, 0) =$ *odd.*

Theorem 1.5 (Even Mapping Theorem). *Let Ω be as in Theorem 1.4, and $F : \Omega \to \mathbb{R}^n$ be an even function, i.e. $F(x) = F(-x)$, $\forall$ $x \in \partial\Omega$. Then we have*

$$\deg(F, \Omega, 0) = \textit{even}.$$

Remark 1.2 The Even Mapping Theorem here will be applied to derive bifurcations from higher-order nondegenerate singularities in Chapter 4 later, which is easily derived from the Sard theorem and Definition 1.1.

1.2.3 *Leray–Schauder degree*

The Leray–Schauder degree is a generalization of the Brouwer degree to mappings defined on an infinite dimensional Banach space. Let X be a Banach space, and $\Omega \subset X$ a bounded open set. A mapping $F : \Omega \to X$ is called compact mapping or completely continuous mapping if F is continuous and maps any bounded set of Ω to a precompact set of X. The following mapping is called completely continuous field

$$f = \mathrm{id} - F : \Omega \longrightarrow X.$$

It is known that for a completely continuous mapping F, there exists finite dimensional spaces $X_n \subset X$ and continuous mappings $F_n : \Omega \to X_n$

such that

$$\sup_{x\in\bar{\Omega}} \|F(x) - F_n(x)\| \longrightarrow 0, \quad \text{as } n \to \infty.$$

Definition 1.2 Let $X_n \subset X$ be finite dimensional spaces, $p \in X_n$, and $F_n : \bar{\Omega} \to X_n$ satisfy

$$\sup_{x\in\bar{\Omega}} \|F(x) - F_n(x)\| < \varepsilon = \text{dist}(p, f(\partial\Omega)).$$

Then, we define the Leray–Schauder degree for the completely continuous field $f = \text{id}-F$ on Ω at p as follows

$$\deg(f, \Omega, p) = \deg(f_n, \Omega, p),$$

where $f_n = \text{id}-F_n$.

It is known that the Leray–Schauder degree is well-defined, and has the following three basic properties.

Theorem 1.6 *For a completely continuous field $f = \text{id} - F$, the Leray–Schauder degree enjoys the properties*

1. NORMALIZATION:

$$\deg(id, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ 0 & \text{if } p \notin \Omega. \end{cases}$$

2. DOMAIN ADDITIVITY: *if $\Omega_1, \Omega_2 \subset \Omega$ with $\Omega_1 \cap \Omega_2 = \emptyset$, $p \notin f(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$, then we have*

$$\deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p).$$

3. HOMOTOPY INVARIANCE: *let $H : \bar{\Omega} \times [0, 1] \to X$ be a compact mapping $h(x, t) = x - H(x, t)$, and $p \notin h(\partial\Omega, t)$ for any $t \in [0, 1]$. Then*

$$\deg(h_0, \Omega, p) = \deg(h_1, \Omega, p),$$

where $h_0 = h(\cdot, 0)$, $h_1 = h(\cdot, 1)$.

The basic properties 5–8 in subsection 1.2 hold true for the Leray–Schauder degree as well, which can derived using Theorem 1.6.

1.2.4 *Indices of isolated singularities*

We consider the equation

$$f(x) = x - F(x) = 0 \tag{1.4}$$

where $F : \Omega \to X$ is a compact operator. Solutions of (1.4) are called singularities of f.

Let $x_0 \in \Omega$ be an isolated solution of (1.4). Then we define the index of $f = \mathrm{id} - F$ at the singular point x_0 by

$$\mathrm{ind}(f, x_0) = \deg(f, U, 0),$$

where U is a small neighborhood of x_0.

We know that if $F : \Omega \to X$ is a differentiable compact mapping, then the derivative operator of F at $x_0 \in \Omega$

$$DF(x_0) : X \longrightarrow X$$

is a linear compact operator. By the Riesz–Schauder theory, the eigenvalues of the following equation are isolated

$$y = \lambda DF(x_0)y \tag{1.5}$$

and the algebraic multiplicity of each eigenvalue λ of (1.5) is finite, which is the dimension of space

$$E_\lambda = \bigcup_{n=1}^{\infty} \{y \in X \mid (\mathrm{id} - \lambda DF(x_0))^n y = 0\}.$$

The following index formula is important for the bifurcation theory.

Theorem 1.7 *Let $x_0 \in \Omega$ be a singular point of a completely continuous field $f = id - F$, and F is differentiable at x_0. If $\lambda = 1$ is not an eigenvalue of $DF(x_0)$, then x_0 is an isolated singular point of f, and*

$$\mathrm{ind}(f, x_0) = \mathrm{ind}(\mathrm{id} - DF(x_0), 0) = (-1)^\beta,$$

where β is the sum of the multiplicities of all eigenvalues of $DF(x_0)$ in $(0, 1)$.

We remark here that the Leray–Schauder degree theory is applicable for the class of nonlinear operators given in the following, which cover many problems in differential equations.

Let H and H_1 be two Hilbert spaces, $H_1 \hookrightarrow H$ a dense and compact inclusion, $A : H_1 \to H$ a linear homeomorphism, and $G : H_1 \to H$ be a compact operator. Assume that there exist a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}^1$ and an eigenvector sequence $\{e_k\} \subset H_1$ of A such that

$$\begin{aligned} &Ae_k = \rho_k e_k, \\ &0 < \rho_1 \leq \rho_2 \leq \cdots, \\ &\rho_k \to \infty \quad (k \to \infty), \end{aligned}$$

and $\{e_k\}$ constitutes a basis of H.

We can define the Leray–Schauder degree for the operator $A+G$ on a bounded open set $\Omega \subset H_1$ at a point $p \in H$, which satisfies the domain additivity property, the homotopy invariance and the normalization. In particular, we have

$$\deg(A, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ 0 & \text{if } p \notin \Omega. \end{cases}$$

In this case, we also call the operator $A + G : H_1 \to H$ a completely continuous field.

1.3 Lyapunov–Schmidt Method

1.3.1 *Preliminaries*

Let X, Y be two Banach spaces, and $L : X \times \mathbb{R}^1 \to Y$ be a linear bounded operator and $G : X \times \mathbb{R}^1 \to Y$ a continuous mapping satisfying

$$G(u, \lambda) = o(\|u\|), \qquad u \in X. \tag{1.6}$$

Consider an operator equation

$$L_\lambda u + G(u, \lambda) = 0. \tag{1.7}$$

It is clear that $(u, \lambda) = (0, \lambda)$ is a trivial solution of (1.7). The bifurcation problem of (1.7) is to seek a nontrivial solution $(u_\lambda, \lambda) \neq (0, \lambda)$ of (1.7) from some point $(0, \lambda_0)$ such that

$$u_\lambda \longrightarrow 0 \quad \text{as } \lambda \to \lambda_0$$

as shown in Figure 1.1.

More precisely, we give the following definition of bifurcation.

Definition 1.3 We say that the equation (1.7) bifurcates from $(0, \lambda_0)$ a solution $(u_\lambda, \lambda) \in X \times \mathbb{R}$ if there exists a sequence of solutions (u_n, λ_n) of

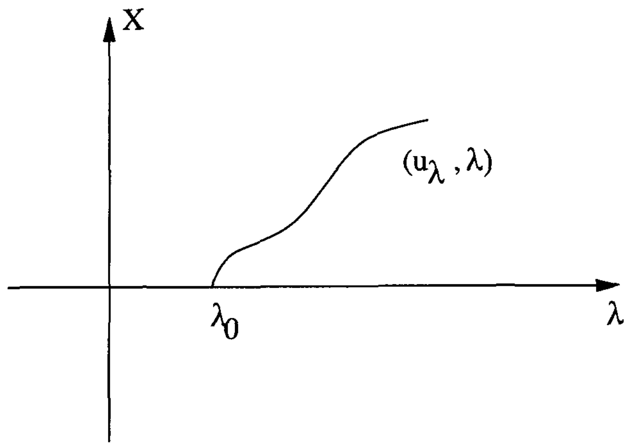

Fig. 1.1

(1.7) which satisfy

$$\lim_{n\to\infty} \lambda_n = \lambda_0, \quad \text{and} \quad \lim_{n\to\infty} \|u_n\| = 0.$$

By the implicit function theorem (Theorem 1.1), one immediately derives the necessary condition of bifurcation for (1.7).

Theorem 1.8 *Under the condition (1.6), if (1.7) has a bifurcation solution from $(0, \lambda_0)$, then the linear operator $L_{\lambda_0} : X \to Y$ is not invertible. Especially, if L_λ is a linear completely continuous field, i.e. $L_\lambda = A_\lambda + B_\lambda$, $A_\lambda : X \to Y$ a linear homeomorphism, and $B_\lambda : X \to Y$ a linear compact operator, then there exist $u \in X$ with $u \neq 0$ such that $L_{\lambda_0} u = 0$.*

1.3.2 *Lyapunov–Schmidt procedure*

We begin with an example to introduce the Lyapunov–Schmidt procedure. Consider a system of two algebraic equations given by

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2) = 0, \\ a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2) = 0, \end{cases} \tag{1.8}$$

where $a_{ij} = a_{ij}(\lambda)$ are continuous functions of λ, and f_1, f_2 are C^∞ functions satisfying that

$$(f_1(x), f_2(x)) = o(|x|).$$

Obviously, $(x, \lambda) = (0, \lambda)$ is a trivial solution of (1.8). We shall use the Lyapunov–Schmidt procedure to investigate the bifurcation of (1.8). For simplicity, we assume that the matrix

$$A_\lambda = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix}$$

has two eigenvalues in a neighborhood of λ_0 as follows

$$\beta_1 = \lambda - \lambda_0, \qquad \beta_2 = 1.$$

Under a coordinate transformation, the equations (1.8) are rewritten as

$$(\lambda - \lambda_0)x_1 + F_1(x_1, x_2) = 0, \tag{1.9}$$

$$x_2 + F_2(x_1, x_2) = 0, \tag{1.10}$$

where $F_i(x) = o(|x|)$, $i = 1, 2$.

By the implicit function theorem–Theorem 1.1, (1.10) has a solution near $x_1 = 0$:

$$x_2 = g(x_1), \quad g(x_1) = o(|x_1|). \tag{1.11}$$

Inserting (1.11) into (1.9) we arrive at

$$\begin{cases} (\lambda - \lambda_0)x_1 + F_1(x_1, g(x_1)) = 0, \\ F_1(x_1, g(x_1)) = o(|x_1|). \end{cases} \tag{1.12}$$

Thus, the bifurcation of (1.8) is equivalent to that of (1.12). By the index theorem–Theorem 1.7, we find

$$\operatorname{ind}((\lambda - \lambda_0)\mathrm{id} + F_1, 0) = \operatorname{ind}((\lambda - \lambda_0)\mathrm{id}, 0) = \begin{cases} 1 & \text{if } \lambda > \lambda_0, \\ -1 & \text{if } \lambda < \lambda_0, \end{cases}$$

which implies that (1.12) has a bifurcation from $(0, \lambda_0)$. Hence, $(0, \lambda_0)$ is a bifurcation point of (1.8).

We now introduce the Lyapunov–Schmidt method in general setting. Consider

$$u - \lambda A u + G(u, \lambda) = 0, \tag{1.13}$$

where $u \in X$, X a Banach space, $A : X \to X$ a linear compact operator, and $G : X \times \mathbb{R} \to X$ a continuous mapping satisfying (1.6).

Assume that λ_0^{-1} is a real eigenvalue of A with algebraic multiplicity $m \geq 1$. Thus, the space X can be decomposed into the direct sum of two invariant subspaces of A as follows

$$\begin{aligned} X &= E_0 \oplus E_1, \\ E_0 &= \bigcup_{n\in\mathbb{N}} \{x \in X \mid (\mathrm{id} - \lambda_0 A)^n x = 0\}, \\ \dim E_0 &= m. \end{aligned}$$

The linear operator $A : X \to X$ can be decomposed into

$$\begin{aligned} A &= A_0 + A_1, \\ A_0 &= A|_{E_0} : E_0 \to E_0, \\ A_1 &= A|_{E_1} : E_1 \to E_1. \end{aligned}$$

Let $P_0 : X \to E_0$ and $P_1 = \mathrm{id} - P_0 : X \to E_1$ be the canonical projections. Then (1.13) is decomposed into

$$x - \lambda A_0 x + P_0 G(x + y, \lambda) = 0, \quad x \in E_0, \tag{1.14}$$

$$y - \lambda A_1 y + P_1 G(x + y, \lambda) = 0, \quad y \in E_1. \tag{1.15}$$

It is clear that $\mathrm{id} - \lambda_0 A_1 : E_1 \to E_1$ is invertible. Based on the implicit function theorem, (1.15) can be solved near $x = 0$ and $y = 0$:

$$y = y(x, \lambda); \qquad y(x, \lambda) = o(\|x\|). \tag{1.16}$$

Putting (1.16) into (1.14), we have

$$x - \lambda A_0 x + P_0 G(x + y(x, \lambda), \lambda) = 0, \quad x \in E_0, \tag{1.17}$$

which is an m–dimensional equation, called the *bifurcation equation* of (1.13). The following theorem verifies the validity of the Lyapunov–Schmidt procedure.

Theorem 1.9 *Let λ_0^{-1} be a real eigenvalue of A with algebraic multiplicity $m \geq 1$. Then the bifurcation problem of (1.13) near $\lambda = \lambda_0$ is equivalent to that of (1.17).*

We remark here that the Lyapunov-Schmidt procedure presented here differs slightly from the one usually presented in classical textbooks. The later is done by decomposing X into the direct sum of the kernel X_0 of $\mathrm{id} - \lambda_0 A$ and its complement X_1:

$$X = X_0 \oplus X_1, \qquad X_0 = \ker(\mathrm{id} - \lambda_0 A).$$

While the Lyapunov-Schmidt procedure presented here is based on the decomposition $X = E_0 \oplus E_1$ with E_0 being generalized eigenspace. As we shall, it is easy to see that this Lyapunov-Schmidt procedure is more nature, and much more convenient to study bifurcation of (1.13). In fact, it is this difference, together with other ingredients presented in this book that made many problems more accessible.

1.3.3 *Normalization*

Let the nonlinear term $G : X \times \mathbb{R}^1 \to X$ be C^∞. Then there is a $k \geq 2$ such that

$$G(u, \lambda) = \sum_{n=k}^{\infty} G_n(u, \lambda),$$

where $G_n(u, \lambda)$ is an n–multilinear mapping, and

$$G_n(\alpha u, \lambda) = \alpha^n G_n(u, \lambda), \quad \forall\, \alpha \in \mathbb{R}^1.$$

By (1.16), $y(x, \lambda)$ is a higher order function of x. Hence, (1.17) has the following form

$$x - \lambda A_0 x + P_0 G_k(x, \lambda) + o(\|x\|^k) = 0, \quad k \geq 2. \tag{1.18}$$

If $x = 0$ is an isolated singular point of $P_0 G_k$, then the bifurcation problem of (1.18) is reduced to that of the following equation:

$$x - \lambda A_0 x + P_0 G_k(x, \lambda) = 0. \tag{1.19}$$

There are many bifurcation problems in science and engineering, where the bifurcation equation (1.19) can be explicitly calculated, providing an important step toward to their bifurcation analysis.

For example, we consider a special case where $A : X \to X$ is a symmetric linear compact operator, X is a Hilbert space, λ_0^{-1} is an eigenvalue of A with multiplicity two, and e_1, e_2 are the eigenvectors corresponding to λ_0, which are orthogonal

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

Let $k = 2$ and G_2 be the second order multilinear operator of G. The equation (1.19) can be expressed as the following system of 2-dimensional algebraic equations

$$\begin{cases} (1 - \lambda\lambda_0^{-1})x_1 + a_{11}^1 x_1^2 + 2a_{12}^1 x_1 x_2 + a_{22}^1 x_2^2 = 0, \\ (1 - \lambda\lambda_0^{-1})x_2 + a_{11}^2 x_1^2 + 2a_{12}^2 x_1 x_2 + a_{22}^2 x_2^2 = 0, \end{cases}$$

where

$$a_{ij}^k = a_{ij}^k(\lambda) = \frac{1}{2}\left[\langle G_2(e_i, e_j, \lambda), e_k\rangle + \langle G_2(e_j, e_i, \lambda), e_k\rangle\right].$$

Remark 1.3 The method that reduces the bifurcation equation (1.17) to its normalized form (1.19), together with the spectrum theorem and the attractor bifurcation theorems developed in [Ma and Wang, 2004e; Ma and Wang, 2004d], is a very useful tool in the bifurcation analysis in many problems. The bifurcation theory with applications developed by the authors is summarized in this book and as well as in a forthcoming book on hydrodynamic stability and bifurcation.

1.4 Krasnosel'ski Bifurcation Theorems

1.4.1 *Bifurcation from eigenvalues with odd multiplicity*

When the eigenvalue λ_0^{-1} of A has odd multiplicity, the following well-known Krasnosel'ski bifurcation theorem says that the equation (1.13) has a bifurcation from $(0, \lambda_0)$.

Theorem 1.10 **(Krasnosel'ski Theorem).** *Under the condition (1.6), if $A : X \to X$ is a linear compact operator, and $\lambda_0^{-1} \in \mathbb{R}^1$ is a real eigenvalue of A with odd algebraic multiplicity, then $(u, \lambda) = (0, \lambda_0)$ is a bifurcation point of (1.13).*

Proof. This theorem can immediately be derived by the index theorem (Theorem 1.7) and the Lyapunov–Schmidt theorem (Theorem 1.9). Indeed, without loss of generality, we assume that $\lambda_0 > 0$. Then the bifurcation equation (1.17) reads

$$\begin{cases} x - \lambda A_0 x + g(x, \lambda) = 0, \\ g(x, \lambda) = P_0 G(x + y(x, \lambda), \lambda) = o(\|x\|). \end{cases} \tag{1.20}$$

By Theorem 1.7, we have

$$\operatorname{ind}(\mathrm{id} - \lambda A_0 + g, 0) = \operatorname{ind}(\mathrm{id} - \lambda A_0, 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_0, \\ -1 & \text{if } \lambda > \lambda_0. \end{cases}$$

It implies that $(x, \lambda) = (0, \lambda_0)$ is a bifurcation point of (1.20). Thus, by Theorem 1.9 we obtain the Krasnosel'ski theorem. $\square$

Consider a special case where the algebraic multiplicity $m = 1$ of λ_0^{-1}, i.e. λ_0^{-1} is a simple eigenvalue of A. The following theorem is due to Crandall and Rabinowitz.

Theorem 1.11 *Assume that the conditions of Theorem 1.10 hold true. If λ_0^{-1} is a simple eigenvalue of A, then the equation (1.13) bifurcates from $(0, \lambda_0)$ to exactly two branches Γ_1 and Γ_2, which are of the following form*

$$\Gamma : (u_\lambda, \lambda), \quad u_\lambda = te + tv(t), \quad \lambda = \lambda_0 + \mu(t),$$

where $v(t) \in X$, $\mu(t) \in \mathbb{R}^1$ are continuous on t with $v(0) = 0$, $\mu(0) = 0$, and e is the eigenvector of A corresponding to λ_0^{-1}.

This theorem can be derived from (1.20). In fact, if $g \equiv 0$, then for any $t > 0$,

$$\begin{aligned} \Gamma_1 &: u_\lambda = te + y(te, \lambda_0), \\ \Gamma_2 &: u_\lambda = -te + y(-te, \lambda_0), \end{aligned}$$

where $y(x, \lambda)$ is given by (1.16).

If $g \not\equiv 0$ in (1.20), and $g(\tau e, \lambda) = \rho(\tau, \lambda)e$, $\rho(\tau, \lambda)$ has the Taylor expansion near $\tau = 0$ as follows

$$\rho(\tau, \lambda) = \alpha(\lambda)\tau^k + o(|\tau|^k), \quad k \geq 2,\ \alpha(\lambda_0) \neq 0,$$

then the equation (1.20) can be written as

$$(1 - \lambda\lambda_0^{-1})\tau + \alpha(\lambda)\tau^k + o(|\tau|^k) = 0,$$

which has two branches of solutions as shown in Figure 1.2.

1.4.2 *Krasnosel'ski theorem for potential operators*

Let X be a Hilbert space. For the equation (1.13) we assume that

(A_1) $A : X \to X$ is a self-adjoint linear compact operator, and
(A_2) there is a functional $F \in C^r(X \times \mathbb{R}^1, \mathbb{R}^1)$, with

$$D_u F(u, \lambda) = G(u, \lambda), \qquad (u, \lambda) \in X \times \mathbb{R}^1,$$

and $G(u, \lambda)$ satisfies (1.6).

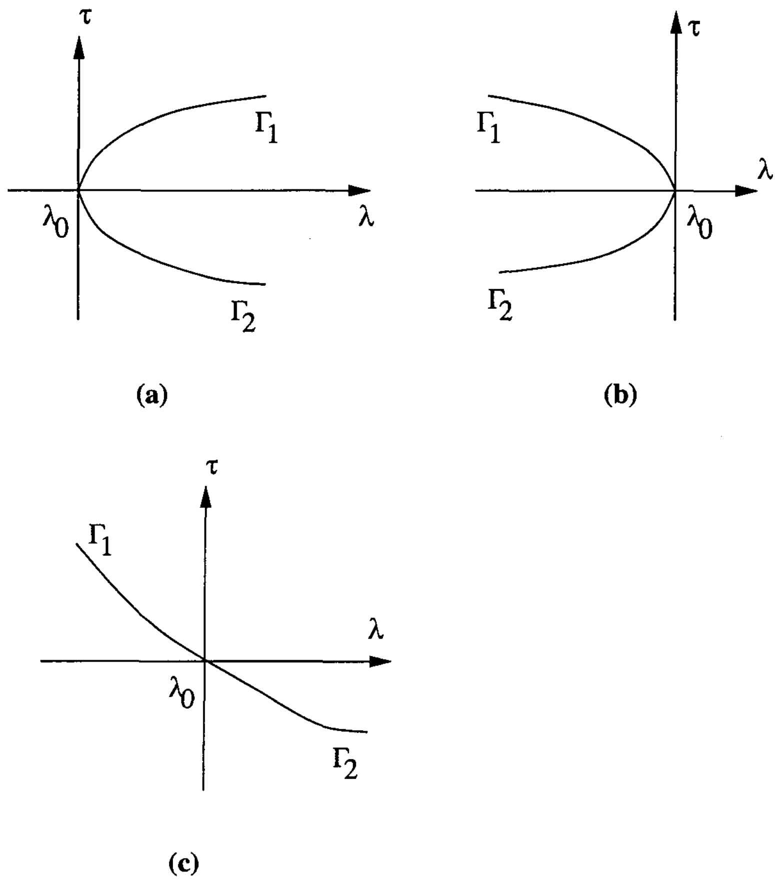

Fig. 1.2 (a) Case: k =odd, $\alpha(\lambda_0) > 0$; (b) case: k =odd, $\alpha(\lambda_0) < 0$; (c) case: k =even.

The following theorem is due to Krasnosel'ski.

Theorem 1.12 *Under the hypotheses (A_1) and (A_2), if $\lambda_0^{-1} \in \mathbb{R}^1$ is an eigenvalue of A, then the equation (1.13) has at least two bifurcated branches from $(u, \lambda) = (0, \lambda_0)$.*

To see this theorem, we assume that the eigenvalue λ_0 has multiplicity $m \geq 1$ with orthogonal eigenvectors $\{e_1, \cdots, e_m\}$, $F = \sum_{n=k+1}^{\infty} F_n$, $k \geq 2$, and F_n is the n-multilinear functional of F. Then, the bifurcation equation of (1.13) is of the form

$$(1-\lambda\lambda_0^{-1})x_i+\frac{\partial}{\partial x_i}F_{k+1}\left(\sum_{j=1}^{m}x_je_j,\lambda\right)+o(|x|^k)=0, \tag{1.21}$$

where $x=(x_1,\cdots,x_m)\in\mathbb{R}^m$.

If $x=0$ is an isolated singular point of $\nabla F_{k+1}(\sum_{j=1}^m x_je_j,\lambda_0)$, it suffices to consider the bifurcation of the following system of equations

$$(1-\lambda\lambda_0^{-1})x+\nabla F_{k+1}\left(\sum_{j=1}^{m}x_je_j,\lambda\right)=0. \tag{1.22}$$

It is clear that the critical points $(x(\lambda),\lambda)$ of the functional

$$\frac{1}{2}(1-\lambda\lambda_0^{-1})|x|^2+F_{k+1}\left(\sum_{j=1}^{m}x_je_j,\lambda\right) \tag{1.23}$$

are the bifurcated solutions of (1.22).

One can obtain the critical points of (1.23) by the following method. Let $(\alpha_1,\cdots,\alpha_m)\in\mathbb{R}^m$ with $|\alpha|^2=1$ satisfy

$$F_{k+1}\left(\sum_{i=1}^{m}\alpha_ie_i,\lambda\right)=\max_{|x|^2=1}F_{k+1}\left(\sum_{k=1}^{m}x_ie_i,\lambda\right).$$

Putting $(x_1,\cdots,x_m)=(\alpha_1t,\cdots,\alpha_mt)$ in (1.23) we get

$$\frac{1}{2}(1-\lambda\lambda_0^{-1})t^2+F_{k+1}\left(\sum_{i=1}^{m}\alpha_ie_i,\lambda\right)t^{k+1}.$$

Without loss of generality, let $F_{k+1}(\sum_{i=1}^m\alpha_ie_i,\lambda_0)>0$. Then, for $\lambda>\lambda_0$ we take

$$t_\lambda^{k-1}=(\lambda\lambda_0^{-1}-1)/(k+1)F_{k+1}\left(\sum_{i=1}^{m}\alpha_ie_i,\lambda\right). \tag{1.24}$$

Thus, this point $x(\lambda)=(\alpha_1t_\lambda,\cdots,\alpha_mt_\lambda)$ is a critical point of (1.23).

In the same fashion, we can obtain another critical point of (1.23) by taking $\alpha\in\mathbb{R}^m$ as follows

$$F_{k+1}\left(\sum_{i=1}^{m}\alpha_ie_i,\lambda\right)=\min_{|x|^2=1}F_{k+1}\left(\sum_{i=1}^{m}x_ie_i,\lambda\right).$$

When the operator $G = DF$ is odd, i.e.

$$G(u,\lambda) = -G(-u,\lambda), \qquad \forall\, u \in X,\ \lambda \in \mathbb{R}^1,$$

we have the following theorem, due to D.C. Clark.

Theorem 1.13 *Under the hypotheses (A_1) and (A_2), if $G(u,\lambda)$ is an odd operator, and λ_0^{-1} is an eigenvalue of A with multiplicity m, then there are at least m distinct pairs of bifurcated branches of (1.13) from $(u,\lambda) = (0,\lambda_0)$.*

The basic idea of finding critical points $x(\lambda)$ of (1.23) is as follows. We take $(\alpha_1,\cdots,\alpha_m) \in \mathbb{R}^m$ such that

$$F_{k+1}\left(\sum_{i=1}^{m} \alpha_i e_i, \lambda\right) = \min_{\mathbb{R}^r \subset \mathbb{R}^m} \max_{x\in\mathbb{R}^r,\,|x|=1} F_{k+1}\left(\sum_{i=1}^{m} x_i e_i, \lambda\right). \tag{1.25}$$

Because F_{k+1} is even, the values $(\alpha_1,\cdots,\alpha_m)$ can be derived for each $1 \le r \le m$, and the m distinct pairs of points

$$x_r(\lambda) = (\pm\alpha_{r1}t_\lambda, \cdots, \pm\alpha_{rm}t_\lambda)$$

are the critical points of (1.23), where $\alpha_r = (\alpha_{r1},\cdots,\alpha_{rm})$ satisfies (1.25) and t_λ satisfies (1.24).

1.5 Rabinowitz Global Bifurcation Theorem

In this section, we introduce the Rabinowitz global bifurcation theorem. Let

$$B = \{(u,\lambda) \in X \times \mathbb{R}^1 \mid u \neq 0,\ (u,\lambda) \text{ satisfy } (1.13)\}.$$

Theorem 1.14 *Let $G : X \times \mathbb{R}^1 \to X$ be a compact operator satisfying (1.6), and $A : X \to X$ a linear compact operator. If λ_0^{-1} is a real eigenvalue of A with odd algebraic multiplicity, then the connected component $\Sigma \subset \overline{B}$ which contains $(0,\lambda_0)$ satisfies one of the following assertions:*

(1) Σ is unbounded in $X \times \mathbb{R}$, or
(2) Σ contains odd number of points $(0,\lambda_i) \neq (0,\lambda_0)$ such that λ_i^{-1} are the eigenvalues of A with odd algebraic multiplicities.

This theorem is, in essence, a corollary of the homotopy invariance of the Leray–Schauder degree. The assertions (1) and (2) of Theorem 1.14 amount to saying that the connected component $\Sigma \subset \overline{B}$ containing $(0,\lambda_0)$ must be one of the two types as shown in Figure 1.3(a) and (b). If otherwise, then Σ must be bounded in $X \times \mathbb{R}^1$ as shown in Figure 1.4. By the basic properties of the Leray–Schauder degree, we immedaitely deduce a contradiction.

More precisely, by the excision property (see Figure 1.4), we have

$$\begin{aligned}\deg(\mathrm{id} - \lambda_2 A + G, B_R, 0) &= \deg(\mathrm{id} - \lambda_2 A + G, B_r, 0) \\ &= \mathrm{ind}(\mathrm{id} - \lambda_2 A, 0) = (-1)^\beta.\end{aligned} \tag{1.26}$$

Then the homotopy invariance of the degree shows that

$$\deg(\mathrm{id} - \lambda_2 A + G, B_R, 0) = \deg(\mathrm{id} - \rho_1 A + G, B_R, 0), \tag{1.27}$$

$$\deg(\mathrm{id} - \lambda_2 A + G, B_r, 0) = \deg(\mathrm{id} - \lambda_1 A + G, B_r, 0). \tag{1.28}$$

It follows then from (1.26)–(1.28) that

$$\deg(\mathrm{id} - \rho_1 A + G, B_R, 0) = \deg(\mathrm{id} - \lambda_1 A + G, B_R, 0) = (-1)^\beta. \tag{1.29}$$

Furthermore, by the index theorem (Theorem 1.7),

$$\deg(\mathrm{id} - \rho_1 A + G, B_R, 0) = \mathrm{ind}(\mathrm{id} - \rho_1 A, 0) = (-1)^{\beta - m}, \tag{1.30}$$

where m = odd is the algebraic multiplicity of λ_0, and $B_R = \{x \in X \mid \|x\| < R\}$. Thus, we arrive at a contradiction from (1.29) and (1.30).

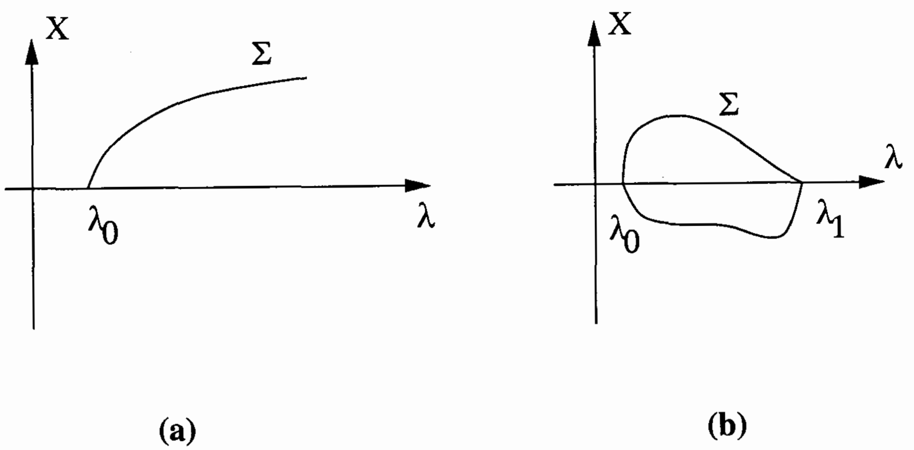

Fig. 1.3 (a) Σ is unbounded; (b) Σ is bounded.

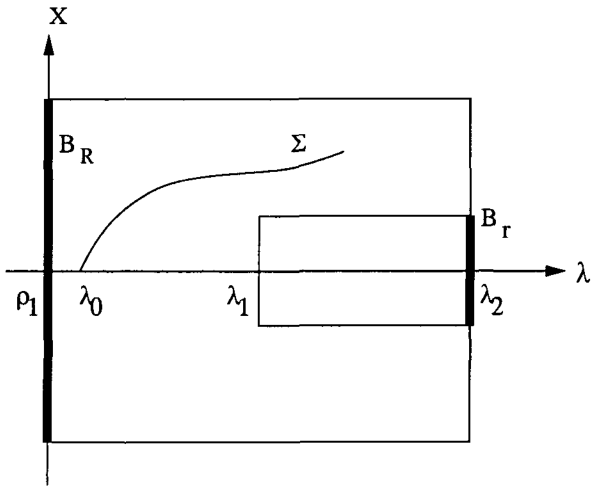

Fig. 1.4 The larger rectangular box is $B_R \times [\rho_1, \lambda_2]$, and the smaller rectangular box is $B_r \times [\lambda_1, \lambda_2]$.

1.6 Notes

1.1 The proof of the classical implicit function theorem can be found in many standard textbooks; see, among others, [Chen, 1981; Chow and Hale, 1982; Nirenberg, 2001; Zhong et al., 1998].

1.2 Both the Brouwer degree theory and the Leray-Schauder degree theory can be found in classical books, including, among many others, [Chen, 1981; Krasnosel'skiĭ, 1956; Nirenberg, 2001; Zeidler, 1986].

1.3 The standard Lyapunov-Schmidt procedures are discussed in [Nirenberg, 2001; Chow and Hale, 1982; Golubitsky and Schaeffer, 1985; Golubitsky et al., 1988].

1.4 Theorems 1.10 and 1.12, are due to the pioneering work by [Krasnosel'skiĭ, 1956]. Theorem 1.11 is due to [Crandall and Rabinowitz, 1971]. Theorem 1.13 is proved by [Clark, 1975], and is generalized by [Rabinowitz, 1977].

1.5 The Rabinowitz global bifurcation theorem, Theorem 1.14, is due to [Rabinowitz, 1971].