Chapter 1

The Number of λ-Colourings and Its Enumerations

1.1 Introduction

Let G be a graph and $\lambda \in \mathbb{N}$. A mapping $f : V(G) \longrightarrow \{1, 2, \cdots, \lambda\}$ is called a *λ-colouring* of G if $f(u) \neq f(v)$ whenever the vertices u and v are *adjacent* in G. Two λ-colourings f and g of G are regarded as *distinct* if $f(x) \neq g(x)$ for some vertex x in G. The number of distinct λ-colourings of G, which is the key notion of this monograph, is denoted by $P(G, \lambda)$.

By convention, $P(G, 0) = 0$. By definition, $P(G, \lambda) \geq 1$ if and only if G is λ-colourable. Thus, $P(G, \chi(G)) \geq 1$ and $P(G, r) = 0$ if $r \in \mathbb{N}_0$ and $r < \chi(G)$; that is,

$$\chi(G) = \min\{\lambda \in \mathbb{N} : P(G, \lambda) \geq 1\}. \tag{1.1}$$

There are six sections in this chapter. In Section 1.2, we shall compute $P(G, \lambda)$ by definition for some classes of graphs. Two more efficient ways of enumerating $P(G, \lambda)$ based on the Fundamental Reduction Theorem and Zykov's K_r-gluing of graphs will be presented in Section 1.3. In Section 1.4, we shall discuss the form in which $P(G, \lambda)$ is expressed in terms of $P(K_i, \lambda)$'s, i.e., $P(G, \lambda) = \sum_{i \geq 1} b_i P(K_i, \lambda)$, and give an interpretation for the coefficients b_i. These results will then be used in Section 1.5 for the introduction of the *umbral product* of polynomials, which enables us to compute $P(G+H,\lambda)$, where $G+H$ is the join of the graphs G and H, in terms of $P(G,\lambda)$ and $P(H,\lambda)$. To end this chapter, we shall introduce in Section 1.6 the tree form of $P(G,\lambda)$, and discuss some features and state some results on $P(G,\lambda)$ in this form.

1.2 Examples

In this section, we shall enumerate $P(G,\lambda)$ for some special graphs G for illustration. We begin with two extremal classes of graphs.

Example 1.2.1 *For the empty graph O_n of order n, it is clear that*

$$P(O_n,\lambda)=\lambda^n. \tag{1.2}$$

More generally, if $G=\bigcup_{i=1}^{k} G_i$, then

$$P(G,\lambda)=\prod_{i=1}^{k} P(G_i,\lambda). \tag{1.3}$$

Example 1.2.2 *For the complete graph K_n of order n, we have*

$$P(K_n,\lambda)=\lambda(\lambda-1)\cdots(\lambda-n+1).$$

Observe that

$$\begin{aligned}
P(K_1,\lambda) &= \lambda,\\
P(K_2,\lambda) &= \lambda(\lambda-1)=\lambda^2-\lambda,\\
P(K_3,\lambda) &= \lambda(\lambda-1)(\lambda-2)=\lambda^3-3\lambda^2+2\lambda,\\
P(K_4,\lambda) &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)=\lambda^4-6\lambda^3+11\lambda^2-6\lambda,\\
&\text{etc.}
\end{aligned}$$

In general, when $P(K_n,\lambda)$ is expressed in terms of the powers of λ, we have

$$P(K_n,\lambda)=\lambda(\lambda-1)\cdots(\lambda-n+1)=\sum_{k=1}^{n} s(n,k)\lambda^k, \tag{1.4}$$

where $s(n,k)$'s are the (signed) *Stirling numbers of the first kind,* which are defined recursively by

$$s(n,k)=s(n-1,k-1)-(n-1)s(n-1,k), \tag{1.5}$$

with the boundary conditions that

$$\begin{cases} s(r,0)=0 & \text{for all } r\in\mathbb{N}, \\ s(r,r)=1 & \text{for all } r\in\mathbb{N}_0. \end{cases}$$

Example 1.2.3 *Let H be a graph containing a K_r as a subgraph, and let G be the graph obtained from H by adding a new vertex w which is linked with each vertex in K_r (and no others) as shown in Figure 1.1. Then we have*

$$P(G,\lambda)=(\lambda-r)P(H,\lambda). \tag{1.6}$$

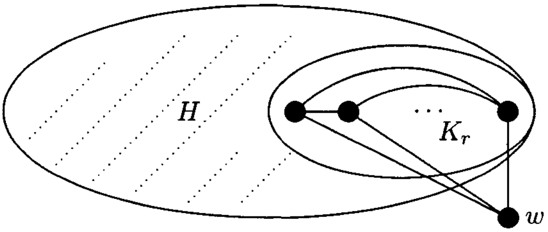


Figure 1.1

Remark It is known that every chordal graph can be constructed recursively using the above method starting with a collection of isolated vertices.

By applying (1.6) together with induction, the following results follow readily.

(i) If G is a chordal graph of order n, then

$$P(G,\lambda)=\lambda^{r_0}(\lambda-1)^{r_1}(\lambda-2)^{r_2}\cdots(\lambda-k)^{r_k}, \tag{1.7}$$

where $k\in\mathbb{N}$ and r_i's are in $\mathbb{N}$ such that $\sum_{i=0}^{k} r_i = n$. Note that $k=\chi(G)-1$.

(ii) If G is a q-tree of order n, then

$$P(G,\lambda)=\lambda(\lambda-1)\cdots(\lambda-q+1)(\lambda-q)^{n-q}. \tag{1.8}$$

(iii) If G is a tree of order n, then

$$P(G,\lambda)=\lambda(\lambda-1)^{n-1}. \tag{1.9}$$

Example 1.2.4 *Let us compute $P(C_4,\lambda)$, where the cycle C_4 is shown in Figure 1.2.*

C_4 :

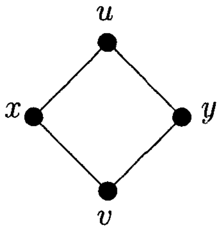


Figure 1.2

Let f be a λ-colouring of C_4.

Case 1: $f(x) = f(y)$. There are $\lambda - 1$ ways to colour the vertices u and v independently, and thus the number of such λ-colourings f is $\lambda(\lambda - 1)^2$.

Case 2: $f(x) \neq f(y)$. There are $\lambda - 2$ ways to colour the vertices u and v independently, and thus the number of such λ-colourings f is $\lambda(\lambda - 1)(\lambda - 2)^2$.

We thus conclude that

$$\begin{aligned} P(C_4, \lambda) &= \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda-1)^4 + (\lambda-1). \end{aligned} \tag{1.10}$$

1.3 Basic results on enumeration of $P(G, \lambda)$

It is known that the problem of evaluating $\chi(G)$ is NP-complete and, by (1.1), the problem of evaluating $P(G, \lambda)$ is at least as hard as that of determining $\chi(G)$. In spite of this, there are results which are useful for evaluating $P(G, \lambda)$ more efficiently for some classes of graphs. Two of them will be introduced in this section.

The first result that we shall introduce provides us with a recursive way to compute $P(G, \lambda)$, and its proof is just an extension of the idea used in counting $P(C_4, \lambda)$ as shown in Example 1.2.4.

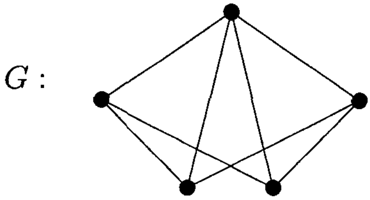
Theorem 1.3.1 *Let x and y be two non-adjacent vertices in a graph G. Then*

$$P(G, \lambda) = P(G + xy, \lambda) + P(G \cdot xy, \lambda). \tag{1.11}$$

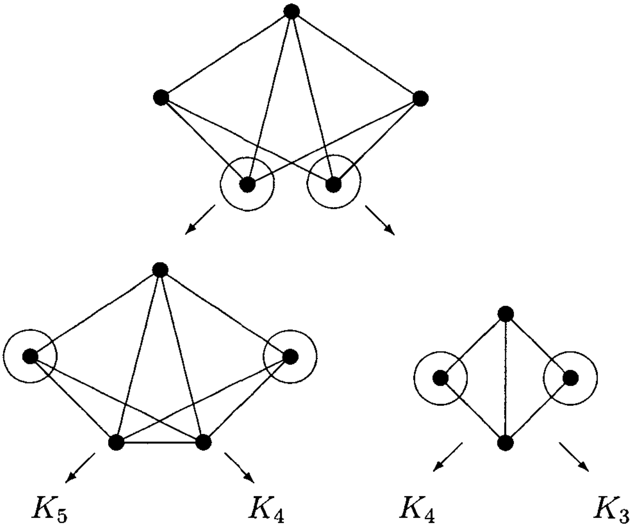
Proof. Let f be a λ-colouring of G. We have either (i) $f(x) \neq f(y)$ or (ii) $f(x) = f(y)$. The number of λ-colourings f of G for which (i) holds equals $P(G + xy, \lambda)$ while the number of λ-colourings f of G for which (ii) holds equals $P(G \cdot xy, \lambda)$. The result thus follows. ■

Throughout this monograph, for convenience, we may sometimes use a drawing to denote its $P(G,\lambda)$.

Example 1.3.1 *Let G be the graph given below.*



By applying (1.11) repeatedly, we have



Thus

$$\begin{aligned} P(G,\lambda) &= P(K_5,\lambda) + 2P(K_4,\lambda) + P(K_3,\lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7) \\ &= \lambda^5 - 8\lambda^4 + 24\lambda^3 - 31\lambda^2 + 14\lambda. \end{aligned}$$

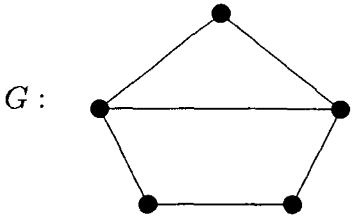
If we treat the graph $G+xy$ in Theorem 1.3.1 as a given graph H, then Theorem 1.3.1 can be equivalently restated as follows:

Let H be a graph and $e \in E(H)$. Then

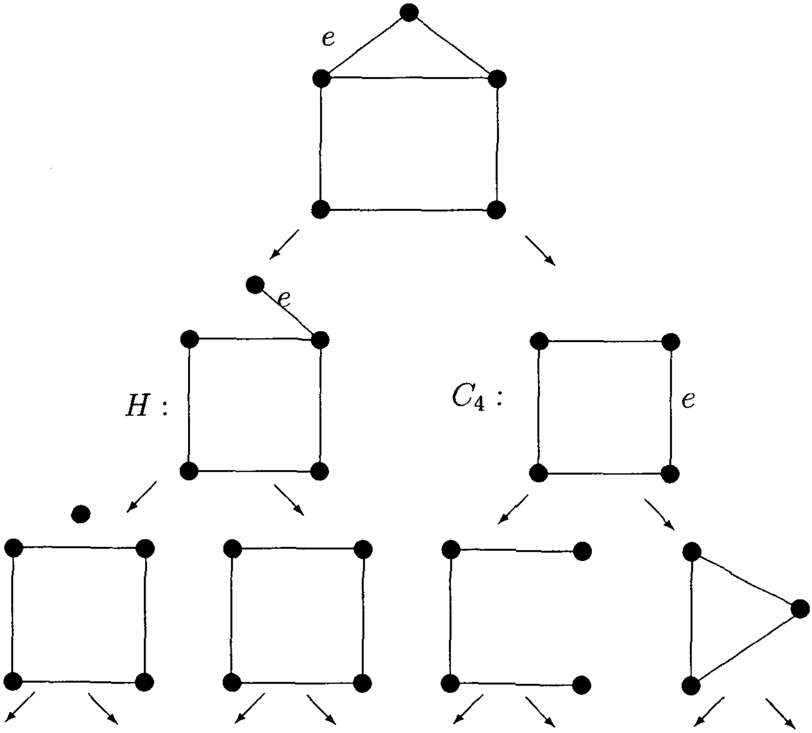
$$P(H,\lambda) = P(H-e,\lambda) - P(H\cdot e,\lambda). \tag{1.12}$$

From now on, both (1.11) and (1.12) will be referred to as the *Fundamental Reduction Theorem* (or simply FRT). As was shown in Example 1.3.1, by applying (1.11) repeatedly, $P(G,\lambda)$ can eventually be expressed in terms of $P(K_r,\lambda)$'s. We now show an example where (1.12) is applied.

Example 1.3.2 *Let G be the graph given below.*



By applying FRT (1.12) repeatedly, we have



It is now obvious that if we proceed further, then $P(G,\lambda)$ could eventually be expressed in terms of $P(O_r,\lambda)$'s. However, to compute $P(G,\lambda)$ in this *case, we actually could stop at level two, and obtain by applying (1.6) and (1.10) that*

$$\begin{aligned} P(G,\lambda) &= P(H,\lambda) - P(C_4,\lambda) \\ &= P(C_4,\lambda)(\lambda-1) - P(C_4,\lambda) \\ &= \left((\lambda-1)^4 + (\lambda-1)\right)(\lambda-2) \\ &= \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda. \end{aligned}$$

Example 1.3.3 *We have computed $P(C_3, \lambda)$ ($= P(K_3, \lambda)$) and $P(C_4, \lambda)$. We shall now show by induction that, in general, for $n \geq 3$,*

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1). \tag{1.13}$$

It is easy to check that (1.13) holds when $n = 3$. Assume that $n \geq 4$ and that the result holds for C_{n-1}. By FRT,

$$P(C_n, \lambda) = P(C_n - e, \lambda) - P(C_{n-1}, \lambda),$$

where $e \in E(C_n)$. Now, by the induction hypothesis and by (1.9),

$$\begin{aligned} P(C_n,\lambda) &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} - (-1)^{n-1}(\lambda-1) \\ &= (\lambda-1)^n + (-1)^n(\lambda-1), \end{aligned}$$

as required.

As shown in Example 1.3.1, if we apply (1.11) repeatedly beginning with a graph G, then we can eventually express $P(G, \lambda)$ in terms of $P(K_r, \lambda)$'s. This way of computing $P(G, \lambda)$ is particularly more efficient if G is dense (almost complete). On the other hand, if G is sparse (containing few edges), then we may apply (1.12) repeatedly, and eventually express $P(G, \lambda)$ in terms of $P(O_r, \lambda)$'s. However, whichever we use, as shown in Example 1.3.2, the procedure could be terminated so long as, for the resulting graphs H's, their $P(H, \lambda)$'s are known.

Let G_1 and G_2 be two graphs, and $r \in \mathbb{N}_0$ with $r \leq \min\{\omega(G_1), \omega(G_2)\}$, where $\omega(H)$ is the clique number of a graph H. Choose a K_r from each G_i, $i = 1, 2$, and form a new graph G from the union of G_1 and G_2 by identifying the two chosen K_r's in an arbitrary manner as shown in Figure 1.3. We call G a *K_r-gluing* of G_1 and G_2, and denote by $\mathcal{G}[G_1 \cup_r G_2]$ the family of all K_r-gluings of G_1 and G_2. When $r = 0$, G is just the disjoint union of G_1 and G_2; when $r = 1$, G is also called a *vertex-gluing* of G_1 and G_2; and when $r = 2$, G is also called an *edge-gluing* of G_1 and G_2.

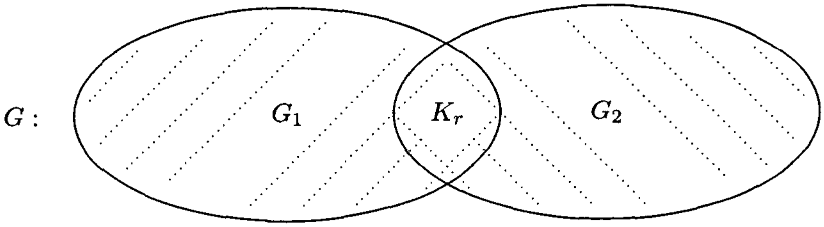


Figure 1.3

The following result, due to Zykov (1949), provides a shortcut for evaluating $P(G,\lambda)$ if G is a K_r-gluing of some graphs.

Theorem 1.3.2 *Let G_1 and G_2 be two graphs and $G \in \mathcal{G}[G_1 \cup_r G_2]$. Then*

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_r,\lambda)}. \tag{1.14}$$

Proof. For $i = 1,2$, a given λ-colouring of K_r gives rise to $\frac{P(G_i,\lambda)}{P(K_r,\lambda)}$ λ-colourings of G_i. Thus

$$P(G,\lambda) = P(K_r,\lambda) \cdot \tfrac{P(G_1,\lambda)}{P(K_r,\lambda)} \cdot \tfrac{P(G_2,\lambda)}{P(K_r,\lambda)} = \tfrac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_r,\lambda)}. \qquad \square$$

Corollary 1.3.1 *If G is a connected graph consisting of k blocks $B_1, B_2, \cdots, B_k$, then*

$$P(G,\lambda) = \tfrac{1}{\lambda^{k-1}} \prod_{i=1}^{k} P(B_i,\lambda). \qquad \square$$

Example 1.3.4 *The graph G in Example 1.3.2 is an edge-gluing of C_3 and C_4. Thus, by Theorem 1.3.2,*

$$\begin{aligned} P(G,\lambda) &= \frac{P(C_3,\lambda)P(C_4,\lambda)}{P(K_2,\lambda)} \\ &= \frac{\lambda(\lambda-1)(\lambda-2)\left((\lambda-1)^4 + (\lambda-1)\right)}{\lambda(\lambda-1)} \\ &= \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda. \end{aligned}$$

Example 1.3.5 *The graph G of Figure 1.1 is a K_r-gluing of H and K_{r+1}. Thus, by Theorem 1.3.2,*

$$P(G,\lambda) = \frac{P(H,\lambda)P(K_{r+1},\lambda)}{P(K_r,\lambda)} = P(H,\lambda)(\lambda - r).$$

By applying FRT and Theorem 1.3.2, Read (1986) obtained the following useful result (see Exercise 1.9(b)).

Theorem 1.3.3 *Let H be a connected graph, and u and v two non-adjacent distinct vertices of H. Let G be the graph obtained from H by adding a $u-v$ path P of length r as shown in Figure 1.4. Then*

$$P(G,\lambda) = \frac{1}{\lambda(\lambda-1)}P(C_{r+1},\lambda)P(H,\lambda) + (-1)^r P(H\cdot uv,\lambda). \qquad (1.15)$$

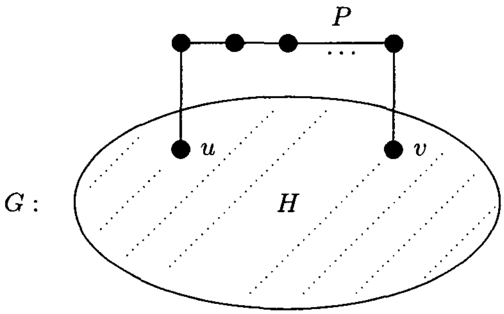
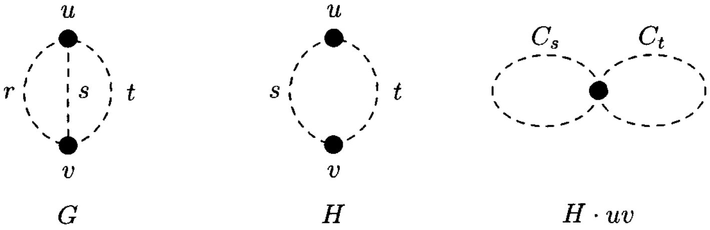


Figure 1.4

Example 1.3.6 *By applying Theorem 1.3.3, we can express $P(G,\lambda)$, where $G = \theta(r,s,t)$ is the generalized θ-graph and $r,s,t \geq 2$, in terms of $P(C_i,\lambda)$'s as shown below.*



$$P(G,\lambda) = \frac{1}{\lambda(\lambda-1)}P(C_{r+1},\lambda)P(C_{s+t},\lambda) + (-1)^r\frac{1}{\lambda}P(C_s,\lambda)P(C_t,\lambda).$$

1.4 $P(G, \lambda)$ in factorial form

Let G be a graph. We have seen that, by applying (1.12) repeatedly, we would have

$$P(G,\lambda) = \sum_{i\geq 1} a_i P(O_i,\lambda) = \sum_{i\geq 1} a_i \lambda^i, \qquad (1.16)$$

where a_i's are some constants. In this situation, we say that $P(G,\lambda)$ is expressed in *power* form.

On the other hand, if we apply (1.11) repeatedly, then we would arrive at

$$P(G,\lambda) = \sum_{i\geq 1} b_i P(K_i,\lambda) = \sum_{i\geq 1} b_i(\lambda)_i, \tag{1.17}$$

where b_i's are some constants and

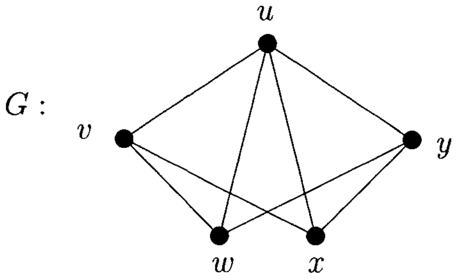
$$(\lambda)_i = \lambda(\lambda-1)\cdots(\lambda-i+1). \tag{1.18}$$

In this situation, we say that $P(G,\lambda)$ is expressed in *factorial* form.

A natural question arises. What can be said about the constants a_i's and b_i's that appear in (1.16) and (1.17) respectively?

While this question for a_i's will be discussed in next chapter, we shall give in this section an interpretation for b_i's.

Consider the following G (see Example 1.3.1) again.



For $i = 1, 2, \cdots, 5$, let $\alpha(G,i)$ denote the number of ways of partitioning $V(G)$ into i independent sets. Observe that, in this case,

$$\{\{u\},\{v\},\{y\},\{w\},\{x\}\}$$

is the only partition with 5 independent sets;

$$\{\{u\},\{v\},\{y\},\{w,x\}\} \quad \text{and} \quad \{\{u\},\{w\},\{x\},\{v,y\}\}$$

are the two partitions with 4 independent sets; $\{\{u\}, \{v,y\}, \{w,x\}\}$ is the unique partition with 3 independent sets; and there are no partitions with i independent sets for $i = 1, 2$. Thus, we have:

$$\alpha(G,1) = \alpha(G,2) = 0,\ \alpha(G,3) = 1,\ \alpha(G,4) = 2 \text{ and } \alpha(G,5) = 1.$$

On the other hand, as shown in Example 1.3.1, $P(G,\lambda) = (\lambda)_5 + 2(\lambda)_4 + (\lambda)_3$. The comparison of these two observations suggests that the constants b_i in (1.17) is equal to $\alpha(G,i)$. It is indeed the case as shown below (see Read (1968), for instance).

Theorem 1.4.1 *Let G be a graph of order n. Then*

$$P(G,\lambda) = \sum_{i=1}^{n} \alpha(G,i)(\lambda)_i, \tag{1.19}$$

where $\alpha(G,i)$ is the number of ways of partitioning $V(G)$ into i independent sets.

Proof. There is a bijection between the family of colourings of G using *exactly* i colours from $\{1,2,\cdots,\lambda\}$ and the family of partitions of $V(G)$ into i independent sets; and further, for any such partition, there are $(\lambda)_i$ λ-colourings of G. Thus the number of λ-colourings of G is given by $\sum\limits_{i=1}^{n} \alpha(G,i)(\lambda)_i$. □

If $\chi(G) = r$, then $\alpha(G,i) = 0$ when $i < r$, and $(r)_i = 0$ when $i > r$. Thus

$$P(G,r) = \sum_{i=1}^{n} \alpha(G,i)(r)_i = \alpha(G,r)r!.$$

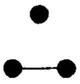
Corollary 1.4.1 *If $\chi(G) = r$, then $\alpha(G,r) = P(G,r)/r!$.* □

An independent set in G corresponds to a clique in $\overline{G}$, and vice versa. Thus, as pointed out in Frucht (1985) and R.Y Liu (1987),

$$\text{$b_i(=\alpha(G,i))$ is the number of spanning subgraphs of $\overline{G}$ consisting of i components, each of which being a complete graph.} \tag{1.20}$$

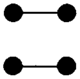
For instance, if $G = P_3$, then

$\overline{G}$:



By (1.20), $b_2 = b_3 = 1$, $b_1 = 0$ and so, by Theorem 1.4.1, $P(G,\lambda) = (\lambda)_3 + (\lambda)_2$. If $H = C_4$, then

$\overline{H}$:



Thus, by (1.20), $b_4 = 1$, $b_3 = 2$, $b_2 = 1$, $b_1 = 0$ and so, by Theorem 1.4.1,

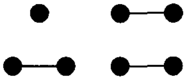
$$P(H, \lambda) = (\lambda)_4 + 2(\lambda)_3 + (\lambda)_2.$$

1.5 The join of graphs and the umbral product

By means of the expression of $P(G, \lambda)$ in factorial form (1.17), we shall introduce in this section a way of expressing $P(G + H, \lambda)$, where $G + H$ is the join of the graphs G and H, in terms of $P(G, \lambda)$ and $P(H, \lambda)$.

Let $G = P_3$ and $H = C_4$ as considered in Section 1.4, and we wish now to express $P(G + H, \lambda)$ in factorial form. Observe that

$\overline{G + H} = \overline{G} \cup \overline{H}$:



Thus, by (1.20), we have $P(G + H, \lambda) = (\lambda)_7 + 3(\lambda)_6 + 3(\lambda)_5 + (\lambda)_4$.

Assume, for any two graphs G and H, that $P(G, \lambda)$ and $P(H, \lambda)$ are expressed in factorial form. Following Read (1968), the *umbral product* of $P(G, \lambda)$ and $P(H, \lambda)$, denoted by $P(G, \lambda) \otimes P(H, \lambda)$, is an expression also in factorial form obtained by performing the usual polynomial multiplication where "factorials" are multiplied as "powers" (i.e., $(\lambda)_i \otimes (\lambda)_j = (\lambda)_{i+j}$).

Thus, for $G = P_3$ and $H = C_4$,

$$\begin{aligned} P(G, \lambda) \otimes P(H, \lambda) &= ((\lambda)_3 + (\lambda)_2) \otimes ((\lambda)_4 + 2(\lambda)_3 + (\lambda)_2) \\ &= (\lambda)_7 + 3(\lambda)_6 + 3(\lambda)_5 + (\lambda)_4 \\ &= P(G + H, \lambda). \end{aligned}$$

In general, we have (see Zykov (1949) and Exercise 1.16):

Theorem 1.5.1 *Let G_1 and G_2 be any two graphs with $P(G_i, \lambda)$ expressed in factorial form, $i = 1, 2$. Then*

$$P(G_1 + G_2, \lambda) = P(G_1, \lambda) \otimes P(G_2, \lambda). \tag{1.21}$$

□

Corollary 1.5.1 *For any graph H,*

$$P(H+K_1,\lambda) = \lambda P(H,\lambda-1). \tag{1.22}$$

Proof. Assume that $P(H,\lambda) = \sum\limits_{i\geq 1} b_i(\lambda)_i$. By Theorem 1.5.1,

$$\begin{aligned} P(H+K_1,\lambda) &= P(H,\lambda)\otimes(\lambda)_1 = \sum_{i\geq 1} b_i(\lambda)_{i+1} \\ &= \lambda\sum_{i\geq 1} b_i(\lambda-1)_i = \lambda P(H,\lambda-1). \end{aligned}$$

□

In particular, for the wheel W_n of order $n \geq 4$,

$$\begin{aligned} P(W_n,\lambda) &= P(C_{n-1}+K_1,\lambda) = \lambda P(C_{n-1},\lambda-1) \\ &= \lambda\left((\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\right). \end{aligned} \tag{1.23}$$

We shall next compute $P(K(p,q),\lambda)$. For this purpose, we first introduce a sequence of numbers $S(n,k)$, called the *Stirling numbers of the second kind*, which are defined recursively as

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.24}$$

where $n, k \in \mathbb{N}$, with the boundary conditions that

$$\begin{cases} S(r,1) = 1 \quad \text{for all } r \in \mathbb{N}, \\ S(r,r) = 1 \quad \text{for all } r \in \mathbb{N}_0, \\ S(r,0) = S(0,r) = 0 \quad \text{for all } r \in \mathbb{N}. \end{cases} \tag{1.25}$$

Combinatorially, $S(n,k)$ counts the number of ways of distributing n distinct objects into k identical boxes such that no box is empty. It is known that when the power λ^n is expressed in terms of the factorials $(\lambda)_k$'s, we have

$$\lambda^n = \sum_{k=1}^{n} S(n,k)(\lambda)_k. \tag{1.26}$$

Thus, by Theorem 1.5.1, we have (see Swenson (1973))

$$\begin{aligned} P(K(p,q),\lambda) &= P(O_p+O_q,\lambda) = P(O_p,\lambda)\otimes P(O_q,\lambda) = \lambda^p \otimes \lambda^q \\ &= \left(\sum_{r=1}^{p} S(p,r)(\lambda)_r\right) \otimes \left(\sum_{s=1}^{q} S(q,s)(\lambda)_s\right) \\ &= \sum_{r=1}^{p}\sum_{s=1}^{q} S(p,r)S(q,s)(\lambda)_{r+s}. \end{aligned} \tag{1.27}$$

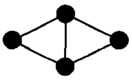
More generally, for the complete m-partite graph $K(p_1, p_2, \cdots, p_m)$, we have (see Laskar and Hare (1975)):

$$P(K(p_1, p_2, \cdots, p_m), \lambda) = \sum_{r_m=1}^{p_m} \cdots \sum_{r_1=1}^{p_1} \prod_{i=1}^{m} S(p_i, r_i)(\lambda)_{r_1+\cdots+r_m}. \qquad (1.28)$$

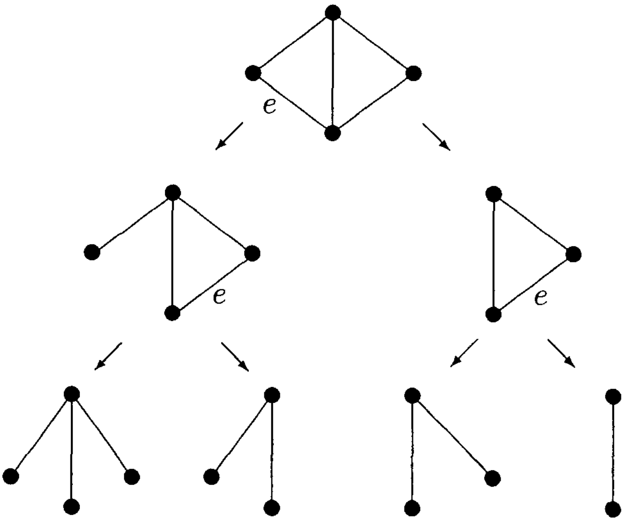
1.6 $P(G, \lambda)$ in tree form

We have seen that for any graph G, $P(G, \lambda)$ can be expressed in either power form (1.16) or factorial form (1.17) (see also Theorem 1.4.1). In this final section, we shall introduce another form for $P(G, \lambda)$ which has its merit from a computational point of view.

Consider the following graph G:



By applying FRT (1.12) repeatedly beginning with G, we may have the following tree of computation:



We note that in each step where (1.12) is applied on a current graph H, an edge e of H is chosen so that $H - e$ is connected, and we end up with four graphs which are trees. Since $P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}$ for any tree T_n

of order n, we arrive at

$$\begin{aligned} P(G,\lambda) &= \lambda(\lambda-1)^3-\lambda(\lambda-1)^2-\lambda(\lambda-1)^2+\lambda(\lambda-1) \\ &= \lambda(\lambda-1)^3-2\lambda(\lambda-1)^2+\lambda(\lambda-1). \end{aligned}$$

In general, given any connected graph G of order n, we have

$$P(G,\lambda)=\sum_{i=1}^{n-1} t_i\lambda(\lambda-1)^i, \tag{1.29}$$

where t_i's are some constants, and we say that $P(G,\lambda)$ is expressed in *tree* form.

The method of computing $P(G,\lambda)$ in tree form was suggested by Nijenhuis and Wilf (1978) (see also James and Riha (1975)). Given a connected graph G with a relatively small number of edges, comparing with the strategy by reducing G to empty graphs, it is clear that the strategy by reducing G to trees takes fewer number of steps. Thus, from a computational point of view, (1.29) has its own advantage.

It had been observed in Tutte (1973) and Nijenhuis and Wilf (1978) that $t_{n-1}=1$ and the coefficients t_i's in (1.29) alternate in sign.

Frank and Shier (1986) showed that the absolute value of t_i, which is the number of trees of order $i+1$ at the end of the computation tree, is an *invariant* regardless of any procedure where FRT (1.12) is repeatedly applied until the resulting graphs are trees.

We state below some results on $P(G,\lambda)$ in tree form for certain classes of graphs, which can be found in Frank and Shier (1986) or Adam and Broere (1993) (see Exercises 1.18 and 1.19).

(1) $G\cong C_n$, where $n\geq 3$, if and only if $|t_i|=1$ for all $i=1,2,\cdots,n-1$.

(2) G is bipartite if and only if $\sum\limits_{i=1}^{n-1} t_i\neq 0$.

(3) $P(W_n,\lambda)=(-1)^{n-1}\lambda(\lambda-1)+\sum\limits_{i=0}^{n-2}(-1)^i\binom{n-1}{i}\lambda(\lambda-1)^{n-i-1}$.

(4) $P(K_n,\lambda)=\sum\limits_{i=1}^{n-1} s(n-1,i)\lambda(\lambda-1)^i$, where $s(n-1,i)$'s are the Stirling numbers of the first kind as introduced in (1.5).

(5) $P(K(2,q),\lambda)=\lambda(\lambda-1)^q+\sum\limits_{i=1}^{q+1}(-1)^{q+1-i}\binom{q}{i-1}\lambda(\lambda-1)^i$.

For more information on computing $P(G, \lambda)$ for more families of graphs, and on certain improved methods or tricks on computing $P(G, \lambda)$, the reader is referred to Loerine (1980), Read (1981, 1987a, 1988b) and Read and Tutte (1988).