Chapter 6

Two-Dimensional Local Average Processes

One of the brightest gems in the New England weather is the dazzling uncertainty of it.

Mark Twain

6.1 Variance Function and Measure of Correlation

Consider the random field $I_A(t_1, t_2)$ derived from a homogeneous two-dimensional random field $X(t_1, t_2)$ by local integration over a rectangular area $A = T_1 T_2$ (see Fig. 6.1):

$$I_A(t_1, t_2) \equiv I_{T_1 T_2}(t_1, t_2) = \int_{t_1 - T_1/2}^{t_1 + T_1/2} \int_{t_2 - T_2/2}^{t_2 + T_2/2} X(t_1, t_2)\, dt_1\, dt_2. \tag{6.1.1}$$

The rectangular window is centered at (t_1, t_2), and its sides (having length T_1 and T_2) remain parallel to the respective coordinate axes. Dividing $I_A(t_1, t_2)$ by the area A yields the 2-D random process of local averages:

$$X_A(t_1, t_2) \equiv X_{T_1 T_2}(t_1, t_2) = \frac{1}{A} I_A(t_1, t_2). \tag{6.1.2}$$

To simplify the notation, we will generally omit reference to the parameters t_1 and t_2. The ratio of the variances of X_A and X is by definition $\gamma(U_1, U_2)$, the variance function of $X(t_1, t_2)$. We may write

$$\mathrm{Var}\,[X_A] \equiv \sigma_{A}^2 \equiv \sigma_{T_1 T_2}^2 = \sigma^2 \gamma(T_1, T_2). \tag{6.1.3}$$

Similarly, $\Delta(T_1, T_2)$ is the variance function associated with the local integral process, defined by the relation:

$$\mathrm{Var}\,[I_{T_1 T_2}] = \sigma^2 \Delta(T_1, T_2), \tag{6.1.4}$$

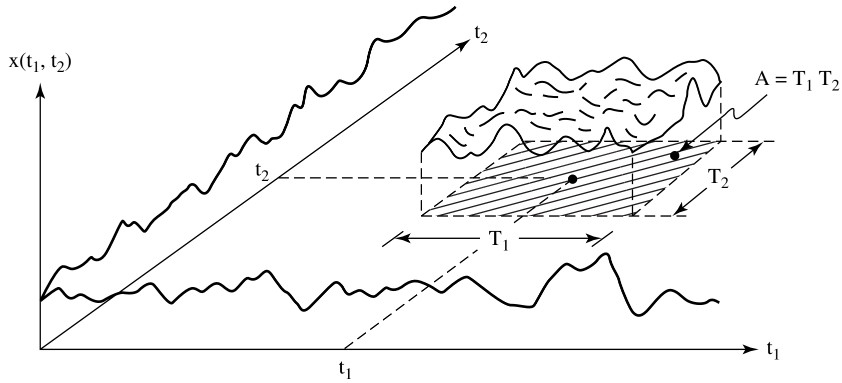


Fig. 6.1 Local averaging over a rectangular area $A = T_1T_2$ in the two-dimensional parameter space (t_1, t_2).

so the two variance functions differ only by the factor $A^2 = (T_1T_2)^2$:

$$\Delta(T_1, T_2) = (T_1T_2)^2\,\gamma(T_1, T_2). \tag{6.1.5}$$

The relationship between $\rho(\tau_1, \tau_2)$ and $\gamma(T_1, T_2)$ follows from Eq. (3.6.47):

$$\gamma(T_1, T_2) = \frac{1}{T_1T_2}\int_{-T_1}^{+T_1}\int_{-T_2}^{+T_2}\left(1 - \frac{|\tau_1|}{T_1}\right)\left(1 - \frac{|\tau_2|}{T_2}\right)\rho(\tau_1, \tau_2)\,d\tau_1\,d\tau_2. \tag{6.1.6}$$

By expanding the product in the above integrand, $\gamma(T_1, T_2)$ can be expressed as a sum of four terms, one of which becomes dominant when T_1 and T_2 increase, *provided* the correlation function $\rho(\tau_1, \tau_2)$ decays sufficiently rapidly. (The exact conditions are examined in the next section.) When both T_1 and T_2 grow large, the variance function tends toward the asymptotic expression

$$\gamma(T_1, T_2) \to \frac{\alpha}{T_1T_2} = \frac{\alpha}{A}, \quad T_1,\ T_2 \text{ large}, \tag{6.1.7}$$

where the proportionality constant α is a "characteristic area" equal to the integral of $\rho(\tau_1, \tau_2)$:

$$\alpha = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho(\tau_1, \tau_2)\,d\tau_1\,d\tau_2. \tag{6.1.8}$$

Table 6.1 Parallel interpretations of the correlation parameters θ and α.

Correlation distance θ	Correlation area α
$\lim_{T\to\infty} T\gamma(T)$	$\lim_{T_1,T_2\to\infty} T_1T_2\gamma(T_1,T_2)$
$\int_{-\infty}^{+\infty} \rho(\tau)\, d\tau$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(\tau_1,\tau_2)\, d\tau_1\, d\tau_2$
$2\pi s(0)$	$4\pi^2 s(0,0)$

The quantity α also has a simple interpretation in the frequency domain. Taking $\omega_1 = \omega_2 = 0$ in the Wiener-Khinchine relation , Eq. (3.3.9), yields

$$s(0,0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(\tau_1,\tau_2)\, d\tau_1\, d\tau_2. \tag{6.1.9}$$

Comparing Eqs. (6.1.8) and (6.1.9) leads to

$$\alpha = 4\pi^2 s(0,0). \tag{6.1.10}$$

Table 6.1 shows the alternate expressions for the characteristic area α in terms of the variance function [Eq. (6.1.7)], the correlation function [Eq. (6.1.8)] and the normalized spectral density function [Eq. (6.1.9)], parallelling those for the scale of fluctuation θ. Note that α cannot be negative, since it is proportional to the s.d.f. (evaluated at $\omega_1 = \omega_2 = 0$).

Behavior of $s(\omega_1, \omega_2)$ Near the Frequency Origin

The validity of the asymptotic expression for the variance function [Eq. (6.1.7)],

$$\gamma(T_1,T_2) \to \frac{\alpha}{T_1T_2}, \quad \text{as} \quad T_1,\ T_2 \to \infty,$$

is subject to certain conditions on the "moments" of the correlation function. As in the one-dimensional case, analysis in the frequency domain permits these conditions to be identified. First, consider how Eq. (6.1.7) can be derived by frequency domain analysis. From Eq. (3.6.43), the relationship between $s(\omega_1, \omega_2)$ and the variance function $\gamma(T_1, T_2)$ is

$$\gamma(T_1, T_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{\sin(\omega_1 T_1/2)}{\omega_1 T_1/2}\right]^2 \left[\frac{\sin(\omega_2 T_2/2)}{\omega_2 T_2/2}\right]^2 s(\omega_1, \omega_2)\, d\omega_1\, d\omega_2. \tag{6.1.11}$$

The function $[(\sin z)/z]^2$ converges toward one when $z \to 0$. Therefore, Eq. (6.1.11) predicts, as expected, that there is no variance reduction when $T_i \to 0$ $(i = 1, 2)$:

$$\gamma(0, 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 = 1. \tag{6.1.12}$$

At the opposite extreme when $T_i \to \infty$ $(i = 1,\ 2)$ in Eq. (6.1.11), assuming the function $s(\omega_1, \omega_2)$ varies smoothly near $\omega_1 = \omega_2 = 0$, it may be replaced by $s(0, 0)$. By introducing a change of variables, $v_i = \omega_i T_i/(2\pi)$ $(i = 1, 2)$, Eq. (6.1.11) becomes

$$\gamma(T_1, T_2) = s(0, 0)\, \frac{2\pi}{T_1} \frac{2\pi}{T_2} \int_{-\infty}^{+\infty} \left[\frac{\sin \pi v_1}{\pi v_1}\right]^2 dv_1 \int_{-\infty}^{+\infty} \left[\frac{\sin \pi v_2}{\pi v_2}\right]^2 dv_2. \tag{6.1.13}$$

Each integral on the right side equals one. Hence, using Eq. (6.1.10), we obtain

$$\gamma(T_1, T_2) = \frac{4\pi^2 s(0, 0)}{T_1 T_2} = \frac{\alpha}{T_1 T_2} = \frac{\alpha}{A}, \quad T_1,\ T_2 \to \infty. \tag{6.1.14}$$

Implicit in the derivation is the assumption that $s(\omega_1, \omega_2)$ varies smoothly near $\omega_1 = \omega_2 = 0$. To examine this, consider the series expansion of $s(\omega_1, \omega_2)$ near the frequency origin:

$$s(\omega_1, \omega_2) = \frac{\alpha}{4\,\pi^2} + b_1 \omega_1 + b_2\, \omega_2 + \frac{d_1}{2}\omega_1^2 + \frac{d_2}{2}\omega_2^2 + d_{12}\, \omega_1 \omega_2 + \cdots, \tag{6.1.15}$$

in which the first term of the series agrees with Eq. (6.1.10). From Eq. (6.1.15), the coefficient b equals the first partial derivative of $s(\omega_1, \omega_2)$ with respect to ω_1, evaluated at the frequency-plane origin,

$$b_1 = \left.\frac{\partial s(\omega_1, \omega_2)}{\partial \omega_1}\right|_{\omega_1 = \omega_2 = 0}. \tag{6.1.16}$$

Expressing $s(\omega_1, \omega_2)$ in terms of $\rho(\tau_1, \tau_2)$ by means of the Wiener-Khinchine relation [Eq. (3.3.7)], taking the partial derivative with respect to ω_1, and letting $\omega_i \to 0$ $(i = 1, 2)$ yields:

$$\begin{aligned} b_1 &= -\frac{1}{(2\pi)^2}\left[\omega_1 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tau_1^2 \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 \right. \\ &\qquad \left. + \omega_2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tau_1 \tau_2 \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 \right] \\ &= -\frac{1}{(2\pi)^2}[\omega_1 \alpha_2^{(1)} + \omega_2 \alpha_{11}], \end{aligned} \tag{6.1.17}$$

in which, in general, the quantities $\alpha_k^{(i)}$ and α_{kl} refer to various *moments* of the 2-D correlation function:

$$\begin{aligned} \alpha_k^{(i)} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tau_i^k \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2, \qquad i = 1, 2,\ k = 1, 2, \ldots, \\ \alpha_{kl} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tau_1^k \tau_2^l \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2, \quad k,\ l = 1, 2, \ldots. \end{aligned} \tag{6.1.18}$$

It follows from Eq. (6.1.17) that $b_1 = 0$, *provided* $|\alpha_2^{(1)}|$ and $|\alpha_{11}|$ are finite. Likewise, we have $b_2 = 0$, *provided* $|\alpha_2^{(2)}| < \infty$ and $|\alpha_{11}| < \infty$.

A similar analysis leads to the following results for the other coefficients in Eq. (6.1.15):

$$\begin{aligned} d_i &= \left.\frac{\partial^2 s(\omega_1, \omega_2)}{\partial \omega_i^2}\right|_{\omega_1=\omega_2=0} = -\frac{\alpha_2^{(i)}}{4\pi^2}, \quad i = 1, 2, \\ d_{11} &= \left.\frac{\partial^2 s(\omega_1, \omega_2)}{\partial \omega_1 \partial \omega_2}\right|_{\omega_1=\omega_2=0} = -\frac{\alpha_{11}}{4\pi^2}. \end{aligned} \tag{6.1.19}$$

In conclusion, *provided the moments of second order of* $\rho(\tau_1, \tau_2)$ *are finite*, $s(\omega_1, \omega_2)$ will obey the following series expansion near $\omega_1 = \omega_2 = 0$:

$$s(\omega_1, \omega_2) = \frac{\alpha}{4\pi^2} - \frac{1}{8\pi^2}(\alpha_2^{(1)}\omega_1^2 + \alpha_2^{(2)}\omega_2^2 - 2\alpha_{11}\omega_1\omega_2) + \cdots. \tag{6.1.20}$$

An equivalent form of the (necessary and sufficient) conditions for the existence of α in the sense implied by Eq. (6.1.7), is that *the first-order partial derivatives of* $s(\omega_1, \omega_2)$ *must vanish at the frequency-plane origin.* Eq. (6.1.20) implies that $s(\omega_1, \omega_2)$ has a local extremum at $\omega_1 = \omega_2 = 0$. Moreover, if the second-order moments of $\rho(\tau_1, \tau_2)$ are positive, $s(0, 0)$ will be a local *maximum*.

Unidirectional Random Variation

The two-dimensional random field $X(t_1, t_2)$ generates many direction-dependent one-dimensional random functions, like those describing the variation along lines parallel to one of the coordinate axes, such as $X(t_1)$. Using the subscript $i = 1$ on t, τ, ω, or T to indicate direction, the second-order properties of $X(t_1)$ may be characterized by:

1. The correlation function

$$\rho(\tau_1) = \rho(\tau_1, 0). \tag{6.1.21}$$

2. The unit-area spectral density function

$$s(\omega_1) = \int_0^{\infty} s(\omega_1, \omega_2)\, d\omega_2. \tag{6.1.22}$$

3. The variance function

$$\gamma(T_1) = \gamma(T_1, 0). \tag{6.1.23}$$

Of course, similar expressions can be stated for $X(t_2)$. Based on the one-dimensional theory developed in Chap. 5, the *scale of fluctuation* of $X(t_i)$ can be expressed in one of the following ways:

$$\theta^{(i)} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \rho(\tau_i)\, d\tau_i, \\ \pi g(\omega_i)|_{\omega_i = 0}, \qquad\qquad i = 1, 2. \\ \displaystyle\lim_{T_i \to \infty} T_i \gamma(T_i), \end{cases} \tag{6.1.24}$$

Recall that the basic condition for the validity of the one-dimensional theory is that the absolute value of the second moment of $\rho(\tau_i)$ be finite:

$$|\theta_2^{(i)}| < \infty, \qquad\qquad i = 1, 2, \tag{6.1.25}$$

where for $k = 1, 2, \ldots,$

$$\theta_k^{(i)} = \int_{-\infty}^{+\infty} \tau_i^k \rho(\tau_i)\, d\tau_i, \qquad\qquad i = 1, 2. \tag{6.1.26}$$

The equivalent condition in the frequency domain is that the first-order derivative of $g(\omega_i)$ must vanish at $\omega_i = 0$:

$$[\dot{g}(\omega_i)]_{\omega_i = 0} = 0.$$

It is useful to introduce the dimensionless parameter c_α which relates the characteristic area α to the product of the "direction-dependent" scales of fluctuation $\theta^{(1)}$ and $\theta^{(2)}$, as follows:

$$c_\alpha = \frac{\alpha}{\theta^{(1)}\theta^{(2)}}. \tag{6.1.27}$$

The parameter c_α plays an important role in the analysis of 2-D random fields, and is evaluated for some cases in the next section.

6.2 Important Special Cases

The Correlation Structure is Quadrant Symmetric

In applications of two-dimensional random field theory it is useful to choose the coordinate axes t_1 and t_2 so as to render the correlation structure of $X(t_1, t_2)$ *quadrant symmetric* (see Sec. 3.2). It then suffices to deal with the quadrant of positive lags ($\tau_1, \tau_2 \geq 0$) and the quadrant of positive frequencies ($\omega_1, \omega_2 \geq 0$). $G(\omega_1, \omega_2)$ is the s.d.f. defined for positive frequencies only, and $g(\omega_1, \omega_2) = \sigma^{-2}G(\omega_1, \omega_2)$ is the associated normalized (unit-area) spectrum. If $X(t_1, t_2)$ is quadrant symmetric (in the weak sense), the correlation measure α may be expressed as follows:

$$\alpha = \pi^2 g(0,0) = 4\int_0^\infty \int_0^\infty \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2. \tag{6.2.1}$$

Likewise, the moments of the 2-D correlation function become

$$\alpha_k^{(i)} = 4\int_0^\infty \int_0^\infty \tau_i^k \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2, \quad i = 1, 2,\ k = 1, 2, \ldots, \tag{6.2.2}$$

and

$$\alpha_{kl} = 4\int_0^\infty \int_0^\infty \tau_1^k \tau_2^l \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2, \quad k,\ l = 1, 2, \ldots. \tag{6.2.3}$$

A number of important special cases, all satisfying the condition of the quadrant symmetry , are examined next.

The Correlation Structure is Separable

In case the correlation structure is separable, the correlation, spectral density, and variance functions can all be expressed as products of the respective one-dimensional functions:

$$\rho(\tau_1, \tau_2) = \rho(\tau_1)\rho(\tau_2),$$
$$g(\omega_1, \omega_2) = g(\omega_1)g(\omega_2), \tag{6.2.4}$$
$$\gamma(T_1, T_2) = \gamma(T_1)\gamma(T_2).$$

The correlation measure α also equals the product of the direction-dependent scales of fluctuation:

$$\alpha = \theta^{(1)}\theta^{(2)}. \tag{6.2.5}$$

Hence, from Eq. (6.1.27),

$$c_\alpha = 1. \tag{6.2.6}$$

Consider below two specific examples of separable correlation functions.

Case 1. The random field is obtained by local averaging of an isotropic purely random (uncorrelated) field, the averaging domain being a rectangle moving parallel to the coordinate axes. Denoting by $2a_1$ and $2a_2$ the sides of the rectangle, the correlation function becomes

$$\rho(\tau_1, \tau_2) = \left(1 - \frac{|\tau_1|}{a_1}\right)\left(1 - \frac{|\tau_2|}{a_2}\right), \quad |\tau_1| \le a_1,\ |\tau_2| \le a_2. \tag{6.2.7}$$

The direction-dependent scales of fluctuation,

$$\theta^{(i)} = a_i, \qquad i = 1, 2, \tag{6.2.8}$$

characterize the 1-D correlation functions $\rho(\tau_1, 0)$ and $\rho(0, \tau_2)$. Eq. (6.2.7) provides a 2-D extension of the 1-D triangular correlation function [see Eq. (5.1.7)]. Note that the derived random field is anisotropic, even when $a_1 = a_2$.

Case 2. The correlation function is given by

$$\rho(\tau_1, \tau_2) = \exp\left\{-\left(\frac{\tau_1}{b_1}\right)^2 - \left(\frac{\tau_2}{b_2}\right)^2\right\}, \tag{6.2.9}$$

an extension of Eq. (5.1.13). The univariate scales of fluctuation are

$$\theta^{(i)} = \sqrt{\pi}\, b_i, \qquad i = 1, 2. \tag{6.2.10}$$

If $b_1 = b_2$, the correlation structure becomes isotropic as well as separable.

Isotropic Correlation Structure

If the random field is isotropic, its correlation structure can be expressed in terms of the "radial" correlation function

$$\rho^R(\tau) = \rho(\tau, 0) = \rho(0, \tau) = \rho(\tau_1, \tau_2), \tag{6.2.11}$$

where $\tau = \sqrt{\tau_1^2 + \tau_2^2}$. The characteristic area α becomes

$$\alpha = 2\pi \int_0^\infty \tau \rho^R(\tau)\, d\tau = \pi\, \theta_1^R, \tag{6.2.12}$$

in which θ_k^R denotes the kth moment of $\rho^R(\tau)$:

$$\theta_k^R = 2 \int_0^\infty \tau^k \rho^R(\tau)\, d\tau, \quad k = 0, 1, 2, \ldots; \tag{6.2.13}$$

note that $\theta_0^R \equiv \theta^R$. Since α cannot be negative [$\alpha = \pi^2 g(0,0)$], it may be concluded from Eq. (6.2.12) that for a correlation function to be acceptable as a model for two-dimensional isotropic random variation, its first moment (for $\tau \geq 0$) must be nonnegative, that is, $\theta_1^R \geq 0$. The scales of fluctuation $\theta^{(i)}$, $i = 1, 2$, do not depend on direction, and we may write

$$\theta^{(1)} = \theta^{(2)} = \theta^R. \tag{6.2.14}$$

Inserting Eqs. (6.2.13) and (6.2.14) into the definition of c_α [Eq. (6.1.27)] yields

$$c_\alpha = \frac{\pi\, \theta_1^R}{(\theta^R)^2}. \tag{6.2.15}$$

Note the relationship $c_\alpha = \pi k_1$, where the dimensionless parameter k_1 is defined in Eq. (5.1.39). Table 6.2 lists the values of c_α for a family of correlation models introduced below, while the parameter $k_1 = c_\alpha/\pi$ is evaluated for four types of correlation functions in Sec. 5.1; all the values of c_α are found to lie between 1 and $\pi/2 \simeq 1.57$.

Ellipsoidal Correlation Structure

By appropriate scaling and rotation of the coordinate axes, random fields with ellipsoidal correlation structure can be made into isotropic random fields. In particular, if the coordinate axes t_1 and t_2 are parallel to the principal axes of the iso-correlation ellipses, the correlation structure of $X(t_1, t_2)$ becomes quadrant symmetric, and the dimensionless parameter

Table 6.2 Coefficient $c_\alpha = \pi k_1$ and scale-related parameter for the family of correlation models defined by Eq. (6.2.16).

m	$c_\alpha = \pi k_1$	$\dfrac{\theta^{(i)}}{(\pi/b_i)}$
$\frac{3}{2}$	$\frac{\pi}{2} \simeq 1.57$	1
2	1.2727	1
3	1.1313	1.5
4	1.0861	1.875
5	1.0639	2.1875
7	1.0421	2.7070
10	1.0277	3.3385
15	1.0176	4.1845

c_α is the same as that of the isotropic process. The value of c_α depends only on the shape of the radial correlation function [Eq. (6.2.15)].

Example. (A Family of Autoregressive Models) It is possible to obtain "closed form" analytical results for an entire family of *autoregressive (Markovian) models* whose spectra are given by

$$G(\omega_1, \omega_2) = G_0 \left[1 + \left(\frac{\omega_1}{b_1}\right)^2 + \left(\frac{\omega_2}{b_2}\right)^2\right]^{-m}, \quad m > 1. \tag{6.2.16}$$

The case $m = 3/2$ corresponds to the exponential (marginal) correlation function. Integrating over all values of ω_1 (from 0 to ∞) yields the marginal s.d.f.

$$G(\omega_2) = \frac{G_0 b_1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(m - \frac{1}{2})}{\Gamma(m)} \left[1 + \left(\frac{\omega_2}{b_2}\right)^2\right]^{-m+1/2}. \tag{6.2.17}$$

The s.d.f. $G(\omega_1)$ has the same form. The variance σ^2 is obtained by integrating Eq. (6.2.17) over ω_2, yielding

$$\sigma^2 = \frac{G_0 b_1 b_2}{4} \frac{[\Gamma(\frac{1}{2})]^2 \Gamma(m-1)}{\Gamma(m)} = \frac{\pi b_1 \, b_2 \, G_0}{4(m-1)}, \quad m > 1, \tag{6.2.18}$$

and the scale of fluctuation $\theta^{(i)}$, $i = 1, 2$, equals

$$\theta^{(i)} = \frac{2\sqrt{\pi}}{b_i} \frac{(m-1)\Gamma(m-\frac{1}{2})}{\Gamma(m)}, \quad m > 1. \tag{6.2.19}$$

If m is an integer, Eq. (6.2.19) may also be written as follows:

$$\theta^{(i)} = \frac{\pi}{b_i} \frac{1 \cdot 3 \cdot 5 \cdots (2m-3)}{(m-2)!\, 2^{m-2}}, \quad m = 2, 3, \ldots. \tag{6.2.20}$$

The ratio $(\theta^{(i)} b_i/\pi)$ is given for a set of values of m in the last column of Table 6.2. The 2-D correlation measure is

$$\alpha = \frac{\pi^2 G_0}{\sigma^2} = \frac{4\pi(m-1)}{b_1 b_2}, \quad m > 1, \tag{6.2.21}$$

and combining Eqs. (6.2.19) through (6.2.21) yields

$$c_\alpha = \frac{\alpha}{\theta^{(1)}\theta^{(2)}} = \begin{cases} \dfrac{[\Gamma(m)]^2}{(m-1)\,[\Gamma(m-\frac{1}{2})]^2}, & m > 1, \\[2ex] \dfrac{4}{\pi} \dfrac{(m-1)!\,(m-2)!\,2^{2m-4}}{[1 \cdot 3 \cdot 5 \cdots (2m-3)]^2}, & m = 2, 3, \ldots. \end{cases} \tag{6.2.22}$$

Some representative values of c_α are listed in Table 6.2.

6.3 Conditional Variance Functions and Scales of Fluctuation

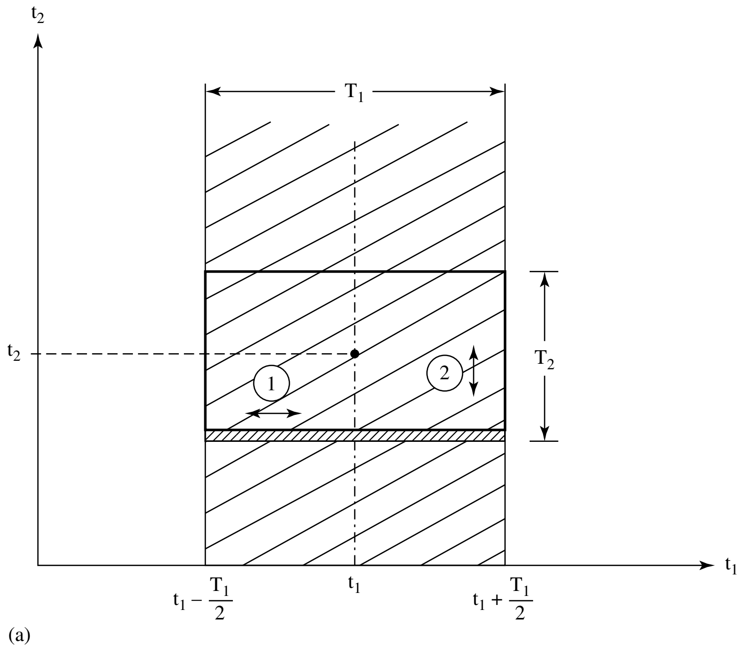
Conditional Variance Functions

The operation of averaging a two-dimensional random field $X(t_1, t_2)$ over a rectangular area $A = T_1 T_2$ may be carried out in two steps, as illustrated in Fig. 6.2. The first step is to integrate $X(t_1, t_2)$ over the distance T_1 along the t_1-axis. This generates a new one-dimensional random function which is the local average of $X(t_1, t_2)$ within a band of width T_1 parallel to the t_2-axis:

$$X_{T_1}(t_2) = X_{T_1}(t_2; t_1) = \frac{1}{T_1} \int_{t_1 - T_1/2}^{t_1 + T_1/2} X(t_1, t_2)\, dt_1. \tag{6.3.1}$$

The variance of $X_{T_1}(t_2)$ equals the product of the "point variance" σ^2 and the 1-D variance function $\gamma(T_1)$,

$$\mathrm{Var}\,[X_{T_1}] \equiv \sigma^2 \gamma(T_1). \tag{6.3.2}$$



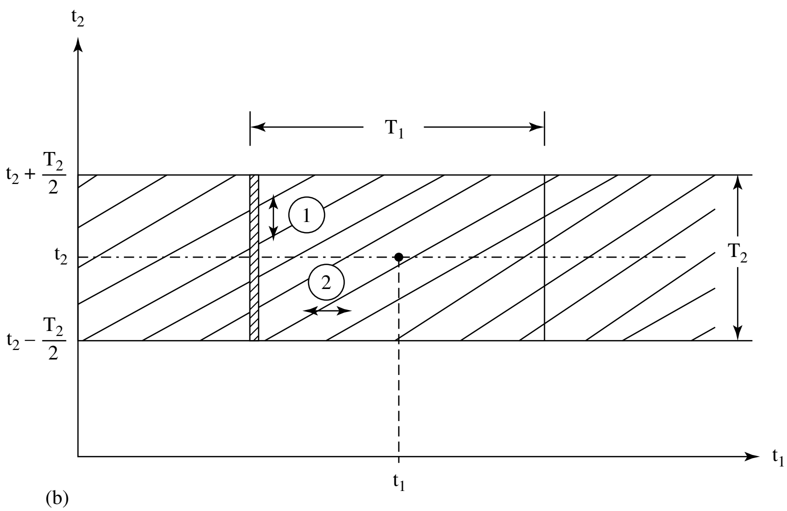


Fig. 6.2 Two ways of averaging $X(t_1, t_2)$ over the rectangular area $A = T_1T_2$: (a) average first with respect to t_1 over a bandwidth T_1, yielding $X_{T_1}(t_2)$, or (b) average first with respect to t_2 over a bandwidth T_2, yielding $X_{T_2}(t_1)$.

The second step is to average $X_{T_1}(t_2)$ over the distance T_2 along the t_2-axis. This leads to further variance reduction, expressed by

$$\sigma_A^2 = \sigma^2\gamma(T_1)\,\gamma(T_2|T_1), \tag{6.3.3}$$

where $\gamma(T_2|T_1)$ is the variance function of $X_{T_1}(t_2)$, or the *conditional variance function* of X given prior averaging over the distance T_1 along the t_1-axis. Note that the "bivariate" variance function may be expressed as the product of a "marginal" and a "conditional" variance function:

$$\gamma(T_1, T_2) = \gamma(T_1)\,\gamma(T_2|T_1). \tag{6.3.4}$$

The two-step procedure may of course be reversed, by averaging first along the t_2-axis and then along the t_1-axis, hence

$$\gamma(T_1, T_2) = \gamma(T_2)\,\gamma(T_1|T_2) = \gamma(T_1)\,\gamma(T_2|T_1). \tag{6.3.5}$$

Similar relations exist between the variance functions defined in terms of local integrals:

$$\Delta(T_1, T_2) = \Delta(T_2)\,\Delta(T_1|T_2) = \Delta(T_1)\,\Delta(T_2|T_1). \tag{6.3.6}$$

If the correlation between of $X(t_1, t_2)$ is separable, the conditional variance function becomes equal to the "marginal" variance function; for example,

$$\gamma(T_1|T_2) = \gamma(T_1). \tag{6.3.7}$$

Also, the 2-D variance function can then be expressed as

$$\gamma(T_1, T_2) = \gamma(T_1)\,\gamma(T_2). \tag{6.3.8}$$

Note the striking similarity between Eqs. (6.3.4) through (6.3.8) and the familiar relationships linking the marginal, conditional, and joint probability density functions of two random variables. Furthermore, in this context, the last two equations, applicable when the correlation structure is separable, evoke the concept of statistical independence between two random variables.

Conditional Scales of Fluctuation

The principal correlation parameter of the random process $X_{T_1}(t_2)$ is the conditional scale of fluctuation $\theta_{T_1}^{(2)}$, a characteristic of the (one-dimensional) conditional variance function $\gamma(T_2|T_1)$. If $T_1 = 0$, there is no averaging along the t_1-axis and the conditional scale reduces to $\theta_0^{(2)} \equiv \theta^{(2)}$.

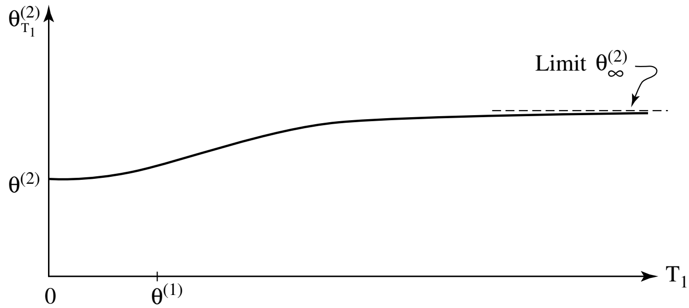


Fig. 6.3 Scale of fluctuation $\theta_{T_1}^{(2)}$ of the derived random process $X_{T_1}(t_2)$ in function of the averaging bandwidth T_1.

As Fig. 6.3 indicates, as T_1 gets larger, $\theta_{T_1}^{(2)}$ tends to increase gradually, converging toward an asymptotic limit $\theta_{\infty}^{(2)}$. Using Eq. (6.3.4), this limit can be evaluated by letting the intervals T_1 and T_2 become large enough so that each variance function can be replaced its asymptotic form:

$$\frac{\alpha}{T_1 T_2} = \frac{\theta^{(1)}}{T_1}\frac{\theta_{T_1}^{(2)}}{T_2}. \tag{6.3.9}$$

The asymptotic value of $\theta_{T_1}^{(2)}$ becomes, for $T_1 \to \infty$,

$$\lim_{T_1\to\infty} \theta_{T_1}^{(2)} \to \theta_{\infty}^{(2)} = \frac{\alpha}{\theta^{(1)}} = c_\alpha \theta^{(2)}. \tag{6.3.10}$$

Likewise,

$$\lim_{T_2\to\infty} \theta_{T_2}^{(1)} \to \theta_{\infty}^{(1)} = \frac{\alpha}{\theta^{(2)}} = c_\alpha \theta^{(1)}, \tag{6.3.11}$$

where α is the volume under the 2-D correlation function, and c_α is the dimensionless factor defined in Eq. (6.1.27), whose typical values range between 1 and $\pi/2$.

In general, the value of $\theta_{T_1}^{(2)}$ can be obtained from Eq. (6.3.5) by taking the limit $T_2 \to \infty$:

$$\gamma(T_1)\frac{\theta_{T_1}^{(2)}}{T_2} = \frac{\theta^{(2)}}{T_2}, \gamma(T_1|T_2 = \infty), \tag{6.3.12}$$

resulting in

$$\theta^{(2)}_{T_1} = \frac{\gamma(T_1|T_2=\infty)}{\gamma(T_1)}\,\theta^{(2)}. \tag{6.3.13}$$

The one-dimensional variance function $\gamma(T_1|T_2=\infty)$ is characterized by the scale $\theta^{(1)}_\infty = \alpha/\theta^{(2)} = c_\alpha\theta^{(1)}$. To see how $\theta^{(2)}_T$ varies with T, consider a specific analytical model for the 1-D variance functions, $\gamma(T_1|T_2=\infty)$ and $\gamma(T_1)$, appearing in Eq. (6.3.13). Adopting the simplest (idealized) model, given by Eq. (5.1.30), means inserting the expressions

$$\gamma(T_1) = \begin{cases} 1, & T_1 \le \theta^{(1)}, \\ \dfrac{\theta^{(1)}}{T_1} & T_1 \ge \theta^{(1)}, \end{cases} \tag{6.3.14}$$

and

$$\gamma(T_1|T_2=\infty) = \begin{cases} 1, & T_1 \le c_\alpha\theta^{(1)}, \\ \dfrac{c_\alpha\theta^{(1)}}{T_1}, & T_1 \ge c_\alpha\theta^{(1)}, \end{cases} \tag{6.3.15}$$

into Eq. (6.3.13). This results in the following piecewise-linear solution (illustrated in Fig. 6.4) for the conditional scale of fluctuation of $X_{T_1}(t_2)$:

$$\theta^{(2)}_{T_1} = \begin{cases} \theta^{(2)}, & T_1 \le \theta^{(1)}, \\ \dfrac{T_1\theta^{(2)}}{\theta^{(1)}}, & \theta^{(1)} \le T_1 \le c_\alpha\theta^{(1)}, \\ c_\alpha\theta^{(2)}, & T_1 \ge c_\alpha\theta^{(1)}. \end{cases} \tag{6.3.16}$$

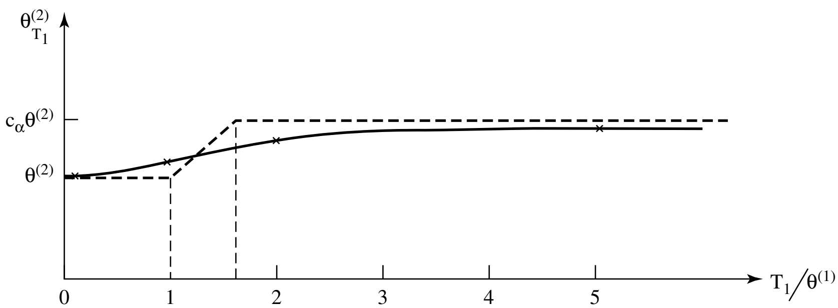


Fig. 6.4 Piecewise-linear approximation of the scale of fluctuation $\theta^{(2)}_{T_1}$. The coefficient c_α equals $\pi/2 \approx 1.57$ for the exponential correlation function (the crosses indicate actual numerical results). The asymptotic upper bound is $\theta^{(2)}_\infty = c_\alpha\theta^{(2)}$.

These values of $\theta^{(2)}_{T_1}$ may now be inserted into that same idealized model for the conditional variance function, namely

$$\gamma(T_2|T_1) = \begin{cases} 1, & T_2 \leq \theta^{(2)}_{T_1}, \\ \dfrac{\theta^{(2)}_{T_1}}{T_2}, & T_2 \geq \theta^{(2)}_{T_1}. \end{cases} \tag{6.3.17}$$

Finally, the corresponding approximation for 2-D variance function $\gamma(T_1, T_2)$ is found by multiplying $\gamma(T_1)$ and $\gamma(T_2|T_1)$ (given by Eqs. (6.3.14) and (6.3.17), respectively). Table 6.3 shows the solution, for an isotropic 2-D random field, for different ranges of the averaging intervals T_1 and T_2.

Table 6.3 Piecewise-linear approximation of the 2-D variance function in case the random field $X(t_1, t_2)$ is isotropic.

	$T_1 \leq \theta$	$\theta \leq T_1 \leq c_\alpha\theta$	$T_1 \geq c_\alpha\theta$
$T_2 \leq \theta$	1	$\dfrac{\theta}{T_1}$	$\dfrac{\theta}{T_1}$
$\theta \leq T_2 \leq c_\alpha\theta$	$\dfrac{\theta}{T_2}$	$\dfrac{\theta}{T_2}$ if $T_1 \leq T_2$ $\dfrac{\theta}{T_1}$ if $T_1 \geq T_2$	$\dfrac{\theta}{T_1}$
$T_2 \geq c_\alpha\theta$	$\dfrac{\theta}{T_2}$	$\dfrac{\theta}{T_2}$	$\dfrac{\alpha}{T_1T_2}$

In general, if a single analytical model is imposed on both the conditional and the marginal variance functions, one must expect a slightly different result for $\gamma(T_1, T_2)$ depending on the order of the computation, that is, $\gamma(T_1)\gamma(T_2|T_1)$ versus $\gamma(T_2)\gamma(T_1|T_2)$. To enhance computational accuracy, one might average the results of the two computations, as follows:

$$\gamma(T_1, T_2) = \frac{1}{2}[\gamma(T_1)\,\gamma(T_2|T_1) + \gamma(T_2)\,\gamma(T_1|T_2)]. \tag{6.3.18}$$

All the quantities on the right side of Eq. (6.3.18) could be evaluated (approximately) using the same (one-dimensional) analytical model for the variance function, for example [see Eq. (5.1.39)],

$$\gamma(T) = \begin{cases} 1 - k_1 \dfrac{T}{\theta}, & 0 \le T \le \theta, \\ \dfrac{\theta}{T}(1 - k_1 \dfrac{\theta}{T}), & T \ge \theta, \end{cases} \tag{6.3.19}$$

where $k_1 = c_\alpha/\pi$ may be set at 1/3 [the default value corresponding to the triangular correlation function, for which Eq. (6.3.19) is exact]. $\gamma(T_1)$ will depend on $\theta^{(1)}$, $\gamma(T_2|T_1)$ on $\theta^{(2)}_{T_1}$, and so on. The conditional scale $\theta^{(2)}_{T_1}$ itself may be computed by combining Eqs. (6.3.13) and (6.3.19).

More Formal Second-Order Descriptions

For completeness we now derive the equations that relate the conditional variance function $\gamma(T_1|T_2)$ to the other second-order descriptions of the derived process $X_{T_2}(t_1)$, first, the conditional correlation function $\rho(\tau_1|T_2)$ and, second, the conditional spectral density function $g(\omega_1|T_2)$.

Conditional Correlation Function

Starting from Eq. (6.1.6), dividing by $\sigma^2\gamma(T_2)$, and accounting for Eq. (6.3.4) yields the following expression for the conditional variance function of the random process $X_{T_2}(t_1)$:

$$\gamma(T_1|T_2) = \frac{1}{T_1}\int_{-T_1}^{+T_1}\left(1 - \frac{|\tau_1|}{T_1}\right)\rho(\tau_1|T_2)\,d\tau_1, \tag{6.3.20}$$

in which

$$\rho(\tau_1|T_2) = \frac{B(\tau_1|T_2)}{\sigma^2\gamma(T_2)}. \tag{6.3.21}$$

The form of these equations implies that $\rho(\tau_1|T_2)$ and $B(\tau_1|T_2)$ are, respectively, the correlation and covariance functions of $X_{T_2}(t_1)$. The conditional scale of fluctuation is

$$\theta^{(1)}_{T_2} = \int_{-\infty}^{+\infty}\rho(\tau_1|T_2)\,d\tau_1. \tag{6.3.22}$$

The following relations involving $B(\tau_1|T_2)$ are notable:

$$\begin{aligned} B(0|\,T_2) &= \sigma^2\gamma(0, T_2) \equiv \sigma^2\gamma(T_2), \\ B(\tau_1|\,0) &= \sigma^2\rho(\tau_1, 0) \equiv \sigma^2\rho(\tau_1). \end{aligned} \tag{6.3.23}$$

Also, for $T_2 \gg \theta^{(2)}$,

$$\rho(\tau_1|\infty) = \frac{1}{\theta^{(2)}} \int_{-\infty}^{+\infty} \rho(\tau_1, \tau_2)\, d\tau_2. \tag{6.3.24}$$

Finally, combining Eqs. (6.3.22) and (6.3.24) confirms

$$\theta_{\infty}^{(1)} = \frac{1}{\theta^{(2)}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 = \frac{\alpha}{\theta^{(2)}}. \tag{6.3.25}$$

Conditional Spectral Density Function

A parallel analysis in the frequency domain starts from Eq (6.1.11); dividing by $\sigma^2\gamma(T_2)$ and accounting for Eq. (6.3.4) yields an expression for the conditional variance function:

$$\gamma(T_1|T_2) = \int_{-\infty}^{+\infty} \left[\frac{\sin(\omega_1 T_1/2)}{\omega_1 T_1/2}\right]^2 s(\omega_1|T_2)\, d\omega_1, \tag{6.3.26}$$

where

$$s(\omega_1|T_2) = \frac{S(\omega_1|T_2)}{\sigma^2\gamma(T_2)}. \tag{6.3.27}$$

The form of these equations implies that $s(\omega_1|T_2)$ is the unit-area spectral density function of $X_{T_2}(t_1)$. The associated scale of fluctuation is

$$\theta_{T_2}^{(1)} = 2\pi\, s(\omega_1|T_2)\big|_{\omega_1=0}, \tag{6.3.28}$$

which for $T_2 \gg \theta^{(2)}$ approaches its upper limit:

$$\theta_{\infty}^{(1)} = 2\pi\, s(0|\infty) = \frac{4\pi^2 s(0,0)}{\theta^{(2)}} = \frac{\alpha}{\theta^{(2)}}. \tag{6.3.29}$$

We conclude that it is possible to obtain, starting from either $B(\tau_1, \tau_2)$ or $S(\omega_1, \omega_2)$, exact expressions for assorted "conditional" second-order properties, such as for the direction-dependent process $X_{T_2}(t_1)$. The correlation function $\rho(\tau_1|T_2)$ and the unit-area s.d.f. $s(\omega_1|T_2)$ constitute a Fourier transform pair and are related to the conditional variance function $\gamma(T_1|T_2)$ by Eqs. (6.3.20) and (6.3.26), respectively. Note that the operations involving variance functions [Eqs. (6.3.4) through (6.3.6)] have the distinct advantage of requiring only simple algebra. Moreover, they have direct physical meaning in terms of variance reduction under local averaging.

6.4 Covariance of Local Averages

Knowledge of the two-dimensional variance function suffices to evaluate, by means of simple algebra, the covariance of local averages (or integrals) of a homogeneous 2-D random field $X(t_1, t_2)$ over rectangular areas whose sides are parallel to the coordinate axes. Consider the areas $A = T_1T_2$ and $A' = T_1'T_2'$ shown in Fig. 6.5. By direct extension of Eq. (5.3.5), the covariance of the local integrals I_A and $I_{A'}$ can be expressed as a linear combination of values of the 2-D variance function $\Delta(T_1, T_2) = \sigma^2\gamma(T_1, T_2)$, as follows:

$$\mathrm{Cov}\,[I_A, I_{A'}] = \frac{\sigma^2}{4}\sum_{k=0}^{3}\sum_{l=0}^{3}(-1)^k(-1)^l\Delta(T_{1k,}, T_{2l}). \tag{6.4.1}$$

The intervals T_{1k} and T_{2l} have the same meaning as in the one-dimensional case. The proof of Eq. (6.4.1) is entirely similar to that of Eq. (5.3.5): it rests on the algebraic identity [Eq. (7.5.3) for $n = 2$] to which the expectation operation is applied. The covariance of the *local averages* $X_A \equiv I_A/A$ and $X_{A'} \equiv I_{A'}/A'$ is

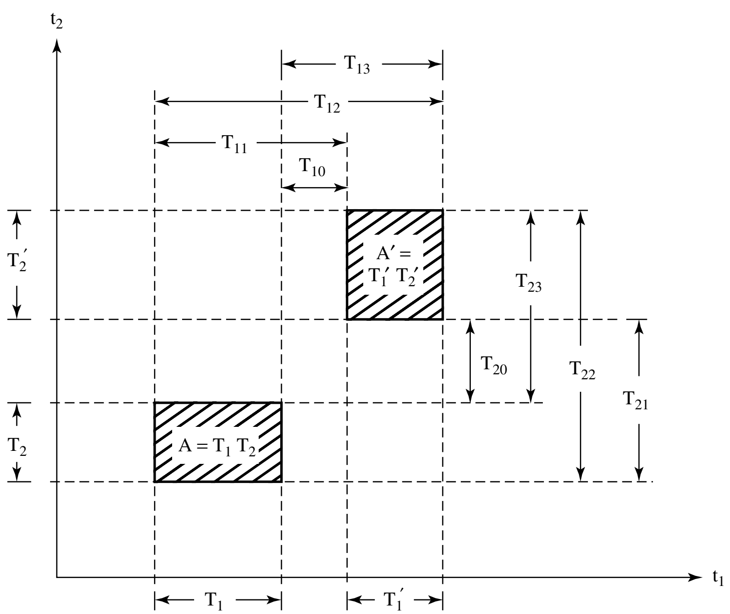


Fig. 6.5 Distances characterizing the relative location of the rectangular areas A and A' in the two-dimensional (orthogonal) parameter space (t_1, t_2).

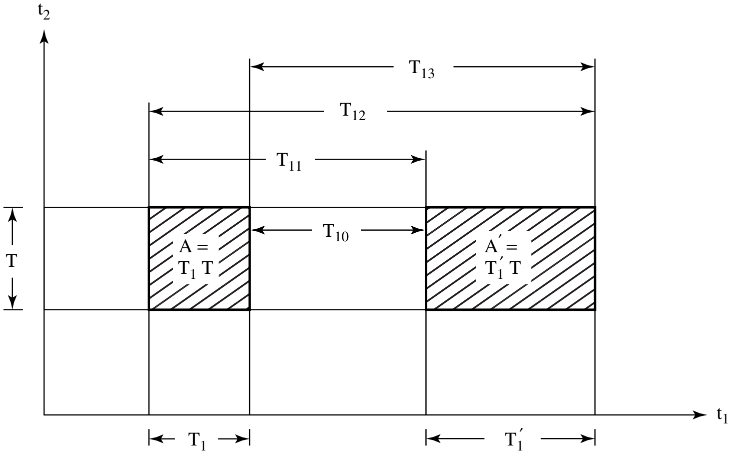


Fig. 6.6 The two averaging areas A and A' share a common dimension.

$$\operatorname{Cov}[X_A, X_{A'}] = \frac{1}{(AA')^2} \operatorname{Cov}[I_A, I_{A'}]. \tag{6.4.2}$$

If the two rectangles are located as shown in Fig. 6.6, namely, if their positions differ only with respect to the coordinate t_2, Eq. (6.4.1) can be simplified to

$$\operatorname{Cov}[I_A, I_{A'}] = \frac{\sigma^2 \Delta(T_1)}{2} \sum_{l=0}^{3} (-1)^l \Delta(T_{2l}|T_1). \tag{6.4.3}$$

The same result can be obtained more directly by applying Eq. (5.3.5) to the derived 1-D process $I_{T_1}(t_2)$.

Stochastic Finite Element Analysis

Repeated use of Eq. (6.4.1) permits evaluation of the covariances of "element properties" (i.e., local spatial averages) for each element of a 2-D finite element mesh, as illustrated in Fig. 6.7. Significant computational savings can be realized if all elements are rectangles of equal size and shape. In case the mesh has $m \cdot n$ finite elements, the covariance matrix has $m^2 \cdot n^2$ elements, but only $m \cdot n$ of these differ, and only $m \cdot n$ different values of the 2-D variance function are needed to evaluate them.

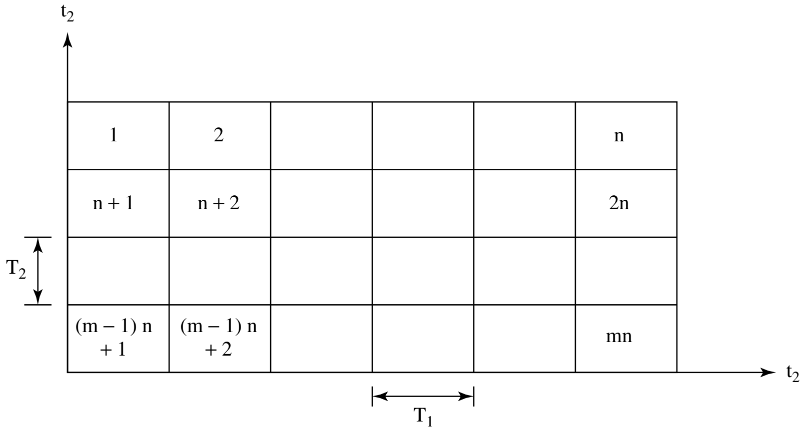


Fig. 6.7 A finite element mesh. In stochastic finite element analysis, key quantities needed are the covariances of the element properties (local averages across each element of the randomly varying property).

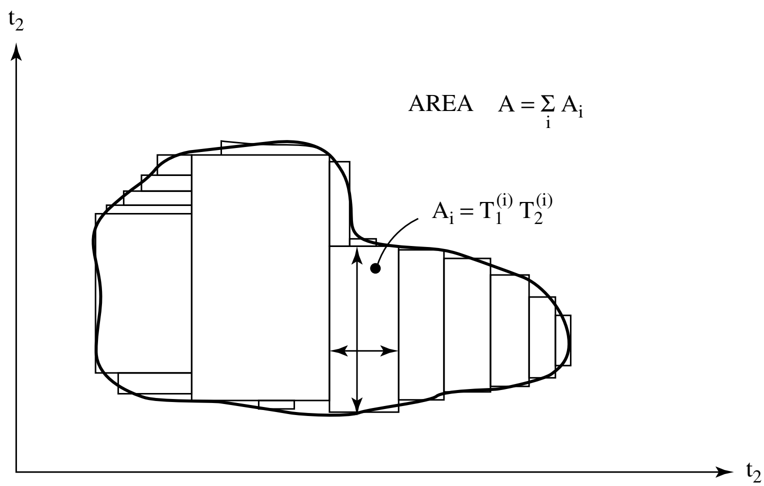


Fig. 6.8 Evaluation of the variance of the integral of a (two-dimensional) random field over an arbitrarily shaped area.

The solution for the covariance of spatial averages over rectangles [Eq. (6.4.1)] serves as a basis for numerical evaluation of: (1) the variances and covariances of spatial averages or local integrals of $X(t_1, t_2)$ over non-rectangular areas; and (2) the variances and covariances of all kinds of "linear transformations" of $X(t_1, t_2)$. Non-rectangular areas may be replaced (to within a prescribed degree of numerical accuracy) by a collection of rectangles, as illustrated in Fig. 6.8. Linear transformations of $X(t_1, t_2)$ may be expressed as follows:

$$Y = \int_A h(t_1, t_2) X(t_1, t_2)\, dt_1\, dt_2 = \sum_{i=1}^{m} h_i X_{A_i}\,, \tag{6.4.4}$$

where $h(t_1, t_2)$ is a deterministic linear operator. The numerical (finite element) procedure requires that the area A be divided into m nonoverlapping rectangles with areas A_i. Eq. (6.4.4) expresses Y as a linear combination of the vector of spatial averages X_{A_i} whose covariances can be found by means of Eq. (6.4.1). The covariance between two such linear combinations, Y_1 and Y_2, may be obtained in the same way. This mode of *stochastic finite element analysis*, not restricted to linear transformations, is applicable to many types of problems involving (locally homogeneous) 2-D random fields.

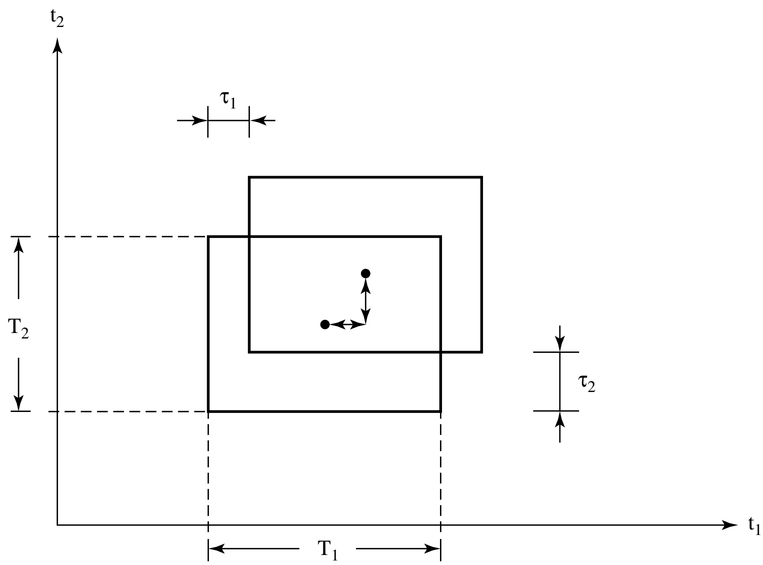


Fig. 6.9 Definition of distances needed to evaluate the covariance function of $X_A(t_1, t_2)$.

Covariance Function of Local Averages

By choosing the intervals T_{1k} and T_{2l} as shown in Fig. 6.9, Eq. (6.4.1) generates an expression for the covariance function of the derived two-dimensional field of local averages $X_A(t_1, t_2)$, where $A = T_1T_2$:

$$\begin{aligned} B_A(\tau_1, \tau_2) &= \frac{\sigma^2}{4T_1^2T_2^2}[\Delta(T_1+\tau_1, T_2+\tau_2) + \Delta(T_1-\tau_1, T_2+\tau_2) \\ &\quad + \Delta(T_1+\tau_1, T_2-\tau_2) + \Delta(T_1-\tau_1, T_2-\tau_2) \\ &\quad - 2\Delta(\tau_1, T_2+\tau_2) - 2\Delta(\tau_1, T_2-\tau_2) - 2\Delta(T_1+\tau_1, \tau_2) \\ &\quad - 2\Delta(T_1-\tau_1, \tau_2) + 4\Delta(\tau_1, \tau_2)]. \end{aligned} \tag{6.4.5}$$

In the limit for T_1, $T_2 \to 0$, Eq. (6.4.5) yields the inverse relationship between the variance function $\gamma(\tau_1, \tau_2)$ and the covariance function $B(\tau_1, \tau_2)$ of a continuous-parameter homogeneous two-dimensional random field:

$$B(\tau_1, \tau_2) = \frac{\sigma^2}{4}\frac{\partial^4}{\partial\tau_1^2\tau_2^2}\Delta(\tau_1, \tau_2). \tag{6.4.6}$$

The derivation requires re-arranging Eq. (6.4.6) in a form similar to the one-dimensional equivalent, Eq. (5.3.24).

Taking $\tau_1 = 0$ in Eq. (6.4.5) leads to the covariance function of the direction-dependent random process $X_A(t_2)$, as well as its slope variance:

$$\mathrm{Var}\,[\dot{X}_A^{(2)}] = \frac{2\sigma^2\gamma(T_1)}{T_2^2}[1 - \rho(T_2|T_1)], \tag{6.4.7}$$

where $\sigma^2\gamma(T_1)$ is the variance of the direction-dependent process $X_{T_1}(t_2)$, while $\rho(\tau_2|T_1)$ is its correlation function evaluated at $\tau_2 = T_2$. Note that Eq. (6.4.7) is a direct extension of Eq. (5.4.11).

Joint Spectral Moment

The most important two-dimensional spectral parameter of the random field $X_A(t_1, t_2)$ is its joint spectral moment

$$\lambda_{11,A} = -\left[\frac{\partial^2 B_A(\tau_1, \tau_2)}{\partial\tau_1\partial\tau_2}\right]_{\tau_1=\tau_2=0}. \tag{6.4.8}$$

We show below that $\lambda_{11,A} = 0$ if the dimensions of the rectangular averaging window differ from zero ($T_1 \neq 0$ and $T_2 \neq 0$), that is, if at least *some* local averaging of $X(t_1, t_2)$ is permitted. $\lambda_{11,A}$ also equals the covariance of the first-order partial derivative $\dot{X}_A^{(1)}$ and $\dot{X}_A^{(2)}$ at a given point (t_1, t_2); and $\lambda_{11,A} = 0$ means that these derivatives are uncorrelated. The proof starts by expressing the series expansion of the two-dimensional covariance function of $X(t_1, t_2)$ near $\tau_1 = \tau_2 = 0$ as follows:

$$B(\tau_1, \tau_2) = \sigma^2 \left[1 + b_1\tau_1 + b_2\tau_2 - d_1\frac{\tau_1^2}{2} - d_2\frac{\tau_2^2}{2} - d_{12}\tau_1\tau_2 + \cdots\right]. \quad (6.4.9)$$

Inserting Eq. (6.4.9) into Eq. (6.1.6) yields the corresponding series for $\gamma(T_1, T_2)$ near $T_1 = T_2 = 0$:

$$\gamma(T_1, T_2) = 1 + \frac{b_1T_1}{3} + \frac{b_2T_2}{3} - \frac{c_1T_1^2}{6} - \frac{c_2T_2^2}{6} - \frac{c_{12}T_1T_2}{9} + \cdots . \quad (6.4.10)$$

A similar analysis in the one-dimensional case [see Eqs. (5.4.2)-(5.4.8)] leads to the conclusion that the term linear in τ vanishes in the corresponding series expansion of $B_T(\tau)$ for $T \neq 0$. Applied to the direction-dependent covariance functions $B_A(\tau_1, 0)$ and $B_A(0, \tau_2)$, this result implies that the terms linear τ_1 and τ_2 in the series expansion of $B_A(\tau_1, \tau_2)$ must also vanish. Furthermore, the constants preceding τ_1^2 and τ_2^2 become the mean square partial derivatives $\mathrm{Var}\,[\dot{X}_A^{(i)}]$, $i = 1, 2$. To see what happens to terms involving $\tau_1\tau_2$, we insert Eq. (6.4.10) into Eq. (6.4.5), accounting $\Delta(T_1, T_2) = T_1^2T_2^2\gamma(T_1, T_2))$, and take the mixed second-order derivative of $B_A(\tau_1, \tau_2)$, as called for on the right side of Eq. (6.4.8). Only the term involving the coefficient d_{12} remains:

$$\frac{\partial^2 B_A(\tau_1, \tau_2)}{\partial\tau_1\partial\tau_2} = \frac{d_{12}\,\sigma^2}{T_1^2\,T_2^2}(\tau_1^2 + 2\tau_1T_1)(\tau_2^2 + 2\tau_2T_2) + \cdots . \quad (6.4.11)$$

If $T_1 \neq 0$ and $T_2 \neq 0$, taking $\tau_1 = \tau_2 = 0$ in Eq. (6.4.11) yields, in combination with Eq. (6.4.8), the desired result:

$$\lambda_{11,A} = 0. \quad (6.4.12)$$

In conclusion, for $T_1 \neq 0$ and $T_2 \neq 0$ the series expansion of the 2-D covariance function of $X_A(t_1, t_2)$ has the following form near $\tau_1 = \tau_2 = 0$:

$$B_A(\tau_1, \tau_2) = \sigma_A^2 - \frac{1}{2}\sum_{i=1}^{n}\tau_i^2\ \mathrm{Var}\,[\dot{X}_A^{(i)}] + \cdots . \quad (6.4.13)$$

The linear terms and the term involving $\tau_1\tau_2$ vanish. For $T_i \to 0$ and $\tau_i \to 0$ ($i = 1, 2$), Eq. (6.4.11) produces the indeterminate result (0/0). In

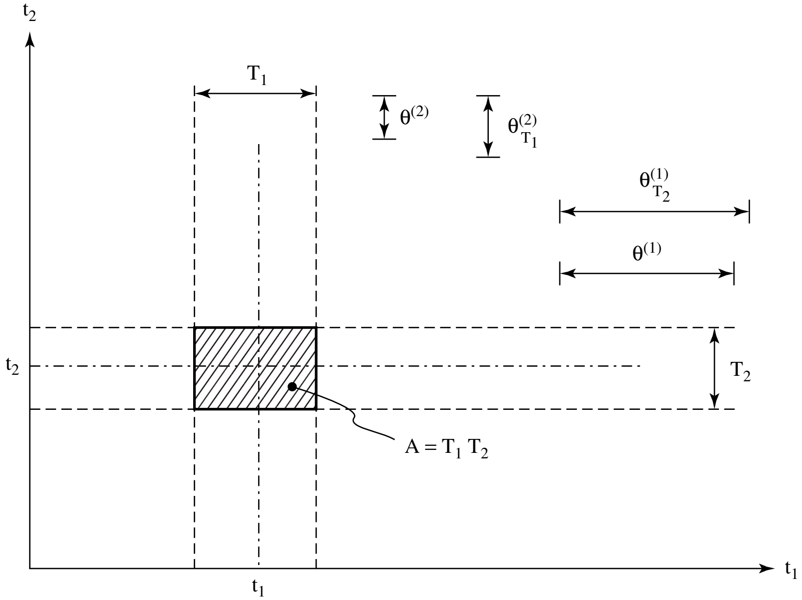


Fig. 6.10 Scales of fluctuation associated with the direction-dependent process $X_{T_1}(t_2)$ and $X_{T_2}(t_1)$.

this case, there is no local averaging, and the joint second-order spectral moment, provided it exists, is given $\lambda_{11} = \lambda_{11,A=0}$.

6.5 Statistics of Level Excursions and Extremes

The theory developed in Chap. 4 for two-dimensional mean square differentiable random processes can now be confidently applied to the random fields $I_A(t_1, t_2)$ and $X_A(t_1, t_2)$ obtained by local aggregation or averaging of the homogeneous random field $X(t_1, t_2)$. These derived random fields will satisfy the conditions for existence of the mean square partial derivatives and (under mild conditions imposed by the Central Limit Theorem) tend to become Gaussian as the size of averaging domain increases.

In this section, interest focuses on excursions of the field $X_A(t_1, t_2)$ above a threshold level b. We first derive some relevant statistics for the direction-dependent one-dimensional processes $X_A(t_1)$ and $X_A(t_2)$, generated by fixing one of the coordinates (see Fig. 6.10).

Level Crossings of Direction-Dependent Local Average Processes

In the same way as the random process $X(t)$ generates the local average process $X_T(t)$, the process $X_{T_2}(t_1)$ gives rise to

$$X_A(t_1) = \frac{1}{T_1} \int_{t_1 - T_1/2}^{t_1 + T_1/2} X_{T_2}(t_1)\, dt_1, \tag{6.5.1}$$

a single observation of which is a local average of $X(t_1, t_2)$ over the rectangular area $A = T_1 T_2$. The variance of $X_A(t_1)$ is

$$\mathrm{Var}\,[X_A^{(1)}] = \mathrm{Var}\,[X_A] = \sigma^2 \gamma(T_2)\gamma(T_1|T_2) = \sigma^2 \gamma(T_1, T_2), \tag{6.5.2}$$

and, by direct extension of Eq. (5.4.11), its mean square derivative is

$$\begin{aligned} \mathrm{Var}\,[\dot{X}_A^{(1)}] &= \frac{2}{T_1^2} \sigma^2 \gamma(T_2)[1 - \rho(T_1|T_2)] \\ &= 2\ \mathrm{Var}\,[X_A] \left\{ \frac{1 - \rho(T_1|T_2)}{\Delta(T_1|T_2)} \right\}. \end{aligned} \tag{6.5.3}$$

If the distances T_1 and T_2 are much larger than their respective scales of fluctuation, $\theta^{(1)}$ and $\theta^{(2)}$, that is, if $T_i \gg \theta^{(i)}$ for $i = 1, 2$, the variances of X_A and the partial derivatives $\dot{X}_A^{(i)}$ $(i = 1, 2)$ become:

$$\mathrm{Var}\,[X_A] = \frac{\sigma^2 \alpha}{A}, \qquad \text{and} \tag{6.5.4}$$

$$\mathrm{Var}\,[\dot{X}_A^{(i)}] = \frac{2\ \mathrm{Var}\,[X_A]}{c_\alpha T_i\, \theta^{(i)}}, \quad T_i \gg \theta^{(i)},\ i = 1, 2. \tag{6.5.5}$$

Based on the theory presented in Sec. 4.4, we can now evaluate the mean rate of b-upcrossings and the mean lengths of excursion above a given level b by the processes $X_A(t_1)$ and $X_A(t_2)$. In particular, the mean rate of b-upcrossings by the process $X_A(t_i)$, $i = 1, 2$, is

$$\nu_{b,A}^{(i)} = f_A(b) \left(\frac{2}{\pi}\ \mathrm{Var}\,[\dot{X}_A^{(i)}] \right)^{1/2}, \quad i = 1, 2, \tag{6.5.6}$$

and the mean length (measured along the t_i axis) of excursions above b by the process $X_A(t_i)$, $i = 1, 2$, is

$$E[\mathscr{T}_{b,A}^{(i)}] = (2\pi\ \mathrm{Var}\,[\dot{X}_A^{(i)}])^{1/2} \frac{F_A^c(b)}{f_A(b)}, \quad i = 1, 2, \tag{6.5.7}$$

where $f_A(\cdot)$ and $F_A^c(\cdot)$ denote, respectively, the p.d.f. and the complementary c.d.f. of the local average X_A.

Mean Size of Regions of Excursion above High Levels

For high threshold levels, one pictures the two-dimensional pattern of excursions (regions where $X_A \geq b$) as a collection of isolated patches. As the level b rises, the patches become more isolated and their sizes shrink. The one-dimensional excursion lengths may be crudely interpreted as the dimensions, parallel to the respective coordinate axes, of the area $\mathscr{A}_b \equiv \mathscr{A}_{b,A}$ of each isolated region of excursion. Since the first-order partial derivatives $\dot{X}_A^{(1)}$ and $\dot{X}_A^{(2)}$ at a given point are uncorrelated ($\lambda_{11,A} = 0$), the mean size of isolated excursion regions above a high level b can be expressed approximately as follows:

$$E[\mathscr{A}_{b,A}] \simeq E[\mathscr{T}_{b,A}^{(1)}]E[\mathscr{T}_{b,A}^{(2)}] = \left(\frac{F_A^c(b)}{f_A(b)}\right)^2 \frac{2\pi}{\sigma_{\dot{X}_A}^2}, \tag{6.5.8}$$

where $\sigma_{\dot{X}_A}^2$ is the geometric mean of the mean-square partial derivatives:

$$\sigma_{\dot{X}_A}^2 = (\mathrm{Var}\,[\dot{X}_A^{(1)}]\mathrm{Var}\,[\dot{X}_A^{(2)}])^{1/2}, \tag{6.5.9}$$

which can also be expressed as in Eq. (6.5.3).

Similar results, paralleling Eq. (4.5.8), can be obtained for the mean area of excursion above b for the "envelope" field $R_A(t_1, t_2)$ (associated with $X_A(t_1, t_2)$):

$$E[\mathscr{A}_{b,R_A}] \simeq \left(\frac{F_{R_A}^c(b)}{f_{R_A}(b)}\right)^2 \frac{2\pi}{\sigma_{\dot{X}_A}|\mathbf{\Delta}_A|^{1/2}}, \tag{6.5.10}$$

in which the determinant of $\mathbf{\Delta}_A$ depends on the bandwidth factors $\delta_A^{(1)}$ and $\delta_A^{(2)}$ of the direction-dependent processes $X_A(t_i)$, $i = 1, 2$.

Mean Frequency of Local Maxima above High Levels

In the same way as the pattern of excursions along any line parallel to a coordinate axis approaches a (one-dimensional) Poisson process, the pattern of high-level excursions in the plane must tend toward a 2-D Poisson process characterized by $\mu_{b,A}$, the mean number of excursions per unit area. From

Eq. (4.6.2), replacing X by X_A and taking $|\mathbf{V}_A| = 1$, we obtain

$$\mu_{b,A} \simeq \frac{1}{2\pi}[f_{A}(b)]^2[F^c_{A}(b)]^{-1}\sigma^2_{\dot{X}_A}. \tag{6.5.11}$$

The mean rate of excursions above a high level b by the corresponding envelope process $R_A \equiv R_A(t_1, t_2)$ follows from Eq. (4.6.4):

$$\mu_{b,R_A} = \frac{1}{2\pi}[f_{R_A}(b)]^2[F^c_{R_A}(b)]^{-1}\sigma_{\dot{X}_A}|\mathbf{\Delta}_A|^{1/2}. \tag{6.5.12}$$

These estimates for the 2-D excursion statistics can now be used to approximate the probability distribution of the maximum value of $X_A(t_1, t_2)$ within an area of given size, in the manner outlined in Sec. 4.6. The general solution for the multi-dimensional case is presented in Sec. 7.7.

6.6 Invariants for 2-D Homogeneous Random Fields

Invariance under Linear Transformation

Assume that a homogeneous random field $X(t_1, t_2)$ is linearly transformed into a new homogeneous random field $Y(t_1, t_2)$. The linear system is characterized by the impulse response function $h(t_1, t_2)$ or the frequency response function $H(\omega_1, \omega_2)$. These two system functions form a Fourier transform pair. In particular, we have

$$H(0,0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1, t_2)\, dt_1\, dt_2. \tag{6.6.1}$$

Also,

$$S_Y(\omega_1, \omega_2) = |H(\omega_1, \omega_2)|^2 S_X(\omega_1, \omega_2). \tag{6.6.2}$$

The expressions for the mean and the variance of $Y(t_1, t_2)$ are, respectively,

$$m_{Y} = H(0,0)\, m_{X}, \tag{6.6.3}$$

and

$$\sigma^2_{Y} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |H(\omega_1, \omega_2)|^2 S_X(\omega_1, \omega_2)\, d\omega_1\, d\omega_2. \tag{6.6.4}$$

By taking $\omega_1 = \omega_2 = 0$ in Eq. (6.6.2), the respective 2-D correlation parameters α_Y and α_X can be related as follows:

$$\alpha_{Y}\sigma^2_{Y} = |H(0,0)|^2\alpha_{X}\sigma^2_{X} = \frac{m^2_{Y}}{m^2_{X}}\alpha_{X}\sigma^2_{X}, \tag{6.6.5}$$

assuming $m_X \neq 0$. The sought-after invariance relation follows:

$$\alpha_Y \left(\frac{\sigma_Y}{m_Y}\right)^2 = \alpha_X \left(\frac{\sigma_X}{m_X}\right)^2. \tag{6.6.6}$$

In case the transformation keeps the mean unchanged (so that $H(0,0) = 1$), or when $m_X = 0$, the invariance relation becomes:

$$\alpha_Y \sigma_Y^2 = \alpha_X \sigma_X^2. \tag{6.6.7}$$

Eqs. (6.6.6) and (6.6.7) express a principle of "conservation of (density of) uncertainty", analogous in form to the 1-D case presented in Sec. 5.6.

Invariance Under Local Spatial Averaging

A special case is $Y \equiv X_A$, when the system function $h(t_1, t_2)$ is a rectangular box covering an area of size $A = T_1 T_2$ and having unit volume. It follows from Eq. (6.6.7) that the correlation parameter of the local average process $X_A(t_1, t_2)$ equals

$$\alpha_A = \frac{\alpha \sigma^2}{\sigma_A^2} = \frac{\alpha}{\gamma(T_1, T_2)}. \tag{6.6.8}$$

If the dimensions T_1 and T_2 vanish, $\alpha_A \to \alpha$, whereas for large averaging areas, when $T_i \gg \theta^{(i)}$, $i = 1$, 2, the correlation parameter becomes $\alpha_A \to T_1 T_2 = A$. The implication is that repeated local averaging causes the direction-dependent scales of fluctuation $\theta_A^{(i)}$, $i = 1$, 2, to increase gradually; and it also leads to increasing values for the spectral bandwidth measures $\delta_A^{(i)}$ and $\varepsilon_A^{(i)}$, $i = 1$, 2, which approach their respective unit upper bounds. One concludes that local averaging produces these quantifiable effects: (1) reduction of the "point variance"; (2) proportional increase in the correlation measures; and (3) increase in the degree of disorder, as measured by the spectral bandwidth factors.

2-D White Noise

The case of two-dimensional white noise is of special interest. Its spectral density function is the same at all points in the 2-D frequency domain:

$$S(\omega_1, \omega_2) = S_0. \tag{6.6.9}$$

To analyze its behavior, we view it as the limit, for $\omega_1^* \to +\infty$ and $\omega_2^* \to \infty$, of a random field with a "band-limited" 2-D spectral density function,

$$S(\omega_1, \omega_2) = S_0, \quad -\omega_1^* \le \omega_1 \le \omega_1^*,\ -\omega_2^* \le \omega_2 \le \omega_2^*, \tag{6.6.10}$$

whose variance is found by integrating $S(\omega_1, \omega_2)$ over all frequencies:

$$\sigma^2 = 4\, S_0\, \omega_1^* \omega_2^*. \tag{6.6.11}$$

The normalized or unit-volume s.d.f. is

$$s(\omega_1, \omega_2) = \frac{S(\omega_1, \omega_2)}{\sigma^2} = \frac{1}{4\, \omega_1^* \omega_2^*}. \tag{6.6.12}$$

From Eq. (6.1.10), the "correlation area" is

$$\alpha = 4\pi^2 s(0,0) = \frac{\pi^2}{\omega_1^* \omega_2^*}. \tag{6.6.13}$$

The ideal-white-noise characteristics emerge when ω_1^* and ω_2^* are both allowed to increase without limit,

$$\sigma^2 \to \infty, \quad \alpha \to 0, \tag{6.6.14}$$

while keeping the product $\alpha\sigma^2$ constant:

$$\alpha\sigma^2 = 4\pi^2 S_0 = \pi^2 G_0, \tag{6.6.15}$$

where $G_0 = 4S_0$ is the spectral density defined for positive frequencies only. Combining Eqs. (6.6.7) and (6.6.15), note that if a field $Y(t_1, t_2)$ represents the homogeneous response of a linear system to zero-mean white noise excitation with spectral intensity S_0, then

$$\alpha_Y \sigma_Y^2 = 4\,\pi^2 S_0. \tag{6.6.16}$$

The 2-D correlation parameter of the response field is easily found from the relation $\alpha_Y = 4\pi^2 S_0/\sigma_Y^2$ once the response variance σ_Y^2 is known.

6.7 Space-Time Processes: Frequency-Dependent Scale of Fluctuation

If the two-dimensional homogeneous random field under study is a *space-time process* $X(u, t)$, one may choose to describe the time variation in the frequency domain without similarly transforming the spatial coordinate u. The nature of the temporal variation is often very different from that of the variation in space, especially in applications (e.g., earthquakes, wind, sea waves) involving waves in random media. The basic "mixed transform" relations are presented in Sec. 3.8. Throughout this section and the next, the correlation structure of the space-time process is assumed to be *quadrant symmetric* (q.s.). This facilitates physical interpretation and practical application of the results, and is easily relaxed when the need arises.

Background

In Sec. 3.8 the space-time covariance function $B(\nu,\tau)$ is introduced as the covariance between two observations at points (u,t) and (u',t') separated by the distance $\nu = u - u'$ and the time lag $\tau = t - t'$. By one-step Fourier transformation [Eq. (3.8.5)] the time lag τ is converted into the frequency ω, thus yielding the space-time cross-spectral density function $C(\nu,\omega)$. This function reduces to the familiar point s.d.f. if the locations u and u' coincide, or $\nu = 0$, namely $C(0,\omega) \equiv S(\omega)$. By "normalizing" $C(\nu,\omega)$ with respect to its value at $\nu = 0$, we obtain the *frequency-dependent spatial correlation function*:

$$\rho_\omega(\nu) = \frac{C(\nu,\omega)}{S(\omega)}, \qquad \text{(from 3.8.8)}$$

where ω is made a subscript to emphasize the dependence on the separation distance ν. The function $\rho_\omega(\nu)$ quantifies the degree of spatial correlation associated with individual sinusoidal components of the space-time process $X(u,t)$. The "composite" spatial correlation function $\rho(\nu)$ may be expressed as follows:

$$\rho(\nu) = \int_{-\infty}^{+\infty} \rho_\omega(\nu) s(\omega)\, d\omega \overset{\substack{\text{q.s.}\\ \downarrow}}{=} \int_0^\infty \rho_\omega(\nu) g(\omega)\, d\omega, \qquad \text{(from 3.8.10)}$$

where $g(\omega)$ is the one-sided unit-area s.d.f. (representing random variation with time at a fixed location). The interpretation is that $\rho(\nu)$ is a weighted combination of the frequency-dependent correlation functions $\rho_\omega(\nu)$, the weights, $g(\omega)\, d\omega$, being the fractional contributions to the variance σ^2.

If the $\nu \to k$ transformation is made in addition to the $\tau \to \omega$ transformation, $C(\nu,\omega)$ generates the two-dimensional s.d.f. $S(k,\omega)$, and $\rho_\omega(\nu)$ the frequency-dependent wave number spectrum $s_\omega(k)$. We may write:

$$s_\omega(k) = \frac{S(k,\omega)}{S(\omega)} = \frac{s(k,\omega)}{s(\omega)}, \qquad \text{(from 3.8.15)}$$

where $s(k,\omega)$ and $s(\omega)$ are normalized with respect to the variance. Also,

$$s(k) = \int_{-\infty}^{+\infty} s_\omega(k) s(\omega)\, d\omega \overset{\substack{\text{q.s.}\\ \downarrow}}{=} \int_0^\infty s_\omega(k) g(\omega)\, d\omega, \qquad \text{(from 3.8.17)}$$

where $s(k)$ is the unit-area wave number spectrum (characterizing spatial variation at a point in time).

Frequency-Dependent Spatial Scale of Fluctuation

A two-dimensional space-time process $X(u,t)$, observed at a given instant t_0, becomes a random spatial pattern $X(u,t_0)$ characterized (like any stationary random function) by its mean m, variance σ^2, and scale of fluctuation θ^u. Provided it exists, θ^u may be expressed in terms of the correlation function $\rho(\nu)$, where $\nu = u - u'$, as follows:

$$\theta^u = \int_{-\infty}^{+\infty} \rho(\nu)\, d\nu = 2\int_0^\infty \rho(\nu)\, d\nu. \tag{6.7.1}$$

Alternately, in terms of the unit-area wave number spectrum $s(k)$, where k is the wave number (or spatial circular frequency), we have

$$\theta^u = 2\pi s(k)|_{k=0}. \tag{6.7.2}$$

The *frequency-dependent spatial scale of fluctuation* can now be defined, first in terms of $\rho_\omega(\nu)$:

$$\theta^u_\omega = 2\int_0^\infty \rho_\omega(\nu)\, d\nu. \tag{6.7.3}$$

The basic relationship between the scales θ^u and θ^u_ω is obtained by combining Eqs. (6.7.1), (6.7.3), and (3.8.10):

$$\theta^u = 2\int_0^\infty d\nu \int_0^\infty \rho_\omega(\nu) g(\omega)\, d\omega = \int_0^\infty \theta^u_\omega\, g(\omega)\, d\omega. \tag{6.7.4}$$

In words, the spatial scale of fluctuation θ^u can be expressed as a weighted combination of the frequency-dependent scales θ^u_ω, the weighting function being the unit-area s.d.f. $g(\omega)$. Also, from Eq. (3.8.19)

$$s_\omega(k) = \frac{1}{\pi}\int_0^\infty \rho_\omega(\nu)\cos \nu k\, d\nu. \tag{6.7.5}$$

Taking $k = 0$ and introducing the definition of θ^u_ω [Eq. (6.7.3)], we obtain

$$\theta^u_\omega = 2\pi s_\omega(k)|_{k=0}, \tag{6.7.6}$$

and it is easy to confirm:

$$\theta^u = 2\pi \int_0^\infty s_\omega(0) g(\omega)\, d\omega = \int_0^\infty \theta_\omega^u\, g(\omega)\, d\omega. \tag{6.7.7}$$

Physical Interpretation

The random process $X(u,t)$ may be expressed as a sum of J independent sinusoidal components $X_j(u,t)$ (assuming zero mean, for convenience):

$$X(u,t) = \sum_{j=1}^{J} X_j(u,t) = \sum_{j=1}^{J} C_j(u) \cos[\omega_j t - \Phi_j(u)]. \tag{6.7.8}$$

The nth component in this expansion is a sinusoid with frequency ω_j, random amplitude $C_j(u)$, and random phase angle $\Phi_j(u)$. At a given instant $t = t_0$, Eq. (6.7.8) expresses the pattern of spatial variation, $X(u,t_0)$, as a sum of independent random contributions $X_j(u,t_0)$. The variance of the nth sinusoidal component is

$$E[X_j^2] = \frac{1}{2} E[C_j^2(u)] = G(\omega_j)\Delta\omega = \sigma^2 g(\omega_j)\Delta\omega. \tag{6.7.9}$$

The scale of fluctuation of the component $X_j(u,t_0)$ is $\theta_{\omega_j}^u$, while the composite scale of fluctuation is θ^u. In general, the scale of fluctuation of a sum of independent random functions may be expressed as a linear combination of the component scales of fluctuation. The coefficients in the linear relationship are the fractional contributions each component makes to the total variance. In the case at hand, the total variance is σ^2 and the fractional contribution of the jth component is $g(\omega_j)\Delta\omega$, and the composite scale θ^u is related to the component scales $\theta_{\omega_j}^u$ as follows:

$$\theta^u = \sum_{j=1}^{J} \theta_{\omega_j}^u g(\omega_j)\Delta\omega. \tag{6.7.10}$$

This relationship is equivalent to Eq. (6.7.4) and converges to it when the limits $J \to \infty$ and $\Delta\omega \to 0$ are taken.

Behavior of the Frequency-Dependent Spatial Scale

How exactly does the scale θ_ω^u depend on the frequency ω? Intuitively, since θ_ω^u characterizes components of spatial variation associated with frequency ω, one expects $\theta_\omega^u \propto \omega^{-1}$. However, for $\omega \to 0$, this inverse proportionality law would imply $\theta^u_\omega \to \infty$, which is inconsistent with the formula [Eq. (6.7.4)] linking θ^u_ω and θ^u.

Combining Eqs. (6.7.2), (6.7.3) and (3.8.20) yields the following expression for the ratio of spatial scales:

$$\frac{\theta^u_\omega}{\theta^u} = \left.\frac{s(k,\omega)}{s(\omega)s(k)}\right|_{k=0}, \tag{6.7.11}$$

where $s(k,\omega)$, $s(\omega)$, and $s(k)$ are all normalized spectra. This result permits evaluation of θ^u_ω directly in terms of $s(k,\omega)$. (Recall that the spectra $s(\omega)$ and $s(k)$ can be obtained by partial integration of $s(k,\omega)$.) Examples of the use of Eq. (6.7.11) are presented at the end of this section.

Behavior near the Frequency Origin

To see what happens at $\omega = 0$, recall that the two-dimensional spectrum $s(k,\omega)$ is related to the 2-D correlation measure α as follows:

$$\alpha = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(\nu,\tau)\,d\nu\,d\tau = (2\pi)^2 s(k,\omega)|_{k=\omega=0}. \tag{6.7.12}$$

Also, the scale of temporal fluctuation of the random process $X(u_0,t)$ is

$$\theta^t = 2\pi s(\omega)|_{\omega=0}. \tag{6.7.13}$$

Hence for $k = \omega = 0$, Eq. (6.7.11) yields

$$\frac{\theta^u_0}{\theta^u} = \left.\frac{s(k,\omega)}{s(k)s(\omega)}\right|_{k=\omega=0} = \frac{\alpha}{\theta^u\,\theta^t} = c_\alpha, \tag{6.7.14}$$

where c_α is the dimensionless 2-D correlation parameter of the space-time process $X(u,t)$. This establishes the following intriguing connection:

$$\lim_{\omega\to 0}\theta^u_\omega = c_\alpha\theta^u \equiv \lim_{T\to\infty}\theta^u_T, \tag{6.7.15}$$

where θ^u_T is the scale of *spatial* fluctuation of the *temporal* average, over the time interval T, of the space-time process $X(u,t)$.

Approximate Functional Form

Further relevant information about the behavior of θ^u_ω at low frequencies is obtainable from the series expansions of $s(k,\omega)$ and $s(\omega)$. From

Eq. (6.1.20), *mutatis mutandis*, we obtain

$$s(k,\omega)|_{k=0} = \frac{\alpha}{(2\pi)^2} - \frac{1}{2(2\pi)^2}\alpha_2^t\omega^2 + \cdots\,, \tag{6.7.16}$$

where, from Eq. (6.1.18),

$$\alpha_2^t = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tau^2\rho(\nu,\tau)\,d\nu\,d\tau. \tag{6.7.17}$$

Likewise, based on Eq. (5.2.12), the series expansion of $s(\omega)$ is

$$s(\omega) = \frac{\theta^t}{2\pi} - \frac{1}{4\pi}\theta_2^t\omega^2 + \cdots\,, \tag{6.7.18}$$

where

$$\theta_2^t = \int_{-\infty}^{+\infty} \tau^2\rho(\tau)\,d\tau. \tag{6.7.19}$$

Combining Eqs. (6.7.11), (6.7.16) and (6.7.18) enables one to express the first few terms of the series expansion for the frequency-dependent spatial scale as follows:

$$\theta_\omega^u = \theta_{\omega=0}^u\left\{1 - \frac{1}{2}\left(\frac{\omega}{\Omega_1^u}\right)^2 + \cdots\right\}, \tag{6.7.20}$$

in which $\theta_{\omega=0}^u = c_\alpha\theta^u$ [see Eq. (6.7.14)] and

$$\Omega_1^u = \left[\frac{\alpha_2^t}{\alpha} - \frac{\theta_2^t}{\theta^t}\right]^{-1/2}, \tag{6.7.21}$$

is a parameter having the dimensions of frequency.

The functional form expressed below, and depicted in Fig. 6.12, for the frequency-dependent spatial scale θ_ω^u is consistent with the information just given, and in particular with the (first few terms in the) above series expansion:

$$\theta_\omega^u = c_\alpha\theta^u\left[1 + \left(\frac{\omega}{\Omega_1^u}\right)^2\right]^{-1/2}, \tag{6.7.22}$$

in which Ω_1^u, given by Eq. (6.7.21), is the "corner frequency" that marks the transition between two modes of functional dependence ($\theta_\omega^u \simeq c_\alpha\theta^u$ and $\theta_\omega^u \propto \omega^{-1}$) of the frequency-dependent scale θ_ω^u. Figure 6.11 indicates that

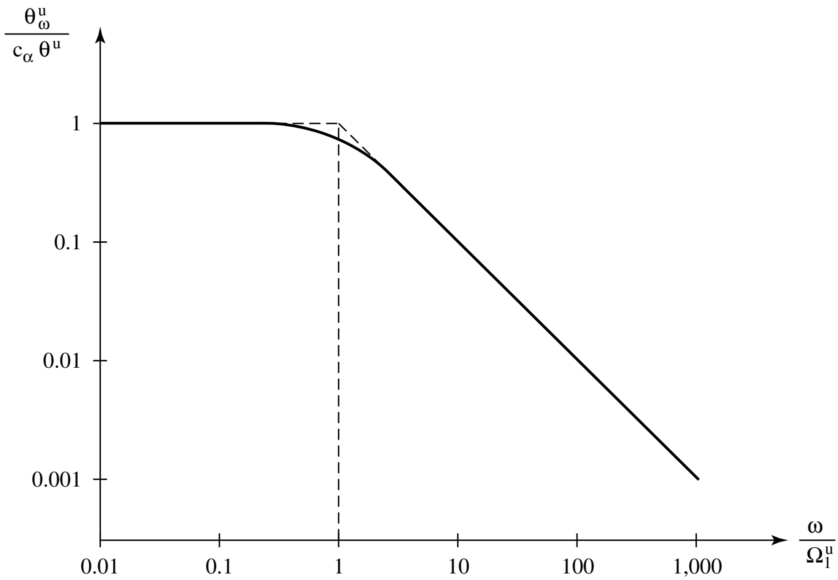


Fig. 6.11 Frequency-dependent scale of fluctuation θ^u_ω of the space-time process $X(u,t)$.

the function is relatively flat until ω reaches Ω^u_1. At higher frequencies, the scale θ^u_ω tends to vary in inverse proportion to ω.

It should be noted that this type of analysis is meaningful only when the 2-D correlation structure is *not* separable. As shown in Case 3 below, the corner frequency Ω^u_1, given by Eq. (6.7.21), becomes infinite when the correlation structure is separable, and in this case $\theta^u_\omega = \theta^u$ for all $\omega \geq 0$. If the correlation structure is not separable, Ω^u_1 is expected to be a low multiple of $1/\theta^t$; a default value, $\Omega^u_1 = 2/\theta^t$, happens to be exact for Case 1 presented below.

Examples of 2-D Space-Time Correlation Structures

Case 1. Exponential Correlation Structure

Consider the exponential correlation function

$$\rho(\nu, \tau) = \exp\{-[(a\nu)^2 + (b\tau)^2]^{1/2}\}, \tag{6.7.23}$$

which corresponds to the 2-D wave number-frequency (k–ω) spectrum

$$s(k,\omega) = \frac{1}{2\,\pi\,a\,b}\left[1 + \left(\frac{k}{a}\right)^2 + \left(\frac{\omega}{b}\right)^2\right]^{-3/2}. \tag{6.7.24}$$

Integrating over ω yields

$$s(k) = \frac{1}{\pi a}\left[1 + \left(\frac{k}{a}\right)^2\right]^{-1}. \tag{6.7.25}$$

Likewise,

$$s(\omega) = \frac{1}{\pi b}\left[1 + \left(\frac{\omega}{b}\right)^2\right]^{-1} \equiv \frac{1}{2}g(\omega). \tag{6.7.26}$$

The scales of spatial and temporal fluctuation are, respectively,

$$\theta^u = \frac{2}{a}, \quad \theta^t = \frac{2}{b}, \tag{6.7.27}$$

and the 2-D correlation parameter is

$$\alpha = \frac{2\pi}{a\,b}. \tag{6.7.28}$$

Hence,

$$c_\alpha = \frac{\alpha}{\theta^u\theta^t} = \frac{\pi}{2} \simeq 1.57. \tag{6.7.29}$$

The expression for the frequency-dependent scale of fluctuation is obtained by inserting Eqs. (6.7.24) and (6.7.26) into Eq. (6.7.11). The result is

$$\begin{aligned}\theta^u_\omega = \left.\frac{2\pi s(k,\omega)}{s(\omega)}\right|_{k=0} &= \frac{\pi}{a}\left[1 + \left(\frac{\omega}{b}\right)^2\right]^{-1/2} \\ &= c_\alpha\theta^u\left[1 + \left(\frac{\omega}{b}\right)^2\right]^{-1/2},\end{aligned} \tag{6.7.30}$$

agreeing exactly with Eq. (6.7.22), with $\Omega^u_1 = b$.

We will now confirm that the result $\Omega^u_1 = b$ can also be obtained directly from Eq. (6.7.21). Since the correlation structure of $X(u,t)$ is ellipsoidal, the parameters needed to evaluate Ω^u_1 can be expressed in terms of the moments of the radial correlation $\rho^R(\tau) = \rho(0,\tau)$ with scale of fluctuation θ^t. In terms of the moments

$$\theta^R_k = 2\int_0^\infty \tau^k\rho^R(\tau)\,d\tau, \tag{6.7.31}$$

the expression Eq. (6.7.21), for the spectral parameter Ω_1^u can be restated as follows:

$$\Omega_1^u = \left[\frac{\theta_3^R}{2\theta_1^R} - \frac{\theta_2^R}{\theta_0^R}\right]^{-1/2}. \tag{6.7.32}$$

In the case at hand, $\rho^R(\tau) = \rho(0, \tau) = \exp\{-b\tau\}$, so that

$$\theta_k^R = 2\int_0^\infty \tau^k \exp\{-b\tau\}\, d\tau = \frac{2k!}{b^{k+1}}, \quad k = 0, 1, 2, \ldots. \tag{6.7.33}$$

Combining Eqs. (6.7.32) and (6.7.33) yields the sought-after confirmation:

$$\Omega_1^u = \left[\frac{6/b^4}{2/b^2} - \frac{2/b^3}{1/b}\right]^{-1/2} = b. \tag{6.7.34}$$

Case 2. Autoregressive Correlation Structure

Consider again the family of autoregressive (Markovian) correlation models examined in Sec. 6.2. To make it fit the notation used to describe 2-D space-time processes, it suffices to replace (ω_1/b_1) and (ω_2/b_2) by (k/a) and (ω/b), respectively, in Eq. (6.2.16):

$$S(k, \omega) = S_0 \left[1 + \left(\frac{k}{a}\right)^2 + \left(\frac{\omega}{b}\right)^2\right]^{-m}, \quad m > 1, \tag{6.7.35}$$

where the case $m = 3/2$ corresponds to the 2-D exponential correlation structure just considered. It is easy to show that the frequency-dependent spatial scale θ_ω^u has exactly the same functional form as in Eq. (6.7.30), so that $\Omega_1^u = b$ for this entire family of models.

Case 3. Separable Correlation Structure

When $\rho(\nu, \tau) = \rho(\nu)\rho(\tau)$, it follows from the Wiener-Khinchine relations that the cross-spectral density function $C(\nu, \omega)$ is also separable:

$$C(\nu, \omega) = S(\omega)\rho(\nu). \tag{6.7.36}$$

Hence, based on Eq. (3.8.8),

$$\rho_\omega(\nu) = \rho(\nu), \tag{6.7.37}$$

and the scale θ_ω^u does not depend on frequency, $\theta_\omega^u = \theta^u$. Also $c_\alpha = \alpha/(\theta^u\theta^t) = 1$, and the expression, Eq. (6.7.21), for the corner frequency

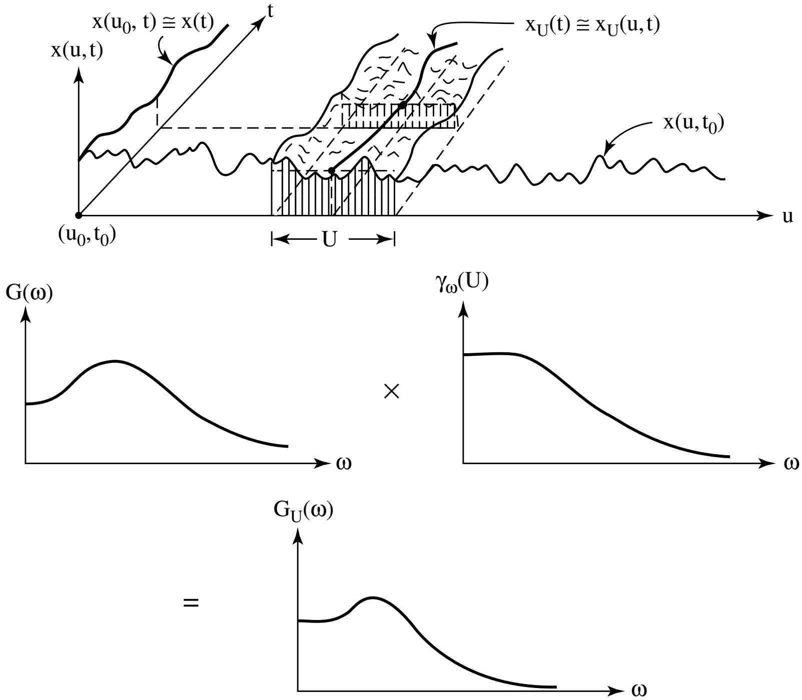


Fig. 6.12 The frequency-dependent variance function $\gamma_\omega(U)$ serves to connect the spectral density function of $X(t)$ and $X_U(t)$. $G_U(\omega)$ equals the product of $G(\omega)$ and $\gamma_\omega(U)$.

yields:

$$\Omega_1^u = \left[\frac{\theta_2^t\,\theta^u}{\theta^t\,\theta^u} - \frac{\theta_2^t}{\theta^t}\right]^{-1/2} = \infty. \tag{6.7.38}$$

Separability implies the lack of any linkage between the patterns of the random variation in time and space.

6.8 Space-Time Processes: Frequency-Dependent Variance Function

In problems involving time-dependent random fields, the analyst may seek information about the spectral content of spatial averages (or integrals) of space-time processes. This section is devoted to the development of a theoretical framework for formulating and solving such problems. The approach is a direct extension of the methodology developed in Chap. 5. The spatial variance function, now permitted to depend on frequency, describes the effect of spatial averaging on components of the "point" variance associated with different narrow bands of frequency. The basic concepts and relations, introduced here for a 2-D space-time process $X(u,t)$, are subsequently extended (in Chap. 7) to time-varying random fields $X(\mathbf{u},t)$, where $\mathbf{u} = (u_1, u_2)$ or $\mathbf{u} = (u_1, u_2, u_3)$. Consider the local spatial average of $X(u,t)$ over a fixed distance U centered at u (see Fig. 6.12):

$$X_U(u,t) = \frac{1}{U}\int_{u-U/2}^{u+U/2} X(u',t)\, du'. \tag{6.8.1}$$

The derived space-time random field $X_U(u,t)$ may be expressed as a sum of sinusoids, in the same way as $X(u,t)$ in Eq. (6.7.8):

$$X_U(u,t) = \sum_{j=1}^{J} X_{U,j}(u,t), \tag{6.8.2}$$

where

$$X_{U,j}(u,t) = \frac{1}{U}\int_{u-U/2}^{u+U/2} X_j(u',t)\, du'. \tag{6.8.3}$$

Clearly, the component processes $X_{U,j}(u,t)$ and $X_j(u,t)$ have frequencies in the same narrow spectral band $\Delta\omega$ centered at ω_j. Also, $X_j(u,t)$ is characterized by the variance $G(\omega_j)\Delta\omega$ and the scale of fluctuation $\theta^u_{\omega_j}$. The variance of the frequency-specific spatial average $X_{U,j}(u,t)$ may be expressed as follows:

$$G_U(\omega_j)\Delta\omega = [G(\omega_j)\Delta\omega]\,\gamma_{\omega_j}(U), \tag{6.8.4}$$

where $\gamma_{\omega_j}(U)$ is by definition the variance function of $X_n(u,t)$. Taking $\omega_j = \omega$, one obtains the frequency-dependent variance function $\gamma_\omega(U)$ that indicates how the "point" spectral density function $G(\omega)$ changes when $X(u,t)$ undergoes local spatial averaging. From Eq. (6.8.4), the point s.d.f. of $X_U(u,t)$ is

$$G_U(\omega) = G(\omega)\,\gamma_\omega(U). \tag{6.8.5}$$

This relationship is illustrated in Fig. 6.12. The variance of $X_U(u,t)$ is found by integrating $G_U(\omega)$ over all frequencies,

$$\sigma_U^2 = \sum_{j=1}^{J} G_U(\omega_n)\Delta\omega \to \int_0^\infty G_U(\omega)\, d\omega = \int_0^\infty G(\omega)\gamma_\omega(U)\, d\omega, \tag{6.8.6}$$

and can also be expressed directly in terms of the variance function $\gamma(U)$ of the composite process,

$$\sigma_U^2 = \sigma^2 \gamma(U). \tag{6.8.7}$$

Combining Eqs. (6.8.6) and (6.8.7) yields the important relationship:

$$\gamma(U) = \int_0^\infty \gamma_\omega(U) g(\omega)\, d\omega. \tag{6.8.8}$$

For $U \to \infty$, this equation reduces to the expression linking the overall and frequency-dependent spatial scales, θ^u and θ_ω^u [see Eq. (6.7.4)].

Finally, the variance function $\gamma_\omega(U)$ can be evaluated in the usual way (as for one-dimensional random processes studied in Chap. 5) by adopting a simple (approximate) functional form that depends only on the scale of fluctuation, in this case θ_ω^u.

There are many opportunities for practical application of the concepts and methodology presented in Secs. 6.7 and 6.8, for instance, in the field of stochastic dynamics of structures subjected to wind forces, earthquake ground motions or ocean waves. A sampling of references to applications is found in Sec. 7.6 on stochastic finite element analysis.