Chapter 1

Two Curious Mathematical Cuneiform Texts from Old Babylonian Mari

Mari was the center of a small kingdom on the middle Euphrates, independent until it was conquered by Hammurabi in 1757 BCE. Important for the discussion below is that Mari's location in the north-western corner of Mesopotamia may have allowed it to promote an exchange of ideas between Mesopotamia and its neighbors to the west, maybe even between Mesopotamia and cities along the coast of the Mediterranean, and ultimately Egypt. Old Babylonian cuneiform texts from a royal archive at Mari are in the process of being published by a team of French scholars. Among already published cuneiform texts from Mari are several texts of mathematical interest, in particular

a) some mathematical table texts published by D. Soubeyran in *RA* 78 (1984), among them a text with 30 terms of a geometric progression (Sec. 1.2 a below),

b) a round hand tablet published by D. Charpin in *MARI* 7 (1993), inscribed with an outline of a city wall and with numbers indicating the volumes of the four sides of the city wall and the sizes of the four teams of workers needed to erect them,

c) a rectangular hand tablet published by M. Guichard in *MARI* 8 (1997), with 30 terms of a geometric progression expressed in three kinds of numbers (Sec. 1.1 b),

d) three metrological texts published by G. Chambon in *FlMar* 6 (2002), among them a clay cylinder of a rare type with a metrological list of weight measures.

The map in Fig. 1.1.1 below shows the location of Mari, as well as of Ebla, another ancient city in Syria, and of Ugarit on the coast of the Mediterranean. Cuneiform texts with metrological tables of Old Babylonian style have been found at Ugarit (Nougayrol, *Ugaritica* 5 (1968)), and interesting mathematical cuneiform texts from the late third millennium BCE have been found at Ebla (Friberg, *VOr* 6 (1986)).

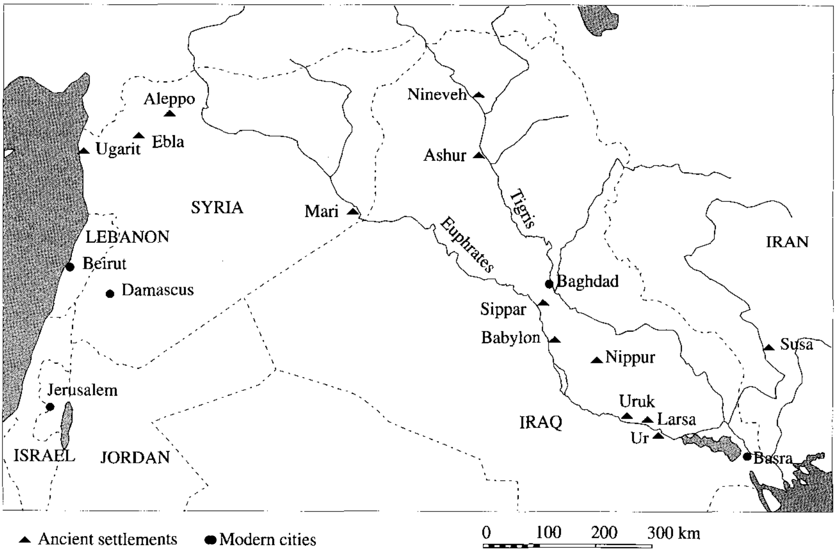


Fig. 1.1.1. A map of Mesopotamia and its neighbors.

1.1. M. 7857. A Fanciful Interpretation of a Geometric Progression

1.1 a. M. 7857. A text with three kinds of counting numbers

M. 7857, the mathematical text from Mari published by Guichard, was poorly understood by him and described as "an account of ants". Actually, the obverse of the clay tablet contains the computation of five terms of a geometric progression, with the first term 99 and the common ratio 9. The computation is carried out twice, first in sexagesimal place value numbers, then in "mixed decimal-sexagesimal" numbers. These were the OB learned and lay ways, respectively, of expressing numbers. The other side of the clay tablet (the reverse) contains a fanciful reformulation of the computation, expressed (not quite successfully) in the "centesimal" numbers used locally at Mari. (This interpretation of the text was discovered independently by the present author and by C. Proust, *FIM* 6 (2002.)

In the Babylonian *sexagesimal place value system*, there are special cuneiform number signs for the ones, from 1 to 9, and for the tens, from 10 to 50. In the OB (non-positional) *mixed decimal-sexagesimal system*, in the local variant used in texts from Mari, there are signs for the number words 'a hundred' (*me,* abbreviation for *mêtum*), 'a thousand' (*līm*, pl. *līmī*), and 'a great' (Sum. gal), meaning either 'ten thousand' or, equivalently, 'a hundred hundred'. Numbers below 100 are written as sexagesimal numbers, with or without the word *šu-ši* 'sixty'. The Mari *centesimal place value system*, on the other hand, operates in the same way as the Babylonian sexagesimal place value system, but with the base 100 instead of 60, and with cuneiform signs not only for the tens from 10 to 50, but also for 60, 70, 80, and 90 (written with from six to nine oblique wedges).

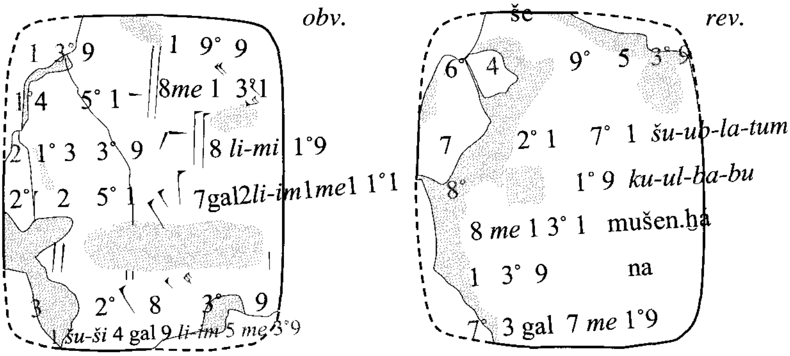


Fig. 1.1.2. M. 7857. An OB mathematical text from Mari, in conform transliteration.

A "conform transliteration" of the text of M. 7857 is presented in Fig. 1.1.2, within an outline of the clay tablet. In such a conform transliteration, the cuneiform signs are replaced by their numerical or phonetic values, placed in the same positions as the original cuneiform signs. The notations 1°, 2°, ···, 9° are used as conform transliterations of the cuneiform signs for the tens, from 10 to 90.

As the figure shows, the surface of the clay tablet is damaged, pieces of it are missing near the corners, and there are traces of a previous, incompletely erased inscription. Anyway, most of the text is preserved, and the lost parts of the text can be reconstructed (grey numbers and letters).

Below is given a standard transliteration of the text in the two columns on the *obverse* of M. 7857 (within the frame), together with a direct translation (underneath the frame). In the standard transliteration, zeros are inserted where needed. Reconstructed parts of the text are placed within straight brackets in the transliteration, but are written with italics in the translation. The exclamation marks indicate corrections: 28 in line 6 should be 25, and 1 99 in line 1 should be 1 39 (= 99), an interesting error.

M. 7857

obv. 1	[1 3]9	||	1 99
2	[1]4 51	||	8 *me* 1 31
3	[2] 13 39	||	8 *li-mi* 19
4	20 02 51	||	7 gal 2 *li-im* 1 *me* 1 11
5	erasure		
6	[3] 28 39		
7	[1] *šu-ši* 4 gal 9 [*li-mi* 5 *me* 39]		

1 39	||	1 3$^{!}$9
*1*4 51	||	8 hundred 1 31
2 13 39	||	8 thousand 19
20 02 51	||	7 great 2 thousand 1 hundred 1 11
erasure		
3 00 25$^{!}$ 39		
1 sixty 4 great 9 *thousand 5 hundred 39*		

The computations producing the numbers in the two columns on the obverse can be explained as follows:

1 39		||	99	
14 51	(= 9 · 1 39)	||	891	(= 9 · 99)
2 13 39	(= 9 · 14 51)	||	8,019	(= 9 · 891)
20 02 51	(= 9 · 2 13 39)	||	72,171	(= 9 · 8,019)
3 00 25 39	(= 9 · 20 02 51)	||	649,539	(= 9 · 72,171)

There is only a single column of text on the *reverse* of M. 7857:

edge	še	
rev. 1	64 95 [39] /	
2	7 21 71	*šu-ub-la-tum* /
3	[80] 19	*ku-ul-ba-bu* /
4	8 *me* 1 31	mušen.ḫa /
5	1 39	na /
6	73 gal 7 *me* 19	

64 95 *39* barley-corns	
7 21 71	ears of barley
80 19	ants
8 hundred 1 31	birds
1 39	people(?)
73 great 7 hundred 19	

The one who wrote the text, probably a school boy, was clearly confused by the bewildering variety of number systems. In line 1 on the obverse he hesitated between writing 99 as 1 39 in the mixed decimal-sexagesimal system or as 99 in the centesimal system, and ended up writing 1 99. In lines 4-6 on the reverse, he forgot that he was supposed to use the centesimal place value system and reverted to the mixed decimal-sexagesimal system used in the right column on the obverse.

The switching to the centesimal number system on the reverse is not the only difference between the reverse and the obverse on M. 7857. On the obverse the terms of the geometric progression increase from 99 to $9^4 \cdot 99 = 649{,}539$, while on the reverse the recorded numbers decrease from 649,539 in line 1 to 99 in line 5. In addition, the numbers of the geometric progression have been given a fanciful interpretation, with a series of appended Sumerian or Akkadian (Babylonian) words. Finally, the number recorded in line 6 on the reverse is the sum of the five terms of the geometric progression, while there is no sum recorded on the obverse. Indeed, in the centesimal system (with the mistakes on the reverse corrected), the sum can be computed as follows:

64 95 39	barley-corns	(Sum. še)
7 21 71	ears of barley	(Akk. *šublātum*)
80 19	ants	(Akk. *kulbābū*)
8 91	birds	(Sum. mušen.ḫá)
+ 99	people	(Sum. na?; the translation is problematic)
73 07 19	diverse items	

It is tempting to try to reconstruct a whimsical story that can have accompanied the text on the reverse. It may have gone like this:

> There were 645, 539 barley corns, 9 barley-corns on each ear of barley, 9 ears of barley eaten by each ant, 9 ants swallowed by each bird, and 9 birds caught by each man. How many were there altogether?

1.1 b. OB texts with ascending or descending geometric progressions

There is no known OB cuneiform text that is a direct parallel to the Mari text M. 7857. There are, however, quite a few known OB clay tablets on which are recorded *ascending* or *descending* geometric progressions of various kinds. The simplest examples of texts with ascending geometric progressions are inscribed with a small number of terms, usually 10, of a "table of powers" (a geometric progression in which the common ratio is equal to the first term). One such text is **Ist. O 3826** (Neugebauer *MKT 1* (1935), 77) inscribed with the first 10 powers of 9, followed by 5 powers of 1 40, from the 6th to the 10th power. Another example is **BM 22706**, in Nissen, Damerow, and Englund, *ABk* (1993), 150 (10 powers of 1 40, followed by 10 powers of 5).

The most recently published text of this kind is **IM 73355**, in Arnaud, *TL* (1994) (10 powers of 3 45, followed by 10 powers of 16, the sexagesimal reciprocal of 3 45). See Fig. 1.1.3 below.

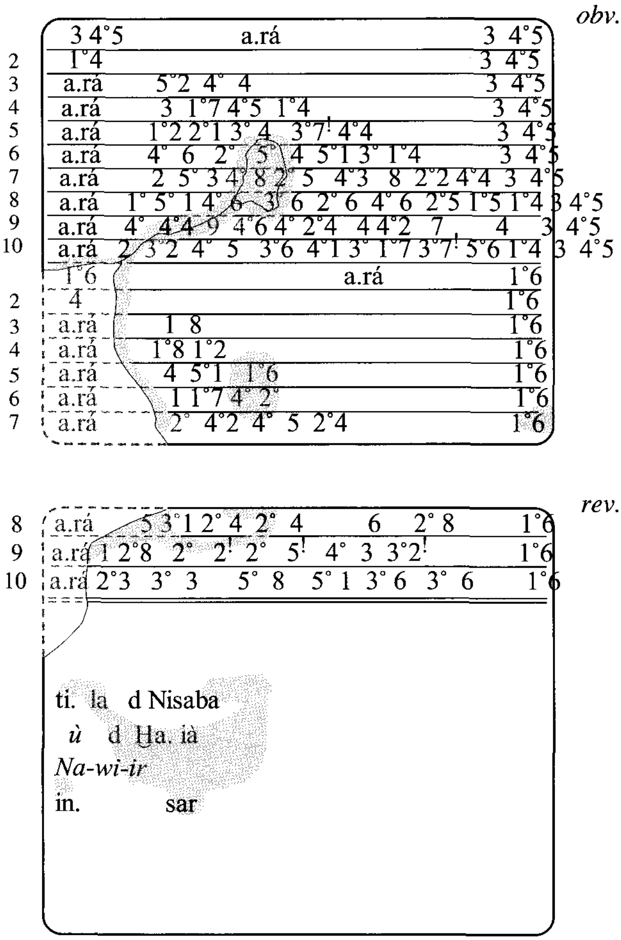


Fig. 1.1.3. IM 73355. Two OB tables of powers, for 3 45 and for its reciprocal 16.

The text of IM 73355 can be translated as:

3 45 times 3 45 /	14	3 45
times	52 44	3 45
times	3 17 45 14	3 45
etc.		

3 45 · 3 45 = 14 03 45
3 45 · 14 03 45 = 52 44 03 45
3 45 · 52 44 03 45 = 3 17 14 03 45
etc.

In the text, a.rá is Sumerian for 'times', literal meaning possibly 'steps'. After the two tables of powers on IM 73355 there follows a brief subscript:

'By the life of (the god) Nisaba and (the god) Haia. Nawir wrote it.'

Note that all the powers of 3 45 are sexagesimal numbers with the "trailing part" 3 45, while all the powers of 16 are numbers with the trailing part 16. For emphasis, all the powers of 3 45 are written with the trailing part 3 45 in each line of the table written close to the right edge of the tablet. All the powers of 16 are written in a similar way, with the trailing part 16 in each line of the table written close to the right edge of the tablet.

There are two known examples of OB clay tablets on which are recorded a finite number of terms of a *descending* geometric progression, both in Friberg, *MCTSC* (2005). One of them is **MS 3037** (*op. cit.*, Fig. 1.4.1), on which is recorded, in descending order, the first 12 powers of 12. The other is **MS 2242** (Fig. 1.1.4 below), with the first 6 powers of 3 45, in descending order. It is likely that these two texts were answers to assignments, namely, to find the factorization of a given many-place regular sexagesimal number through successive elimination, one at a time, of factors visible as the trailing parts of the successively computed sexagesimal numbers. (See the discussion of "regular sexagesimal numbers" and of "the trailing part algorithm" in Friberg, *RlA 7* (1990), Secs. 5.2 b and 5.3.)

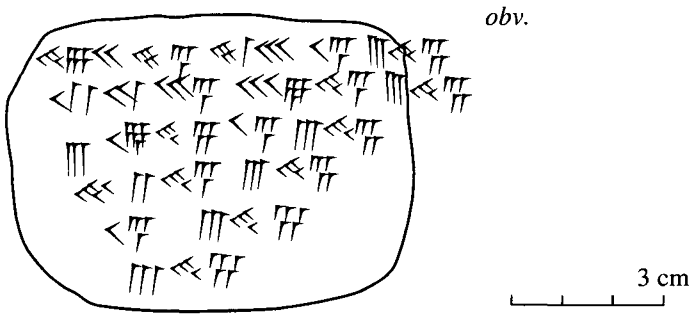


Fig. 1.1.4. MS 2242, *obv.* (*rev.* blank). An OB descending table of six powers of 3 45.

1.1 c. A Late Babylonian text using the trailing part algorithm

In the discussion above of the descending table of powers on the OB clay tablets MS 3037 and MS 2242 it was suggested that the computations in those texts were applications of a certain *trailing part algorithm*. Repeated applications of the same trailing part algorithm can be found also in a *Late Babylonian* (LB) text, the round clay tablet **W 23021** (Friberg, *BaM* 30 (1999), *BaM* 31 (2000)). In W 23021, the trailing part algorithm is used to compute the reciprocals of eight "many-place" regular sexagesimal numbers, all between 52 40 29 37 46 40 and 49 00 07 12.

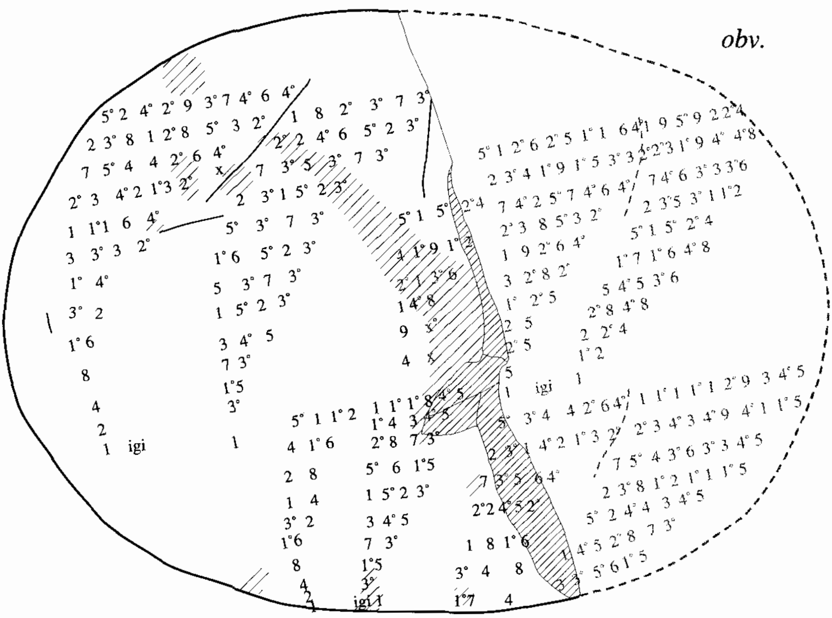


Fig. 1.1.5. W 23021. A LB text with eight applications of the trailing part algorithm.

1.1 d. The sum of a geometric progression in an Old Babylonian text

MS 1844 is a massive OB round hand tablet with what appears to be the numerical solution algorithm for an inheritance problem (Friberg, *MCTSC* (2005), Fig. 7.4.2). The tablet is inscribed with 6 terms of a *decreasing* geometric progression with the *last term* 2, and with the *constant ratio* between the terms equal to $1 - 1/7$. The constant ratio is described with the following words in a somewhat cryptic subscript in the last line of the text:

igi 7.gál.bi tur.šè   for a 7th-part less.

The sum of the geometric progression is inscribed in the first line of the text. It is, for a certain reason, given in the following curious form:

23 15 20 36
12 08 53 20.

There is a numerical error in the number recorded in line 3. It is easy to check that this error is *propagated upwards*, to the numbers recorded in lines 1 and 2. This means that the numbers in the algorithm table were computed *in reverse order*, beginning with the number '2' in line 8.

obv.

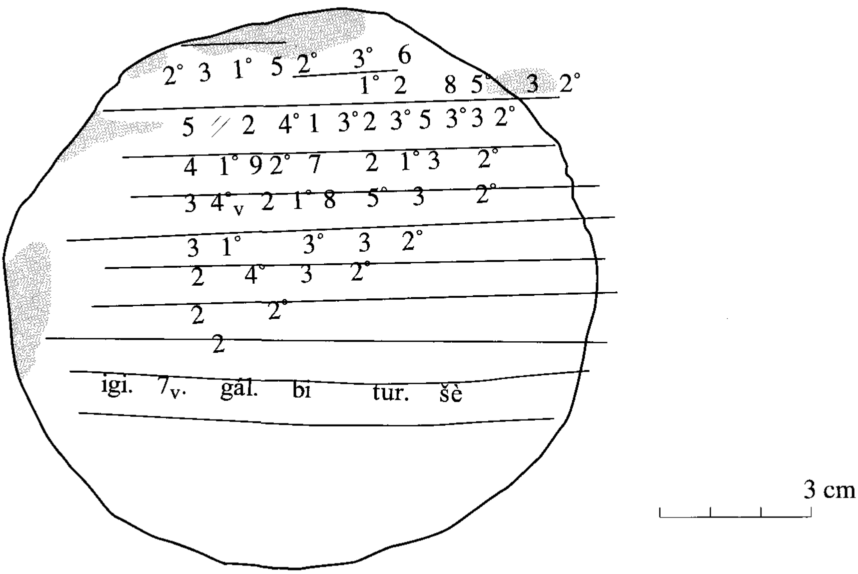


Fig. 1.1.6. MS 1844. An OB computation of the sum of a geometric progression.

The reason for the reverse order of computation is obvious. Since 7 is not a regular sexagesimal number, Babylonian mathematicians would have found it difficult to *count* with a number like $1 - 1/7$. On the other hand, it is known through explicit examples (see Sec. 3.1 e below) that they were familiar with a counting rule of the type

if $b = a - a \cdot 1/7$, then $a = b + b \cdot 1/6$.

The correctness of this counting rule is obvious, at least in the case when a is a multiple of 7. Indeed,

if $a = n \cdot 7$ and $b = a - a \cdot 1/7$, then
$b = n \cdot 7 - n = n \cdot 6$, and $b + b \cdot 1/6 = n \cdot 6 + n = n \cdot 7 = a$.

In view of this simple counting rule, the requirement that each number in the algorithm table shall be equal to the number in the preceding line, diminished by 1/7 of its value, can be replaced by the equivalent requirement that each number in the table shall be equal to number below it, increased by 1/6 of its value. This reformulation of the requirement is a great simplification, since 6 is a regular sexagesimal number with the reciprocal 10. (In modern notations, this means that 1/6 = 10/60.) Therefore, increasing a given number by 1/6 of its value is equivalent to multiplying the given number by 1 10 in Babylonian relative (floating) place value notation, or by 1;10 in absolute place value notation. So it would be easier for the author of the text to count upwards, multiplying with the factor '1 10' in each step of the algorithm, than to count downwards, subtracting seventh parts. That is also precisely what he did.

1.1 e. The sum of a geometric progression in a Seleucid text

AO 6484 (Neugebauer, *MKT 1* (1935), 96) is a mathematical cuneiform text of mixed content from the Seleucid period, the last third of the first millennium BCE. The first exercise in that text shows through an example how to compute the sum of a geometric progression:

AO 6484 # 1

1 ta 1 en 10 gar a.na 2-*ma* bal-*it* gar.gar-*ma* 8 [32 gar]
2 [1 ta 8 32 lá-*ma*] / *re-ḫe* 8 31
8 31 *a-na* 8 32 tab-*ma* 17 03 [···]

From 1 to 10 set, always by 2 surpass, heap (add together), then *8 32 set.*
1 from 8 32 subtract, then 8 31 remains.
The 8 31 to 8 32 repeat (add on), then 17 03 ···.

The phrasing is quite obscure, and the translation above is only tentative. Anyway, what is going on here seems to be that 10 numbers are given. Each number is twice the one before it, and the first number is 1. What is then the sum of the 10 numbers, from 1 to (2^9 = 512 =) 8 32?

This means that the 10 numbers form a geometric progression with the first term 1 and the common ratio 2. How the sum *S* of the 10 terms is computed in the text can possibly be explained (in modern terms) as follows:

$S = 1 + 2 + \cdots + 8\ 32.$
$2 \cdot S = 2 + 4 + \cdots + 8\ 32 + 17\ 04 = S + 17\ 04 - 1.$
$S = 17\ 04 - 1 = 8\ 32 + (8\ 32 - 1).$

1.1 f. *P.Rhind* # 79: a parallel to M. 7857 in a hieratic papyrus

The largest and best known Egyptian hieratic mathematical papyrus is ***P.Rhind***. The only exercise in that text concerned with the sum of a geometric progression is *P.Rhind* # 79. In Fig. 1.1.7 below is shown a copy of the original hieratic text of that exercise, borrowed from Chase, *et al. RMP* (1929), together with a conform transliteration. The conform translation is in the form of a *mirror image* of the hieratic text, which is written from right to left. Moreover, since the decimal numbers in the hieratic text are written with non-positional number notations, it is appropriate to let this be apparent also in the conform translation. The way this is done in Fig. 1.1.7, and in the following, is by use of *subscripts*, 't' for tens, 'h' for hundreds, 'th' for thousands, and 'tth' for ten thousands.

The text in column *ii* of *P.Rhind* # 79 can be explained as follows:

houses	7	7	(= 1 · 7)
cats	$4_t\ 9$	49	(= 7 · 7)
mice	$3_h\ 4_t\ 3$	343	(= 49 · 7)
emmer	$2_{th}\ 3_h\ 1$	2,401!	(= 343 · 7)
heqats	$1_{tth}\ 6_{th}\ 8_h\ 7$	+ 16,807	(= 2,401 · 7)
total	$1_{tth}\ 9_{th}\ 6_h\ 7$	19,607	(= 2,801 · 7)

Column *i* contains the computation 7 · 2,801 = 19,607. In the usual *binary arithmetic* of *P.Rhind*, the product is computed as follows:

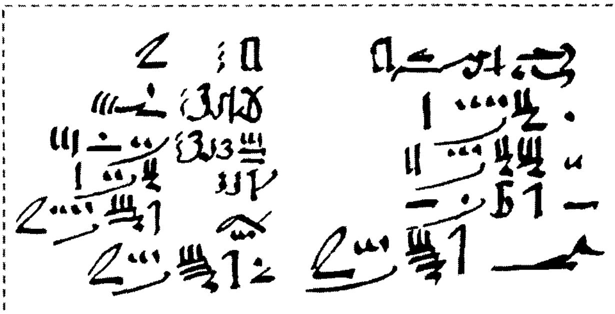
7 · 2,801 = (1 + 2 + 4) · 2,801 =	1	2,801
	2	5,602
	4	+ 11,204
	total	19,607

The computation can be explained as follows:

Presumably, it was known beforehand that $1 + 7 + \cdots + 2{,}401 = 2{,}801$.
Consequently, $7 + \cdots + 16{,}807 = 7 \cdot (1 + 7 + \cdots + 2{,}401) = 7 \cdot 2{,}801 = 19{,}607$.

Essentially, *P.Rhind* # 79 is the computation of the sum of a geometric progression of five terms with the first term 7 and the common ratio 7. However, this computation has been given a fanciful interpretation, reminding very much of the fanciful interpretation in M. 7857 of the sum of another geometric progression. One can try to reconstruct the whimsical story associated with the text of *P.Rhind* # 79. It may have gone like this:

> There were 7 houses, in each house 7 cats, each cat caught 7 mice, each mouse ate 7 bags of emmer, and each bag contained 7 *heqat*. How many were there altogether?



a house inventory (?)	
•	$2_{th}\ 8_h\ 1$
2	$5_{th}\ 6_h\ 2$
4	$1_{tth}\ 1_{th}\ 2_h\ 4$
total:	$1_{tth}\ 9_{th}\ 6_h\ 7$

houses:	7
cats:	$4_t\ 9$
mice:	$3_h\ 4_t\ 3$
ears:	$2_{th}\ 4_h\ 1$
heqat:	$1_{tth}\ 6_{th}\ 8_h\ 7$
total:	$1_{tth}\ 9_{th}\ 6_h\ 7$

Fig. 1.1.7. *P.Rhind* # 79. Col. *i*: The sum, computed as 7 times 2,801.
Col. *ii*: The five terms and their sum.

1.1 g. Summary. The Mesopotamian roots of a Mother Goose riddle

The mathematical cuneiform text M. 7857 from Mari (Fig. 1.1.2) seems to be a crucial link in a chain of related texts that begins with some OB algorithm texts and ends with a nursery rhyme still familiar today. The OB algorithm texts in question are several known examples of tables of powers, either ascending like the one on IM 73355 (Fig. 1.1.3) or descending like the one on MS 2242 (Fig. 1.1.4). In Mesopotamia proper, the last manifestation of a text of this kind is W 23021, a Late Babylonian round clay tablet from Uruk (Fig. 1.1.5) with eight applications of the trailing part algorithm.

In addition to such OB and LB purely *abstract and numerical* tables of powers there is also MS 1844 (Fig. 1.1.6 above), an OB example of "applied" or "practical" mathematics, where a descending geometric progression and its sum are interpreted as the progressively smaller shares of a given sum of silver divided between seven partners or brothers.

In OB Mari, in the north-western periphery of Mesopotamia, the text genre seems to have been transformed into something else, according to the testimony of M. 7857. In that text, which features both ascending and descending geometric progressions, counting with Babylonian sexagesimal numbers is replaced by counting with decimal-sexagesimal or even centesimal numbers. Moreover, what starts out on the obverse of the clay tablet as a no-nonsense abstract computation of a geometric progression with five terms is turned on the reverse into a whimsical computation with *barley-corns, ears, ants, birds, and people*, and a totally nonsensical summation of those five disparate categories.

A similar whimsical interpretation of a geometric progression with five terms can be found in the hieratic *P.Rhind* # 79 (Fig. 1.1.7 above), which even, just like M. 7857, ends with a meaningless summation, this time of *houses, cats, mice, ears, and grains*. Although both the objects counted and the numbers are different in the two exercises, the texts are so similar in spirit that it is inconceivable that they were devised independently.

As is well known, another appearance of the text genre is in Leonardo Pisano's 13th century treatise ***Liber abaci*** (fol. 138 *recto*), in a problem which starts with the words *Septem uetule uadunt romam* '7 old women go to Rome', and ends asking for *summa omnium predictorum* 'the sum of all those mentioned above' (old women, mules, sacks, breads, knives, and sheaths).

For a survey of further appearances of the text genre, see Tropfke, *GE* (1980), Sec. 4.2.4.2.1.

The final reappearance of the same topic, with an added twist, is the ***Mother Goose*** riddle "As I was going to St. Ives, I met a man with seven wives. Each wife had seven sacs, each sack had seven cats, each cat had seven kits. Kits, cats, sacks, and wives, how many were going to St. Ives?"

Note that, just as in *P.Rhind* # 79, both the first term and the common ratio are 7 in the *Liber abaci* problem and in the *Mother Goose* riddle, although the number of terms is not the same in the three texts.

1.1. M. 8631. Another Curious Mathematical Text from OB Mari

1.1 a. M. 8631. A fanciful interpretation of 30 doublings

M. 8631 (Fig. 2.1.1) is a fragment of an OB mathematical text from Mari, published by Soubeyran in *RA* 78 (1984), together with some multiplication tables and a table of squares.

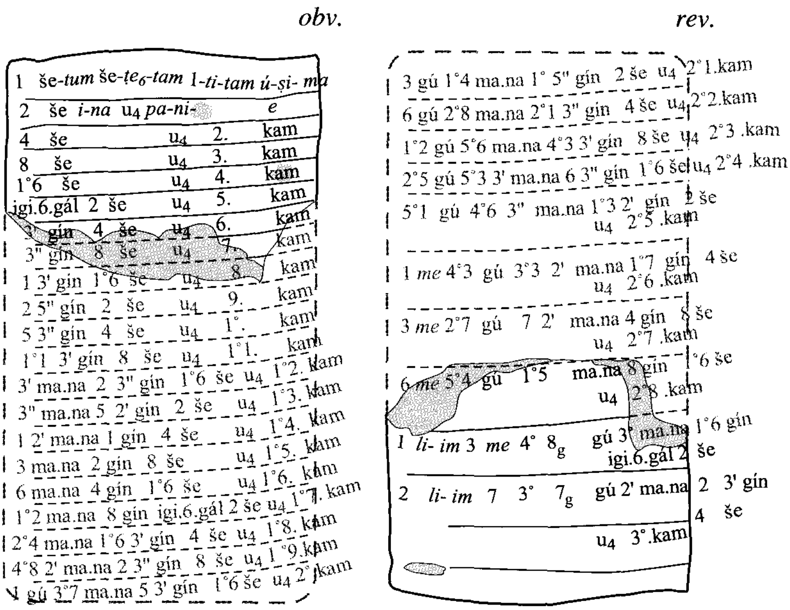


Fig. 1.1.1. M. 8631. An initial capital of 2 barley-corns, doubled 29 times.

The text on M. 8631 begins with the following introductory phrase in line 1 of the obverse:

1 še-*tum* <*a-na*> še-*ṭe*$_6$-*tam* 1-*ti-tam ú-ṣi-ma*

1 barley-corn (to) a single barley-corn I added, then ···

Here barleycorn (Sum. še, Akk. *uṭṭetum*) is written as a Sumerian logogram še followed by an Akkadian phonetic complement *-tum*. The corresponding inflected form is *uṭṭetam*, written as še followed by the Akkadian phonetic complement *-ṭe₆-tam*. Similarly, 1-*ti-tam* is an inflected form (feminine, accusative) of the Akkadian word *ištēnum* 'one', written as the digit 1 (also a Sumerian logogram!) followed by an Akkadian phonetic complement. The Akkadian verb hiding behind the inflected form *ú-ṣi-(ma)* is *(w)aṣābum* 'to add (something) to (something)'.

The meaning of the whole phrase is, as shown by the ensuing text, that an initial capital of 1 barley-corn is doubled each day for a month of 30 days. The barley-corn (appr. 0.05 g) was the smallest unit in the Sumerian/ Old Babylonian system of weight measures. The relation of the barley-corn to the higher units of the system can be described by the following chain of equations:

1 gú (talent, man's-load) = 60 ma.na,
1 ma.na (mina, c. 500 g) = 60 gín,
1/3 gín (shekel) = 60 še (barley-corns).

The curious form of the third equation, which defines the size of the barley-corn, is due to the fact that originally (in the Early Dynastic IIIa period in Mesopotamia in the third millennium BCE) the barley-corns and a unit equal to 60 barley-corns belonged to a system of weights suitable for the weighing of small quantities of some precious metal (silver or gold), while the higher units belonged to another system of weights, suitable for measuring a less valuable metal like copper, or other heavy objects. (See Friberg *JCS* 51 (1999), 133, Friberg, *CDLJ* (2005)). Since at that time silver was 180 times more valuable than copper, 1 barley-corn of silver was worth as much as 1 shekel of copper.

It is likely that the author of M. 8631 started his work by constructing a preliminary table in abstract numbers for a geometric progression of sexagesimal place value numbers with the first term 20 (because 1 barley-corn = 1/3 of 1/60 shekel = ;00 20 shekel) and the common ratio 2.
Here follows first a standard transliteration of the 21 lines of text on the *obverse* of M.8631, with an explanation of how the successive weight numbers may have been computed. Reconstructed parts of the text are as usual within brackets in the transliteration, but in italics in the translation.

M. 8631, ***obv.***

1 še-*tum* še-*ṭe*$_6$-*tam* 1-*ti-tam ú-ṣi-ma* /					
2 še	*i-na* u$_4$ *pa-ni-e* /				
4 še				u$_4$	2.kam /
8 še				u$_4$	3.kam /
16 še				u$_4$	4.kam /
igi.6.gál	2 še			u$_4$	5.kam /
[3'] gín	4 še			u$_4$	6.kam /
[3" gín	8 še			u$_4$]	7.[kam] /
[1 3' gín	16 še			u$_4$	8.kam] /
[2 6" gín	2 še			u$_4$	9.kam] /
[5 3" gín	4 še			u$_4$	10.kam] /
[11 3' gín	8 še			u$_4$	11.kam] /
[3' ma.na		2 3" gín	16 še	u$_4$	12.kam] /
[3" ma.na		5 2' gín	2 še	u$_4$	13.kam] /
[1 2' ma.na		1 gín	4 še	u$_4$	14.kam] /
[3 ma.na		2 gín	8 še	u$_4$	15.kam] /
[6 ma.na		4 gín	16 še	u$_4$	16.kam] /
[12 ma.na 8 gín igi.6.gál			2 še	u$_4$	17.kam] /
[24 ma.na		16 3' gín	4 še	u$_4$	18.kam] /
[48 2' ma.na		2 3" gín	8 še	u$_4$	19.kam] /
[1 gú 37 5/6 ma.na 3' gín			16 še	u$_4$	20.kam] /

1 barley-corn to a single barley-corn I added, then			
2 b.c.	in the first day		(= ;00 40 shekel)
4 b.c.		the 2nd day	(= ;01 20 sh.)
8 b.c.		the 3rd day	(= ;02 40 sh.)
16 b.c.		the 4th day	(= ;05 20 sh.)
1/6 shekel 2 b.c.		the 5th day	(= ;10 40 sh.)
1/3 shekel 4 b.c.		the 6th day	(= ;21 20 sh.)
2/3 shekel 8 b.c.		the 7*th day*	(= ;42 40 sh.)
1 1/3 shekels 16 b.c.		*etc.*	(= 1;25 20 sh.)
2 5/6 shekels 2 b.c.			(= 2;50 40 sh.)
5 2/3 shekels 4 b.c.			(= 5;41 20 sh.)
11 1/3 shekels 8 b.c.			(= 11;22 40 sh.)
1/3 mina 2 2/3 shekels 16 b.c.			(= 22;45 20 sh.)
2/3 minas 5 1/2 shekels 2 b. c			(= 45;30 40 sh.)
1 1/2 minas 1 shekels 4 b.c.			(= 1 31;01 20 sh.)
3 minas 2 shekels 8 b.c.			(= 3 02;02 40 sh.)
6 minas 4 shekels 16 b.c.			(= 6 04;05 20 sh.)
12 minas 8 1/6 shekels 2 b.c.			(= 12 08;10 40 sh.)
24 minas 16 1/3 shekels 4 b.c.			(= 24 16;21 20 sh.)
48 1/2 minas 2 2/3 shekels 8 b.c.			(= 48 32;42 40 sh.)
1 talent 37 minas 5 1/3 shekels 16 b.c.			(= 1 37 05;25 20 sh.)

In the transliteration above, the following notations are used for the four Sumerian/Old Babylonian "basic fractions": 3' (= 1/3), 2' (= 1/2), 3" (= 2/3), and 6" (= 5/6). These notations are intended to turn the readers' attention to the fact that special Sumerian/Old Babylonian cuneiform signs existed only for these fractions. All other fractions were of the form igi *n* 'the opposite of n' (= 1/*n*), *n* being a regular sexagesimal number. (A sexagesimal number *n* is called "regular" if there exists another sexagesimal number *n*', such that *n* and *n*' is a "pair of reciprocals" in the sense that *n* · *n*' = '1' in floating sexagesimal place value notation, that is if *n* · *n*' = a power of 60.) Fractions of the form *n*/*m* do not appear in OB cuneiform texts.

On the reverse of M. 8631 there were, originally, ten more lines for days 21 through 30. In the same way as in the case of the text on the obverse, the computation of the weight numbers in those ten lines on the reverse can be explained as follows, by use of sexagesimal arithmetic:

3 talents	*14 minas*	*10 5/6 sh.*	*2 b.c.*	day 21	(= 3 14 10;50 40 sh.)
6 talents	*28 1/ 3minas*	*1 2/3 sh.*	*4 b.c.*	day 22	(= 6 28 21;41 20 sh.)
12 talents	*56 2/3 minas*	*3 1/3 sh.*	*8 b.c.*	day 23	(= 12 56 43;22 40 sh.)
25 talents	*53 1/3 minas*	*6 2/3 sh.*	*16 b.c.*	day 24	(= 25 53 26;45 20 sh.)
51 talents	*46 5/6 minas*	*3 1/2 sh.*	*2 b.c.*	day 25	(= 51 46 53;30 40 sh.)
1 43 talents	*33 2/3 minas*	*7 sh.*	*4 b.c.*	day 26	(= 1 43 33 47;01 20 sh.)
3 27 talents	*7 1/2 minas*	*4 sh.*	*8 b.c.*	day 27	(= 3 27 07 34;02 40 sh.)
6 54 talents	15 minas	*8 sh.*	*16 b.c.*	day 28	(= 6 54 15 08;05 20 sh.)
13 48 talents	30 *minas*	*16* 1/6 sh.	2 b.c.	day 29	(= 13 48 30 16;10 40 sh.)
27 37 talents	1/2 mina	2 1/3 sh.	4 b.c.	day 30	(= 27 37 00 32;21 20 sh.)

As noticed already in Soubeyran's original publication of M. 8631, there are some surprising notations in the last two lines of the cuneiform text. Thus, the weight numbers recorded for days 29-30 are

1 *li-im* 3 *me* 48_g gú 30 [ma.na 16 gín] / igi.6.gál 2 še
2 *li-im* 7 37_g gú 2' ma.na 2 3' gín / 4 še

(The subscripts in the numbers 48_g and 7 37_g before gú 'talent' are meant to be reminders of the fact that in the OB cuneiform script the digits 1 through 9 are written with *horizontal* wedges when preceding the cuneiform sign for the weight unit gú (or the capacity unit gur), but with *vertical* wedges in most other circumstances.)

What is remarkable here is that in the line for day 29 the weight number '13 48 talents' is written as '1 thousand 3 hundred 48 talents', that is with the OB notation for 'a thousand' used to denote 'ten sixties', and with the

OB sign for 'a hundred' used to represent 'sixty'. Similarly in the line for day 30 the weight number '23 37 talents' is written as '2 thousand 7 37 talents', again with the normal notation for 'a thousand' used to represent 'ten sixties'. It is not clear if these curious deviations from the OB norm were mistakes or if they were intentional. Anyway, just like the use of mixed decimal-sexagesimal and centesimal numbers alongside with sexagesimal numbers in M. 7857, so the use of decimal notation for sexagesimal numbers in M. 8631 testifies that the scribes in Mari were not comfortable with the use of sexagesimal numbers, and that they, possibly for that reason, were experimenting with new ways of writing numbers in cuneiform.

1.2 b. The Old Babylonian doubling and halving algorithm

The 30 weight numbers recorded in the Mari text M. 8631 form a geometric progression of a special kind, 30 successive *doublings* of an initial weight number. Indirect parallels are several well known OB algorithm tables in which an initial regular sexagesimal number is doubled a number of times. The best known example of such a text is **UM 29.13.21** (Neugebauer and Sachs, *MCT* (1945), 13; Friberg, *MCTSC* (2005), App. 3), a small fragment of a large table text from Nippur with several applications of the OB "doubling and halving algorithm" (Sachs, *JCS* 1 (1947), Friberg, *RlA 7* (1990) Sec. 5.3 b). A reconstruction of most of that text is possible. A conform transliteration of the reconstructed text within an outline of the clay tablet is shown in Fig 1.2.2 below.

The *doubling and halving algorithm* is based on the observation that if n and n' are a pair of reciprocals, then $2 \cdot n$ and $1/2 \cdot n'$ are another pair of reciprocals. Successive doublings of n and halvings of n' will produce a never ending sequence of pairs of reciprocals. In the first application of this algorithm in UM 29.13.21, a table with 30 pairs, the first pair is chosen as $n = 2\ 05$ (the 3rd power of 5) and $n' = 28\ 48$ (the 3rd power of 12). The other 29 pairs can be characterized as *29 doublings of the initial number 2 05*, together with the corresponding reciprocals, *29 successive halvings of the initial reciprocal number 28 48*. Thus, the table proceeds from

2 05 / igi.bi 28 48
(2 05, its reciprocal is 28 48)

in line 1 to

1 25 20 X 58 09 11 06 40 / igi.bi 41 42 49 22 21 12 39 22 30
(1 25 20 X 58 09 11 06 40, its reciprocal is 41 42 49 22 21 12 39 22 30)

in line 30. The cuneiform sign represented here by an X is a separation sign, probably introduced in line 20 of the algorithm table as a kind of sign for zero. The scribe's inability to handle this zero correctly in lines 21-30 led to an error in line 30, where the number written 1 25 20 X 58 09 11 06 40 should rightly be 1 25 20+58 09 11 06 40 = 1 26 18 09 11 06 40.

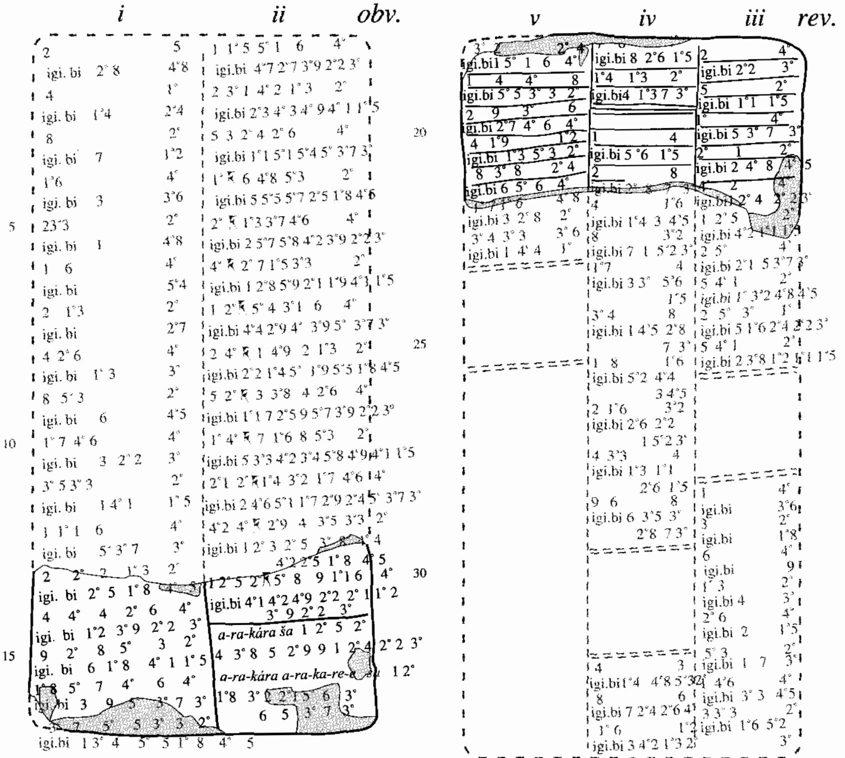


Fig. 1.2.2. UM 29.13.21. Five applications of the OB doubling and halving algorithm.

1.2 c. P.IFAO 88: A parallel to M. 8631 in a Greek-Egyptian papyrus

P.IFAO 88 is a Greek-Egyptian papyrus fragment of unknown date and origin (IFAO = *Institut Français d'Archéologie Orientale du Caire*), published by Boyaval in *ZPE* 7 (1971), in the form of a murky photograph accompanied by an only partly successful interpretation. Improved interpretations were presented later, by Rea in *ZPE* 8 (1971), and by Boyaval himself in *ZPE* 14 (1974).

The inscription on *P.IFAO 88* consists of Greek alphabetic number signs for decimal numbers, in the usual Greek non-positional decimal number notation, with α, β, γ, ··· for 1, 2, 3, ···, ι, κ, λ, ··· for 10, 20, 30, ···, ρ, σ, τ, ··· for 100, 200, 300, ··· , and α´, β´ γ´, ··· for 1,000, 2,000, 3,000, ··· . From line 14 on, numbers smaller than 6,000 are preceded by a special sign, presumably standing for *drachma*, a small Greek unit of weight. Referring to a suggestion by Youtie, Boyaval (*op. cit.*, 1974) reads the sign as χ^{α}, meaning χα(λκου) 'copper (drachma)'. Multiples of 6,000 are counted separately and are preceded by another special sign, known to represent the *talent*. Customarily, 1 *talent* = 6,000 *drachmas*.

The use of the *talent* and the copper *drachma* as monetary units is a clue to the date of the text. Those units were used in Egypt in the Ptolemaic and early Roman periods. Boyaval (*op. cit.*, 1974) dates the text to not later than the 1st century CE.

In this connection it may be worth mentioning that Greek multiplication tables for fractions, such as, for instance[9], those on the first three pages of the mathematical papyrus codex ***P.Akhmim*** (Baillet, *PMA* (1892); 7th c. CE; Sec. 4.5 below), or on the obverse of the mathematical wooden tablet ***Michael. 62*** (Crawford, *Aeg.* 33 (1953); 6th? c. CE; Sec. 4.6 below), begin by listing the fraction in question applied to 6,000, 'the number' (ἀριθμός), as in the following examples:

3" to 'the number'	4,000	3' to 'the number'	2,000
of 1 the 3"	3"	of 1 the 3'	3'
of 2	1 3'	of 2	3"
of 3	2	of 3	1
... ...	... ...	... ...	... ...
of 1 (myriad)	6,666 3"	of 1 (myriad)	3,333 3'.

About this, Crawford (*op. cit.*, 227) has the following to say:

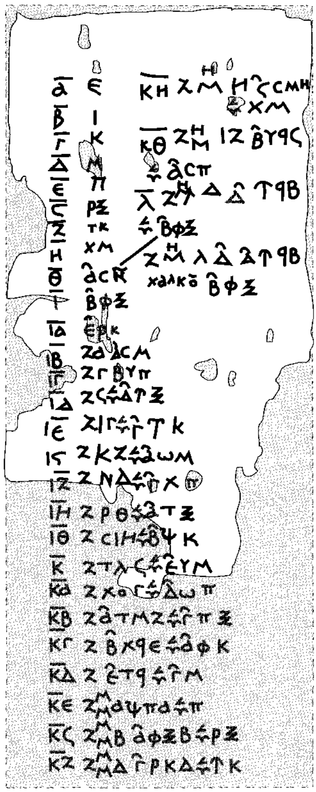
> "The number 6,000, the fraction of which figures in the first or second place in all versions of the tables, can only be, as has been recognized, the number of *drachmae* in the *talent*. The implication is that the tradition of starting the tables in this way dates from a time when money was always counted in *drachmae* and *talents*; in fact it must have been firmly established before the introduction of the gold-standard

9. Cf. the catalogue of published tables in Fowler, *MPA* (1987; 1999), Sec. 7.5.

coinage. Now when this tablet (*Michael. 62*) and P.Akhmim were written and probably even when P.Mich. 621 was written (if it is the 4th Century), monetary values were normally reckoned in *nomismatia* and *keratia* ··· ···. We may therefore be confident that tables in this particular form go back at least to the 3rd Century A. D. They may of course be much earlier still."

Thus, in *Michael. 62*, which contains multiplication tables on the obverse beginning with fractions of 6,000 (*drachmas*), the problem texts on the reverse nevertheless count in terms of *nomismatia* and *keratia*. (See the text of *Michael. 62* # 2 in Sec. 4.6 below.)

		should be
1. 5	28. T M 8 6,248	(T M **11 1,848**)
2. 10	*dr.* 640	
3. 20	29. T M 17 2,496	(T M **22 3,696**)
4. *40*	*dr.* 1,280	
5. 80		
6. 160	30. T M *34* 4,992	(T M **44 7,392**)
7. 320	dr. 2,560	
8. 640	T M 34 4,992	(T M **44 7,392**)
9. 1,280	copper 2,560	
10. 2,560		
11. *5,120*		
12. T 1 4,240		
13. T 3 2,480		
14. T 6 dr. 4,960		
15. T 13 *dr.* 3,920		
16. T 27 dr. 1,840		
17. T 54 dr. *3,680*		
18. T 109 dr. 1,360		
19. T 218 dr. 2,720		should be
20. T 336 dr. 5,440		(T 436, etc.)
21. T 673 dr. 4,880		(T **8** 73, etc.)
22. T 1,347 dr. 3,760		(T 1747, etc.)
23. T 2,695 dr. 1,520		(T **3,495**, etc.)
24. T 5,390 dr. 3,040		(T **6,990**, etc.)
25. T M 1 781 dr. 80		(T M **1 3,9**81, etc.)
26. T M 2 1,562 dr. 160		(T M 2 **7,9**62, etc.)
27. T M 4 3,124 dr. 320		(T M **5 5,9**24, etc.)



α	β	γ	δ	ε	ϛ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	ο	π	ϙ	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900

∸ dr. (drachmas)?   Ϫ T (talents = 6,000 drachmas) $\overset{H}{M}$ M (myriads = 10,000)

Fig. 1.2.3. *P.IFAO 88.* An initial capital of 5 copper drachmas, doubled 29 times.

The lines of *P.IFAO 88* are numbered, making it easy to determine the original extent of the text. Thus, the line numbering begins with $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ (= 1, 2, 3) in the left column and continues with $\overline{\kappa\eta}$, $\overline{\kappa\theta}$, $\overline{\lambda}$ (= 28, 29, 30) in the right column. The end result in line 30 is repeated in a final line without line number.

It is easy to see that the recorded numbers form a geometric progression of 30 terms, with the first term equal to 5 (*copper drachmas*) and with the common ratio 2. In other words, the geometric progression has been given an interpretation as an initial weight of 5 *copper drachmas*, doubled 29 times. The regular form of a geometric progression makes it easy to reconstruct the missing parts of the text, as shown in Fig. 1.2.3.

What makes the text particularly interesting is that there seems to be a trivial error in the non-preserved line 20, namely 336 talents instead of 436 talents, possibly due to the fact that the two Greek letters T = 300 and Y = 400 can be mistaken for each other. The effect of the error is avalanching through lines 21-30. The end result, in line 30, is given as

> 34 myriad 4,992 talents 2,560 drachmas (=344,992 talents 2,560 drachmas = 2,069,954,560 drachmas)

(Remember that 1 myriad = a hundred hundred = 10,000, 1 talent = 6,000 drachmas.) The correct result should be $2^{10} \cdot 100$ talents = 10 myriads 2,400 talents more than that, or altogether

> 44 myriad 7,392 talents 2,560 drachmas (= 447,392 talents 2,560 drachmas = 2,684,354,560 drachmas).

Note: In the hand copy above of *P.IFAO 88*, reconstructed parts of the text are grey. In the transliteration reconstructed parts are written in italics.

1.2 d. Summary. The Mesopotamian roots of a well known legend

Much like M. 7857, the Mari text M. 8631 (Fig. 1.2.1) seems to be a crucial link in a chain of related texts beginning with an important category of OB algorithm tables and possibly ending with a legend that is still well known today. The OB texts in question are various applications of doubling or doubling-and-halving algorithms, notably the fragment UM. 29.13.21 (Fig. 1.2.2) from Nippur (see the map in Fig.1.1.1) which begins with 30 doublings and halvings of the pair of reciprocals (2 05, 28 48).

In M. 8631, a table of 30 doublings was imaginatively reinterpreted as the growth of an initially given capital (1 barley-corn) over a month of 30 days, with a daily doubling of the capital.[10]

Then follows a large gap in the supposed chain of related texts, because there is no known hieratic or even demotic Egyptian text mentioning 30 doublings. Instead, there is the Greek-Egyptian *P.IFAO 88* (Fig. 1.2.3) with its 30 lines of doublings of an initial amount of 5 drachmas, where the final amount is expressed in terms of myriads of talents. There is no explicit mention of days in that text, only line numbers. Note, by the way, that the Sumerian/Old Babylonian talent of 60 · 60 shekels (60 shekels = 1 mina) clearly is the ancestor of the Greek talent of 60 · 100 drachmas (100 drachmas = 1 mina). Hence, the assumption of a connection between a Greek-Egyptian papyrus and a clay tablet from OB Mari is not as far-fetched as it may seem at first sight. (The circumstance that the Greek 'myriad' = 100 · 100 is a decimal number unit formed in the same way as the Mari 'great' is interesting but may have no historical significance.)

Another appearance of the theme is in the form of the legend about the reward granted by some Indian king to the inventor of the game of chess, who demanded one grain of rice on the first square of a chess board and then twice as much on each consecutive square.[11] It may be noted, in this connection, that the ancient Egyptian game of Senet had a game board with 10 · 3 = 30 squares. So, maybe, the missing link in the chain is a hieratic mathematical text that simply has not happened to be preserved.

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10. Note, for comparison, that the standard interest rate in OB mathematical texts was 1/5 or 1/3 of the capital, not per day but per year or per transaction. See, for instance, the mixed theme text **YBC 4698** (Fig. 2.1.17) § 1 (## 1-2), where the interest on 1 gur of barley (1 gur = 5 00 sìla = appr. 300 liters) is given as 1 barig (= 1 00 sìla) in the first example, and as 1 barig 4 bán (= 1 40 sìla) in the second example. However, there are no known OB "interest tables" for the growth of a capital, based on standard interest rates. On the other hand, there is a known Neo-Sumerian table for the regular growth of a herd of cows and bulls during a period of 10 years (**AO 5499**: see Nissen/Damerow/Englund, *ABk* (1993), Figs. 76-79).

11. A variant of this legend, mentioned by Rea in *ZPE* 8 (1971), is "the tale of the crafty blacksmith who offered to shoe the king's horse at one penny for the first nail, twopence for the second, fourpence for the third, and so on till all the nails were in. Since a horse needs twenty-eight or thirty (!) nails to keep its shoes in place, the result was a formidable accumulation". For an extensive survey of appearances of the theme, see Tropfke *GE* (1980), Sec. 4.2.4.2.2.

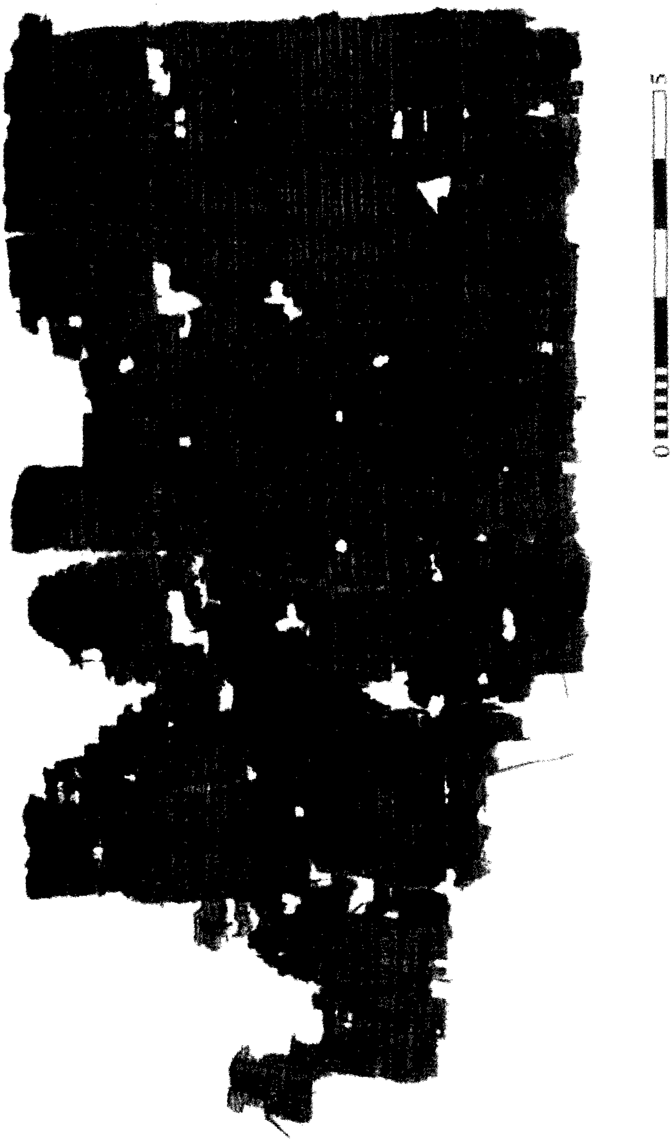

Fig. 1.2.4. *P. IFAO 88*.