Chapter 1

INTRODUCTION

1.1 Where do ODEs arise

The theory of ordinary differential equations deals with the large time behavior of the solution $x(t, x_0)$ of the initial value problem (I.V.P.) of the first order system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, x_2, \cdots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, \cdots, x_n) \\ x_i(0) &= x_{i0}, \quad i = 1, 2, \cdots, n \end{aligned}$$

or in vector form

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)$, $f = (f_1, \cdots, f_n)$, $f : D \to R^n$, D is open in $R \times R^n$.

If the right-hand side of (1.1) is independent of time t, i.e.,

$$\frac{dx}{dt} = f(x), \quad x \in \Omega \subseteq R^n, \tag{1.2}$$

then we say that (1.2) is an autonomous system. In this case, we call f

a vector field on its domain Ω. If the right-hand side depends on time t, we say that (1.1) is a nonautonomous system. The most important nonautonomous system is the periodic system, i.e., $f(t,x)$ satisfies

$$f(t+w,x)=f(t,x)$$

for some $w>0$ (w is called the period). If $f(t,x)=A(t)x$ where $A(t)\in R^{n\times n}$, then we say that

$$\frac{dx}{dt}=A(t)x \tag{1.3}$$

is a linear system of differential equations. It is easy to verify that if $\varphi(t),\psi(t)$ are solutions of (1.3), then $\alpha\varphi(t)+\beta\psi(t)$ is also a solution of linear system (1.3) for $\alpha,\beta\in R$. The system

$$\frac{dx}{dt}=A(t)x+g(t) \tag{1.4}$$

is called linear system with nonhomogeneous part $g(t)$. If $A(t)\equiv A$, then

$$\frac{dx}{dt}=Ax \tag{1.5}$$

is a linear system with constant coefficients. A system (1.1) which is not linear is called a nonlinear system. It is usually much harder to analyze nonlinear systems than the linear ones. The main difference is that linear systems can be broken down into parts. Through the superposition principle, Laplace transform, Fourier analysis, we find a linear system precisely equals to the sum of its parts. However, nonlinear systems are the most phenomena in our daily life and, do not have a superposition principle. In the following we present some important examples of differential equations from physics, chemistry and biology.

Example 1.1.1 $m\ddot{x}+c\dot{x}+kx=0.$

This describes the motion of a spring with damping and restoring forces. Applying Newton's law, $F=ma$, we have

$$ma=m\ddot{x}=F=-c\dot{x}-kx=\text{Friction}+\text{restoring force}.$$

Let $y=\dot{x}$. Then we obtain

$$\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{c}{m}y-\frac{k}{m}x.\end{cases}$$

Example 1.1.2 $m\ddot{x} + c\dot{x} + kx = F\cos wt$. This describes the motion of a spring with external periodic force. It can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{c}{m}y - \frac{k}{m}x + F\cos wt \end{pmatrix}.$$

If $c = 0$ and $w = \sqrt{k/m}$, then we have "resonance". ([BD], p. 184)

Example 1.1.3 Electrical Networks ([BD], p. 184)

Let $Q(t)$ be the charge on s capacitor at time t. Use the Kirchoff's 2nd law: In a closed circuit, the impressed voltage equals the sum of the voltage drops in the rest of the circuit.

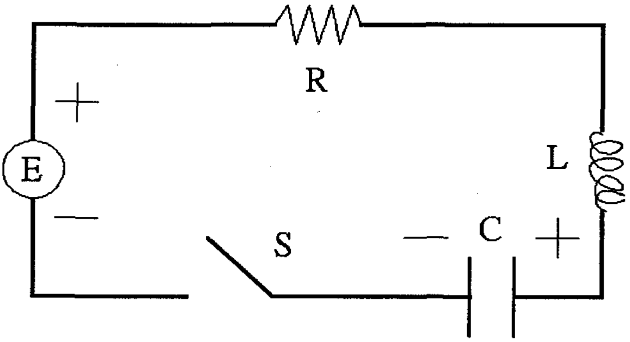

Fig. 1.1

1. The voltage drop across a resistance of R(ohms) equals RI (Ohm's law)
2. The voltage drop across an inductance of L(henrys) equals $L\frac{dI}{dt}$
3. The voltage drop across a capacitance of C(farads) equals Q/C

Hence

$$E(t) = L\frac{dI}{dt} + RI + \frac{Q}{C}.$$

Since $I(t) = \frac{dQ}{dt}$, it follows that

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

The equation of electric network is similar to that of mechanical vibration with $L_1 \to m$, $R \to c$ and $\frac{1}{C} \to k$.

Example 1.1.4 Van der Pol Oscillator ([K], p. 481, [HK], p. 172)

$$u'' + \epsilon u'(u^2 - 1) + u = 0, \quad 0 < \epsilon \ll 1.$$

Let $E(t) = \frac{u'^2}{2} + \frac{u^2}{2}$ be the energy. Then

$$\begin{aligned} E'(t) &= u'u'' + uu' = u'(-\epsilon u'(u^2 - 1) - u) + uu', \\ &= -\epsilon(u')^2(u^2 - 1) = \begin{cases} < 0, & |u| > 1 \\ > 0, & |u| < 1. \end{cases} \end{aligned}$$

Hence the oscillator is "self-excited".

Example 1.1.5 Van der Pol Oscillator with periodic forcing

$$u'' + \epsilon u'(u^2 - 1) + u = A\cos wt.$$

This is the equation Cartwright and Littlewood studied in 1945 and it led Smale to construct Smale's horseshoe in 1960. It is one of the model equations in chaotic dynamical systems [Sm].

Example 1.1.6 Second order conservative system

$$\ddot{x} + g(x) = 0,$$

or equivalently

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -g(x_1). \end{aligned}$$

The energy $E(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1)$, where $V(x_1) = \int_0^{x_1} g(s)ds$, is the potential. Then the energy E satisfies

$$\frac{d}{dt}E = 0.$$

Example 1.1.7 Duffing's equation

$$\ddot{x} + (x^3 - x) = 0.$$

The potential $V(x) = -(1/2)x^2 + (1/4)x^4$ is a double-well potential.

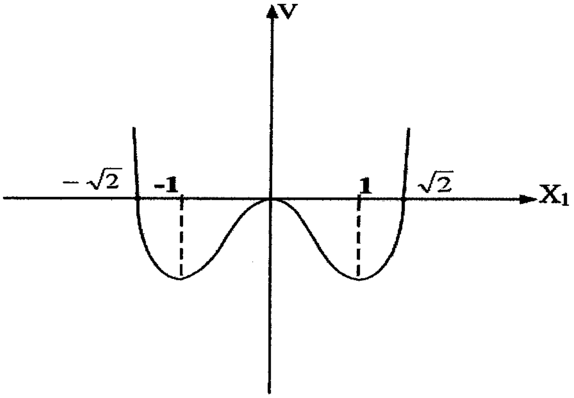

Fig. 1.2

Example 1.1.8 Duffing's equation with damping and periodic forcing.

$$\ddot{x} + \beta\dot{x} + (x^3 - x) = A\cos wt.$$

This is also a typical model equation in chaotic dynamical systems.

Example 1.1.9 Simple pendulum equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

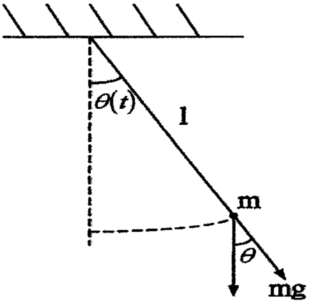

Fig. 1.3

$$\begin{aligned} F &= ma \\ -mg\sin\theta &= m\ell \cdot \frac{d^2\theta}{dt^2}. \end{aligned}$$

Example 1.1.10 Lorentz equation ([S], p. 301, [V])

$$\begin{aligned} &\dot{x} = \sigma(y-x) \\ &\dot{y} = rx - y - xz \quad \text{where } \sigma, r, b > 0. \\ &\dot{z} = xy - bz \end{aligned}$$

When $\sigma = 10,\ b = \frac{8}{3},\ r = 28,\ (x(0), y(0), z(0)) \approx (0,0,0)$, we have the "butterfly" phenomenon.

Example 1.1.11 Michaelis-Menten Enzyme Kinetics ([K], p. 511), ([LS], p. 302)
Consider the conversion of a chemical substrate S to a product P by enzyme catalysis. The reaction scheme

$$E + S \underset{k-1}{\overset{k_1}{\rightleftharpoons}} ES \overset{k_2}{\rightarrow} E + P$$

was proposed by Michaelis and Menten in 1913. The law of Mass action states that the rate of reaction is proportional to the concentrations of reactants. Then according to the law of mass action, we have the following equations:

$$\begin{aligned} &\frac{d}{dt}[E] = -k_1[E][S] + k_{-1}[ES] + k_2[ES], \\ &\frac{d}{dt}[S] = -k_1[E][S] + k_{-1}[ES], \\ &\frac{d}{dt}[ES] = k_1[E][S] - k_{-1}[ES] - k_2[ES], \\ &\frac{d}{dt}[P] = k_2[ES], \end{aligned}$$

with initial concentrations

$$[E](0) = E_0,\ [S](0) = S_0,\ [ES](0) = [P](0) = 0.$$

Since

$$\frac{d}{dt}[ES] + \frac{d}{dt}[E] = 0,$$

then

$$[ES] + [E] \equiv E_0.$$

Let

$$u = [ES]\Big/C_0,\ v = [S]\Big/S_0,\ \tau = k_1 E_0 t,$$

and

$$\kappa = (k_{-1} + k_2)\Big/k_1 S_0,\ \epsilon = C_0/S_0,\ \lambda = \frac{k-1}{k_{-1}+k_2},\ C_0 = E_0/1+\kappa.$$

Then we have the equation of singular perturbation

$$\begin{aligned} &\epsilon\frac{du}{d\tau} = v - \frac{(v+\kappa)u}{1+\kappa}, \quad 0 < \epsilon \ll 1 \\ &\frac{dv}{d\tau} = -v + (v + \kappa\lambda)u\Big/(1+\kappa) \\ &u(0) = 0,\ v(0) = 1. \end{aligned}$$

We shall study this example in Chapter 9.

Example 1.1.12 Belousov-Zhabotinskii Reaction [M]

$$\begin{aligned} &\epsilon\frac{dx}{dt} = qy - xy + x(1-x) \\ &\delta\frac{dy}{dt} = -qy - xy + 2fz \\ &\frac{dz}{dt} = x - z \end{aligned}$$

where ϵ, δ, q are small, $f \approx 0.5$. This is an important oscillator in chemistry.

Example 1.1.13 Logistic equation.
Let $x(t)$ be the population of a species. Then $x(t)$ satisfies

$$\frac{dx}{dt} = rx - bx^2 = rx\left(1 - \frac{x}{K}\right), \quad r, b > 0$$

where K is the carrying capacity of the species and r is the intrinsic growth rate.

Example 1.1.14 The Lotka-Volterra model for Predator-Prey interaction. Let $x(t), y(t)$ be the population of a prey and a predator at time t respectively. ([M], pp. 124 and 62). Then we have the following two predator-prey models of Lotka-Volterra type:

$$\begin{aligned} \tfrac{dx}{dt} &= ax - bxy, \\ & \qquad\qquad a, b, c, d > 0, \\ \tfrac{dy}{dt} &= cxy - dy, \end{aligned}$$

and

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - bxy, \\ \frac{dy}{dt} = cxy - dy. \end{cases}$$

The first model assumes the prey species grows exponentially in the absence of predation while the second assumes the prey species grows logistically with carrying capacity K.

Example 1.1.15 The Lotka-Volterra two species competition model. ([M]) Let $x_i(t)$, $i = 1, 2$, be the population of i-th competing species. We assume that the i-th species grows logistically in the absence of competition with intrinsic rate r_i and carrying capacity K_i. In the following model, α, $\beta > 0$ are called the competition coefficients. The model takes the form:

$$\begin{aligned} &\frac{dx_1}{dt} = r_1x_1\left(1 - \frac{x_1}{K_1}\right) - \alpha x_1x_2 \\ &\frac{dx_2}{dt} = r_2x_2\left(1 - \frac{x_2}{K_2}\right) - \beta x_1x_2 \\ &x_1(0) > 0, x_2(0) > 0. \end{aligned}$$