Chapter 1

Basic Equations for Electromagnetic Fields

Maxwell's equations, and the integral theorems derived from them, with appropriate boundary conditions, constitute the basic laws underlining propagation and scattering of light and other electromagnetic waves.

In this chapter we briefly review Maxwell's equations, as well as the wave equations that govern the propagation, scattering, diffraction and radiation of electromagnetic fields in optics, and in general in electromagnetic theory. We shall also quote the conservation laws and discuss the arbitrariness in the definition of density of flow of energy. Further, we shall discuss certain fundamental theorems that constitute the basis of modern theories of wave propagation, such as the integral theorem of Helmholtz and Kirchhoff and the extinction theorem, together with the integro-differential equations obtained by means of the Hertz vectors.

1.1. Maxwell's Equations

Electromagnetic fields are characterized by their *electric vector* **E** and *magnetic vector* **H**. In material media, the response to the excitation produced by these fields is described by the *electric displacement* **D** and the *magnetic induction* **B**. These quantities satisfy Maxwell's equations, which in Gaussian units are:

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \tag{1.1a}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.1b}$$

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{1.1c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.1d}$$

where c is the velocity of light in vacuum, and $\mathbf{j}$ and ρ denote the *electric current density* and the *charge density*, respectively. Equations (1.1) imply the conservation of charge by means of the equation of continuity:

$$\boldsymbol{\nabla} \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}.$$

The vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$ are related by the expressions:

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}, \tag{1.2a}$$

$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}, \tag{1.2b}$$

where $\mathbf{P}$ and $\mathbf{M}$ are the *polarization* and *magnetization* vectors, respectively. They characterize the effect of the interaction of the electromagnetic field with the material. The connection of these quantities with the electric and magnetic vectors is given by the so-called *constitutive relations*, which in general are nonlinear equations; however, for most media, and when the fields are not too intense, they reduce to the linear relationships:

$$\mathbf{P} = \chi \mathbf{E}, \tag{1.3a}$$

$$\mathbf{M} = \eta \mathbf{H}, \tag{1.3b}$$

χ and η being scalar quantities that are called *dielectric* and *magnetic susceptibility*, respectively.

From Eqs. (1.2) and (1.3) a direct relation between $\mathbf{D}$ and $\mathbf{E}$, and between $\mathbf{B}$ and $\mathbf{H}$ is obtained, namely:

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{1.4a}$$

$$\mathbf{B} = \mu \mathbf{H}; \tag{1.4b}$$

where the quantity ε, called *dielectric permittivity*, is defined by:

$$\varepsilon = 1 + 4\pi\chi;$$

and the *magnetic permeability* μ is given by:

$$\mu = 1 + 4\pi\eta.$$

In addition to Eqs. (1.4), the electric current density $\mathbf{j}$ is related to the electric field by:

$$\mathbf{j} = \sigma_c \mathbf{E}, \tag{1.5}$$

σ_c being the *specific conductivity* of the material. In many problems encountered in optics and electromagnetic theory one has to solve Maxwell's equations in regions where there exist interfaces separating different media. Across an interface the electromagnetic field satisfies certain *continuity* or *saltus conditions*; namely, let $\mathbf{n}$ be the local unit normal to the surface separating two media 1 and 2, and pointing from region 1 into region 2, these conditions are:

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \tag{1.6a}$$

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \frac{4\pi}{c}\mathbf{J}, \tag{1.6b}$$

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 4\pi\sigma, \tag{1.6c}$$

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0. \tag{1.6d}$$

Equations (1.6) express the continuity of the tangential component of $\mathbf{E}$ and of the normal component of $\mathbf{B}$ across the interface, as well as the discontinuity of both the tangential component of $\mathbf{H}$ and the normal component of $\mathbf{D}$. In these equations the subscripts 1 and 2 are to be understood as the limiting values of the fields as the interface is approached from mediums 1 and 2, respectively. $\mathbf{J}$ and σ represent the electric current density and the charge density at the interface, respectively; they are always zero, except when one of the media is a perfect conductor.

1.2. Wave Equations

By taking the curl in Eqs. (1.1a) and (1.1b) and eliminating $\mathbf{D}$ and $\mathbf{B}$ by means of Eqs. (1.2), one easily obtains the following wave equations that satisfy the time dependent fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$:

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{4\pi}{c}\frac{\partial}{\partial t}\left[\frac{1}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\right], \tag{1.7a}$$

$$\nabla \times \nabla \times \mathbf{H} + \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \frac{4\pi}{c}\left[\nabla \times \mathbf{j} + \nabla \times \frac{\partial \mathbf{P}}{\partial t} - \frac{1}{c}\frac{\partial^2 \mathbf{M}}{\partial t^2}\right]. \tag{1.7b}$$

For monochromatic fields all vector quantities appearing in Eqs. (1.7) have a time dependence factor $\exp(-i\omega t)$, ω being the *frequency*. It is straightforward to obtain from Eqs. (1.7) the following wave equations for the time independent parts in this case:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = \mathbf{F}_e(\mathbf{r}), \tag{1.8a}$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2\mathbf{H}(\mathbf{r}) = \mathbf{F}_m(\mathbf{r}); \tag{1.8b}$$

where we have defined the wavenumber k:

$$k = \frac{\omega}{c}. \tag{1.9}$$

Equations (1.8) are vector inhomogeneous Helmholtz equations. The right-hand sides $\mathbf{F}_e$ and $\mathbf{F}_m$ play the role of source terms characterizing the generation of electromagnetic waves. These terms are:

$$\mathbf{F}_e(\mathbf{r}) = 4\pi \left[\frac{ik}{c}\mathbf{j}(\mathbf{r}) + k^2\mathbf{P}(\mathbf{r}) + ik\boldsymbol{\nabla} \times \mathbf{M}(\mathbf{r})\right], \tag{1.10a}$$

$$\mathbf{F}_m(\mathbf{r}) = 4\pi \left[\frac{1}{c}\boldsymbol{\nabla} \times \mathbf{j}(\mathbf{r}) - ik\boldsymbol{\nabla} \times \mathbf{P}(\mathbf{r}) + k^2\mathbf{M}(\mathbf{r})\right]. \tag{1.10b}$$

It should be emphasized that, although it is convenient to represent the fields by complex quantities for calculation purposes, only their real parts constitute the actual physical fields.

Often the time average of the product of two quantities is needed. It is straightforward to see that if they have harmonic time dependence, namely, if two quantities $A(\mathbf{r}, t)$ and $B(\mathbf{r}, t)$ can be expressed as:

$$A(\mathbf{r}, t) = A(\mathbf{r}) \exp(-i\omega t), \tag{1.11a}$$

$$B(\mathbf{r}, t) = B(\mathbf{r}) \exp(-i\omega t), \tag{1.11b}$$

then the time average $\overline{(\mathrm{Re}\ A(\mathbf{r}, t))(\mathrm{Re}\ B(\mathbf{r}, t))}$ is:

$$\overline{(\mathrm{Re}\ A(\mathbf{r}, t))(\mathrm{Re}\ B(\mathbf{r}, t))} = \frac{1}{2}\mathrm{Re}[A(\mathbf{r})B^*(\mathbf{r})]. \tag{1.12}$$

(Re stands for "real part.")

In deriving the relation (1.12) we have used the following results: $\overline{\cos^2 \omega t} = \overline{\sin^2 \omega t} = \frac{1}{2}$; $\overline{\sin \omega t \cos \omega t} = 0$.

1.3. Conservation Laws

By recalling the vector identity:

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}), \tag{1.13}$$

and making use of Eqs. (1.1a) and (1.1b), one easily obtains:

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) + \frac{1}{c}\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{1}{c}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = -\frac{4\pi}{c}\mathbf{j} \cdot \mathbf{E}. \tag{1.14}$$

By integrating Eq. (1.14) in an arbitrary volume V and applying the Gauss theorem, we obtain:

$$\frac{c}{4\pi}\int_S (\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}\,ds + \frac{1}{8\pi}\frac{\partial}{\partial t}\int_V (\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})dv = -\int_V \mathbf{j}\cdot\mathbf{E}\,dv, \quad (1.15)$$

where S is the surface enclosing the volume V and $\mathbf{n}$ is the unit outward normal.

Equation (1.15) is the relation that governs the balance of *electromagnetic energy*. The vector $\mathscr{S}$:

$$\mathscr{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) \quad (1.16)$$

is the *Poynting vector* and represents the *density of flow of energy*. The quantity:

$$\mathscr{W} = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}) \quad (1.17)$$

is the *density of electromagnetic energy* inside the volume V. The *electric* and *magnetic energies* are given by the first and second terms of Eq. (1.17), respectively. On the other hand, the right-hand side of Eq. (1.15) constitutes the *dissipation of energy* due to Joule heating.

The density of flow of energy $\mathscr{S}$ in Eq. (1.15) can be, however, defined with certain freedom. It is evident from the first term of this equation that any other quantity obtained by adding to $\mathscr{S}$ the *curl* of an arbitrary vector, also satisfies Eq. (1.15) (this fact has been remarked in many textbooks, e.g. Ref. 1.1 or 1.2). This has given rise to alternative definitions of density of flow of energy by choosing appropriate Lagrangians (in this connection, see for instance, Ref. 1.3 or 1.4). The observable quantity is however the *time average* flow of energy integrated over a finite region.

Due to the high frequency of optical signals, detection is always done by averaging the flow of energy over a time interval. According to what was said in Sec. 1.2, and to Eq. (1.12), the *time averaged* Poynting vector is:

$$\bar{\mathscr{S}}(\mathbf{r}) = \frac{c}{8\pi}\mathrm{Re}[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})]. \quad (1.18)$$

It is customary to identify the modulus of the vector $\bar{\mathscr{S}}$ with the *optical intensity*.

Using the constitutive relations (1.4), the *time averaged* density of electromagnetic energy may be written as:

$$\bar{\mathscr{W}}(\mathbf{r}) = \frac{1}{16\pi}[\varepsilon\mathbf{E}(\mathbf{r})\cdot\mathbf{E}^*(\mathbf{r}) + \mu\mathbf{H}(\mathbf{r})\cdot\mathbf{H}^*(\mathbf{r})]. \tag{1.19}$$

1.4. Scalar Theory of Optical Problems

The ambiguity in the definition for the density of flow of energy is even deeper than the discussion in Sec. 1.3 reveals, and also exists for the density of electromagnetic energy. Green and Wolf [1.5] obtained alternative expressions for these quantities in terms of a single complex scalar function, leaving the vectors **E** and **H** unchanged.

Diffraction and scattering problems in optics and electromagnetics are often approached by means of a scalar wavefield, or *complex disturbance* $U(\mathbf{r}, t)$, that satisfies the scalar wave equation:

$$\nabla^2 U(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(\mathbf{r}, t) = -4\pi\rho(\mathbf{r}, t), \tag{1.20}$$

$\rho(\mathbf{r}, t)$ representing a source distribution of the wavefield. For monochromatic fields, in a similar manner as discussed in Sec. 1.2, the spatial part of this wavefunction, $U(\mathbf{r})$, satisfies the inhomogeneous Helmholtz equation:

$$\nabla^2 U(\mathbf{r}) + k^2 U(\mathbf{r}) = -4\pi\rho(\mathbf{r}), \tag{1.21}$$

where $\rho(\mathbf{r}, t) = \rho(\mathbf{r})\exp(-i\omega t)$.

However, the meaning and origin of the wavefunction U that approximates the actual electromagnetic field by a scalar quantity are not explained. In problems in which no depolarization takes place, like two-dimensional problems with the electric vector transversal to the plane in which propagation is considered, or when no multiple interactions occur in the scattering or diffraction process, it is possible to characterize the electromagnetic field by a scalar which corresponds to one transversal component of either the electric or the magnetic vector. However, this is not the only case in which a scalar theory has been employed; even then, it is not obvious why the usual definition of intensity:

$$I(\mathbf{r}) = |U(\mathbf{r})|^2$$

agrees with experiments, and hence, why it is in accordance with the time averaged Poynting vector defined in Eq. (1.18).

Green and Wolf [1.5] have pointed out the freedom in the definition of the density of flow of energy and density of electromagnetic energy, and the corresponding invariance of the observable quantities; namely, of the total flow and energy over finite regions. They introduced a definition for those densities in free space in terms of a single scalar complex wavefunction $U(\mathbf{r}, t)$ by expressions analogous to those for the probability current and the probability density in quantum mechanics:

$$\mathscr{S}(\mathbf{r}, t) = -\frac{1}{8\pi}\left(\frac{\partial U^*}{\partial t}\nabla U + \frac{\partial U}{\partial t}\nabla U^*\right), \tag{1.22a}$$

$$\mathscr{W}(\mathbf{r}, t) = \frac{1}{8\pi}\left(\frac{1}{c^2}\frac{\partial U^*}{\partial t}\frac{\partial U}{\partial t} + \nabla U \cdot \nabla U^*\right), \tag{1.22b}$$

where the function $U(\mathbf{r}, t)$ satisfies the homogeneous wave equation associated with Eq. (1.20) (namely, with $\rho(\mathbf{r}, t) = 0$). This theory was subsequently extended by Roman [1.6] to regions with sources. Other kinds of representations by scalar functions were also considered by Whittaker [1.7].

The definitions (1.22) were shown later by Wolf [1.8] to justify rigorously the use of a scalar theory under a wide variety of conditions encountered in optical systems, provided the wavefunction $U(\mathbf{r}, t)$ of Eqs. (1.22) was identified with the optical wavefield, and its spatial part $U(\mathbf{r})$ was used to define the optical intensity as $I(\mathbf{r}) = C|U(\mathbf{r})|^2$, with C being a constant, thus placing the scalar theory of optics in a formal basis. In this connection, an interesting discussion can also be found in Chapter 7 of the book by Marathay [1.9], where the use of the scalar theory is analyzed by illustrating the interpretation of the well-known Wiener's experiment [1.10]–[1.12] on standing waves formed by the interference between an incident and a reflected wave at a metallic mirror.

1.5. Lorentz's Reciprocity Theorem

Let us consider monochromatic electromagnetic fields in an isotropic, time independent medium. From Eqs. (1.1a) and (1.1b) and the constitutive relations, one has for the spatial parts of the field vectors one solution that satisfies:

$$\boldsymbol{\nabla} \times \mathbf{H}_1 + ik\varepsilon\mathbf{E}_1 = \frac{4\pi}{c}\sigma_c\mathbf{E}_1, \tag{1.23a}$$

$$\boldsymbol{\nabla} \times \mathbf{E}_1 - ik\mu\mathbf{H}_1 = 0. \tag{1.23b}$$

Let us take the scalar product of (1.23a) and (1.23b) with the vectors $\mathbf{E}_2$ and $\mathbf{H}_2$ of another solution, respectively. Conversely, let us write equations like (1.23) for $\mathbf{E}_2$ and $\mathbf{H}_2$ and perform the scalar product with $\mathbf{E}_1$ and $\mathbf{H}_1$, respectively. Making use of the identity (1.13) one obtains in a straightforward manner:

$$\boldsymbol{\nabla} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{H}_2 \cdot (\boldsymbol{\nabla} \times \mathbf{E}_1) + \mathbf{E}_2 \cdot (\boldsymbol{\nabla} \times \mathbf{H}_1) - [\mathbf{E}_1 \cdot (\boldsymbol{\nabla} \times \mathbf{H}_2) + \mathbf{H}_1 \cdot (\boldsymbol{\nabla} \times \mathbf{E}_2)], \quad (1.24)$$

which is easy to see, from the first pair of Maxwell's equations, becomes:

$$\boldsymbol{\nabla} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0. \quad (1.25)$$

On making use of the Gauss theorem, Eq. (1.25) becomes:

$$\int_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{n}\, dS = 0. \quad (1.26)$$

S being a closed surface and $\mathbf{n}$ its unit outward local normal. Equation (1.26) constitutes *Lorentz's reciprocity theorem* and, as will be seen in Chapter 5, it forms the basis of interesting reciprocity relations in scattering theory.

1.6. Integral Equations for the Electromagnetic Field. The Extinction Theorem

The solution to the wave equations with appropriate boundary conditions in optics and electromagnetics is, in many cases, successfully sought from integral theorems that are consequences of Maxwell's equations. These integral equations also contain physical insights about the phenomenon under study.

In this section we shall discuss the *extinction theorem* and other laws such as the *Helmholtz–Kirchhoff integral theorem*. The extinction theorem was originally investigated by Ewald [1.13] and Oseen [1.14] in the domain of molecular optics; an account in this context may be found in Chapter 2 of the book by Born and Wolf [1.2]. However, in recent years it has been shown that this theorem can be obtained in the framework of Maxwell's equations due to the studies of Sein [1.15], Pattanayak [1.16], Pattanayak and Wolf [1.16], [1.18], de Goede and Mazur [1.19]. On the other hand, Waterman [1.20] derived independently a theorem from the wave equations, which is known as the *extended boundary condition*. Later, Agarwal [1.21] showed that both theorems are in fact two different forms of writing the same law.

The extinction theorem essentially describes the suppression inside a material medium of the field incident from vacuum, and its replacement by another field radiated by the dipoles induced in the medium. In addition, Pattanayak and Wolf [1.16]–[1.18] proved that this formula can be used as a nonlocal boundary condition in diffraction and scattering problems.

The Helmholtz–Kirchhoff integral theorem and related expressions form the basis of diffraction and scattering formulations. In this section we shall outline these integral theorems. We shall use the notation used by Wolf [1.17]. The details of the derivation can be found in Wolf's tutorial paper which is reproduced in Appendix 1.1 for convenience of the reader.

1.6.1. *The vector form of Green's theorem*

We shall introduce the dyadic Green function $\mathscr{G}(\mathbf{r},\mathbf{r}')$, which is defined by the expression:

$$\mathscr{G}(\mathbf{r},\mathbf{r}') = \left(\mathscr{I} + \frac{1}{k^2}\nabla\nabla\right) G(\mathbf{r},\mathbf{r}'), \tag{1.27}$$

$\mathscr{I}$ being the unit dyadic. $G(\mathbf{r},\mathbf{r}')$ is the scalar Green function of the Helmholtz equation (1.21) represented by the outgoing spherical wave:

$$G(\mathbf{r},\mathbf{r}') = \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.28}$$

which, as is well known (see for instance Chapter 7 of Ref. 1.22), satisfies the equation:

$$(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \tag{1.29}$$

δ being Dirac's delta function. $G(\mathbf{r},\mathbf{r}')$ also fulfils *Sommerfeld's radiation condition*:

$$\lim_{r\to\infty} r\left(\frac{\partial G}{\partial r} - ikG\right) = 0, \tag{1.30}$$

where $r = |\mathbf{r}|$.

By means of Eqs. (1.27) and (1.29) it is straightforward to prove that $\mathscr{G}(\mathbf{r},\mathbf{r}')$ satisfies:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathscr{G}(\mathbf{r},\mathbf{r}') - k^2\mathscr{G}(\mathbf{r},\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}')\mathscr{I}; \tag{1.31}$$

$\mathscr{G}$ also holds the *vector form* of *Sommerfeld's radiation condition*:

$$\lim_{r\to\infty} r(\nabla \times \mathscr{G} - ik\hat{\mathbf{r}} \times \mathscr{G}) = 0, \tag{1.32}$$

$\hat{\mathbf{r}}$ being the unit vector in the direction of $\mathbf{r}$.

The vector form of *Green's theorem* reads (see e.g. Chapter 13 of Ref. 1.22):

Let $\mathbf{P}$ and $\mathbf{Q}$ be two vector functions of position that are well behaved, together with their first and second derivatives, then:

$$\int_V (\mathbf{Q} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{P} - \mathbf{P} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{Q})dv$$
$$= \int_S (\mathbf{P} \times \boldsymbol{\nabla} \times \mathbf{Q} - \mathbf{Q} \times \boldsymbol{\nabla} \times \mathbf{P}) \cdot \mathbf{n}\, dS, \tag{1.33}$$

V being a certain volume, S denoting the surface enclosing it and $\mathbf{n}$ being the unit outward normal.

1.6.2. *Integral theorems*

Let us consider a material medium occupying a volume V limited by a surface S, $\mathbf{r}_<$ and $\mathbf{r}_>$ denoting the position vectors of a generic point inside and outside V, respectively.

Applying Eq. (1.33) to $\mathbf{P} = \mathscr{G}(\mathbf{r}, \mathbf{r}')\mathbf{C}$ ($\mathbf{C}$ being a constant vector) and to $\mathbf{Q} = \mathbf{E}(\mathbf{r})$, one obtains, depending on whether $\mathbf{r}$ and $\mathbf{r}'$ are inside or outside V:

$$\mathbf{E}(\mathbf{r}_<) = \frac{1}{4\pi}\int_V \mathbf{F}e(\mathbf{r}') \cdot \mathscr{G}(\mathbf{r}_<, \mathbf{r}')d^3r' - \frac{1}{4\pi}\sum\nolimits_e^{(-)}(\mathbf{r}_<), \tag{1.34}$$

when $\mathbf{r}$ and $\mathbf{r}'$ belong to V.

$$\mathbf{E}^{(i)}(\mathbf{r}_<) + \frac{1}{4\pi}\mathbf{S}_e = 0, \tag{1.35}$$

when $\mathbf{r}$ belongs to V and $\mathbf{r}'$ is outside V.

$$\mathbf{E}(\mathbf{r}_>) = \mathbf{E}^{(i)}(\mathbf{r}_>) + \frac{1}{4\pi}\mathbf{S}_e(\mathbf{r}_>), \tag{1.36}$$

when $\mathbf{r}$ and $\mathbf{r}'$ are outside V, and:

$$0 = \frac{1}{4\pi}\int_V \mathbf{F}_e(\mathbf{r}') \cdot \mathscr{G}(\mathbf{r}, \mathbf{r}')d^3r' - \frac{1}{4\pi}\sum\nolimits_e^{(-)}(\mathbf{r}_>), \tag{1.37}$$

when $\mathbf{r}$ is outside V and $\mathbf{r}'$ is inside V.

In Eqs. (1.35) and (1.36) $\mathbf{E}^{(i)}$ is the incident field upon the volume V, and $\mathbf{S}_e$ is given by:

$$\mathbf{S}_e(\mathbf{r}) = \int_{S^-} \left[\left(\mathbf{n} \times (\nabla \times \mathbf{E} - 4\pi ik\mathbf{M}) + \frac{4\pi ik}{c}\mathbf{J} \right) \cdot \mathscr{G}(\mathbf{r}, \mathbf{r}') + (\mathbf{n} \times \mathbf{E}) \cdot \nabla \times \mathscr{G}(\mathbf{r}, \mathbf{r}') \right] dS; \tag{1.38}$$

and:

$$\sum\nolimits_e^{(-)}(\mathbf{r}) = \int_{S^-} [(\mathbf{n} \times \nabla \times \mathbf{E}) \cdot \mathscr{G}(\mathbf{r}, \mathbf{r}') + (\mathbf{n} \times \mathbf{E}) \cdot \nabla \times \mathscr{G}(\mathbf{r}, \mathbf{r}')] dS. \tag{1.39}$$

The symbol S^- in Eqs. (1.38) and (1.39) indicates that the limiting values of the integrands on S are taken by approaching the surface S from inside the volume V; both the surface element dS and the unit outward normal $\mathbf{n}$ depend on $\mathbf{r}'$.

Equation (1.34) expresses the field at an arbitrary point $\mathbf{r}_<$ of a volume V in terms of the limiting values of the tangential component of the electric and magnetic vectors at the boundary S, when sources are present in V and they are represented by the term $\mathbf{F}_e(\mathbf{r})$.

When $\mathbf{F}_e(\mathbf{r}) = 0$, Eq. (1.34) acquires one of the several equivalent forms of the vector formulation of the theorem of Helmholtz and Kirchhoff of diffraction theory, which constitutes the mathematical representation of Huygens' principle. We discuss further this theorem in Chapter 6. Detailed studies can be found in the works by Baker and Copson [1.23] and by Hönl *et al.* [1.24] (brief summaries are given in textbooks like Chapter 8 of Ref. 1.2).

Equation (1.35) is satisfied at every point $\mathbf{r}_<$ *inside* the volume V, and constitutes one of the forms of the *extinction theorem* describing the cancellation of the field $\mathbf{E}^{(i)}$, incident upon the volume V, by the field induced inside V which is given by the integral term $\mathbf{S}_e$. According to Pattanayak and Wolf [1.25], this theorem can be envisaged as a *nonlocal* boundary condition for the field at the surface S. The exterior field can then be calculated from Eq. (1.36).

Similar equations can be derived for the magnetic vector. They can be found in Appendix 1.1.

In addition, alternative forms of Eqs. (1.35) and (1.36) can be obtained (see Refs. 1.16 and 1.17), by means of the alternative expression for $\mathbf{S}_e(\mathbf{r})$

(see also Problem 1.2):

$$\mathbf{S}_e(\mathbf{r}) = \nabla \times \nabla \times \left[\frac{1}{k^2} \int_{S^-} \left(\mathbf{E}\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial \mathbf{E}}{\partial \mathbf{n}} \right) dS \right.$$
$$\left. - \frac{4\pi}{k^2} \int_{S^-} (\mathbf{n}\nabla \cdot \mathbf{P}) G(\mathbf{r},\mathbf{r}') dS - \frac{4\pi i}{k} \int_{S^-} (\mathbf{n} \times \mathbf{M}) G(\mathbf{r},\mathbf{r}') dS \right]. \tag{1.40}$$

If the medium is nonmagnetic, nonconductive and spatially nondispersive, $\mathbf{S}_e$ reduces to the first integral of Eq. (1.40) and Eqs. (1.35) and (1.36) adopt simple familiar forms in terms of the boundary values of both $\mathbf{E}$ and G, and their normal derivatives, on the surface. Similarly, when the medium occupying the volume V is a perfect conductor, $\mathbf{S}_e$ becomes:

$$\mathbf{S}_e(\mathbf{r}) = \frac{4\pi i}{kc} \nabla \times \nabla \times \int_{S^-} \mathbf{J}(\mathbf{r}') G(\mathbf{r},\mathbf{r}') dS; \tag{1.41}$$

and Eqs. (1.35) and (1.36) become equal to well known expressions used to solve scattering problems in electromagnetics; this will be discussed in Chapter 7.

Equations (1.34) and (1.37) can be expressed in terms of the *electric* and *magnetic Hertz vectors*, $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ [1.18], which are defined as:

$$\mathbf{\Pi}_e(\mathbf{r}) = \int_V \left[\mathbf{P}(\mathbf{r}') + \frac{i}{kc}\mathbf{j}(\mathbf{r}') \right] G(\mathbf{r},\mathbf{r}') d^3r' + \frac{i}{kc} \int_{S^-} \mathbf{J}(\mathbf{r}') G(\mathbf{r},\mathbf{r}') dS, \tag{1.42}$$

and:

$$\mathbf{\Pi}_m(\mathbf{r}) = \int_V \mathbf{M}(\mathbf{r}') G(\mathbf{r},\mathbf{r}') d^3r'. \tag{1.43}$$

From Eq. (1.34) one obtains:

$$\mathbf{E}(\mathbf{r}_<) = \mathbf{E}^{(i)}(\mathbf{r}_<) + \nabla \times \nabla \times \mathbf{\Pi}_e(\mathbf{r}_<) + ik\nabla$$
$$\times \mathbf{\Pi}_m(\mathbf{r}_<) - 4\pi \mathbf{P}(\mathbf{r}_<) - \frac{4\pi i}{kc}\mathbf{j}(\mathbf{r}_<), \tag{1.44}$$

and a similar equation for the magnetic field:

$$\mathbf{H}(\mathbf{r}_<) = \mathbf{H}^{(i)}(\mathbf{r}_<) + \nabla \times \nabla \times \mathbf{\Pi}_m(\mathbf{r}_<) - ik\nabla \times \mathbf{\Pi}_e(\mathbf{r}_<) - 4\pi \mathbf{M}(\mathbf{r}_<). \tag{1.45}$$

On the other hand, from Eq. (1.37) the following expression is derived:

$$\mathbf{E}(\mathbf{r}_>) = \mathbf{E}^{(i)}(\mathbf{r}_>) + \nabla \times \nabla \times \mathbf{\Pi}_e(\mathbf{r}_>) + ik\nabla \times \mathbf{\Pi}_m(\mathbf{r}_>), \tag{1.46}$$

with the corresponding expression for the magnetic vector:

$$\mathbf{H}(\mathbf{r}_>) = \mathbf{H}^{(i)}(\mathbf{r}_>) + \nabla \times \nabla \times \mathbf{\Pi}_m(\mathbf{r}_>) - ik\nabla \times \mathbf{\Pi}_m(\mathbf{r}_>). \tag{1.47}$$

Equations (1.44)–(1.47) are integro-differential equations for the electric and magnetic vectors of application in radiation, scattering, and diffraction problems.

1.6.3. *Integral theorems for scalar fields*

The integral theorems discussed in Sec. 1.6.2 can be also established for scalar waves [1.25]. As far as light fields are concerned, they correspond to the classical way in which a scalar description of light is employed, as discussed in Sec. 1.4, even if one imposes somewhat idealized continuity conditions for the wavefields across the surface separating two different media. This excludes the case of a perfect conductor. Such continuity conditions are considered to be:

$$U_2 = U_1; \tag{1.48a}$$

$$\frac{\partial U_2}{\partial \mathbf{n}} - \frac{\partial U_1}{\partial \mathbf{n}} = 0, \tag{1.48b}$$

where the subindices 1 and 2 denote each of the two media separated by the surface.

More details on the scalar theory, to be discussed next, can be found in Appendix 1.1.

Let the spatial part of the field wavefunction satisfy the inhomogeneous Helmholtz equation, Eq. (1.21), with the source density $\rho(\mathbf{r})$ being given by:

$$\rho(\mathbf{r}) = -\frac{1}{4\pi}F(\mathbf{r})U(\mathbf{r}), \tag{1.49}$$

where the *scattering potential* $F(\mathbf{r})$ is defined by:

$$F(\mathbf{r}) = \begin{cases} -k^2[n^2(\mathbf{r}) - 1], & \text{if } \mathbf{r} \text{ belongs to } V, \\ 0, & \text{if } \mathbf{r} \text{ is outside } V, \end{cases} \tag{1.50}$$

$n(\mathbf{r})$ being the refractive index of the material occupying the volume V: $n = \sqrt{\varepsilon\mu}$. Equation (1.49) will be discussed later in Sec. 3.10.

Let a plane wave $U^{(i)}(\mathbf{r}) = \exp(ik\mathbf{n}_0 \cdot \mathbf{r})$, with propagation direction given by the unit vector $\mathbf{n}_0$, be incident from vacuum upon the medium occupying volume V, then, by using *Green's theorem* applied to $U(\mathbf{r})$ and

$G(\mathbf{r},\mathbf{r}')$, one obtains, in a similar manner to that in Sec. 1.6.2, for points either inside and outside V:

$$U^{(i)}(\mathbf{r}_<) + \frac{1}{4\pi}\sum(\mathbf{r}_<) = 0, \tag{1.51}$$

for $\mathbf{r}$ in V and $\mathbf{r}'$ outside V.

$$U(\mathbf{r}_>) = U^{(i)}(\mathbf{r}_>) + \frac{1}{4\pi}\sum(\mathbf{r}_>), \tag{1.52}$$

for $\mathbf{r}$ and $\mathbf{r}'$ outside V.

$$U(\mathbf{r}_<) = U^{(i)}(\mathbf{r}_<) - \frac{1}{4\pi}\int_V U(\mathbf{r}')F(\mathbf{r}')G(\mathbf{r}_<,\mathbf{r}')d^3r', \tag{1.53}$$

for $\mathbf{r}$ and $\mathbf{r}'$ inside V.

$$U(\mathbf{r}_>) = U^{(i)}(\mathbf{r}_>) - \frac{1}{4\pi}\int_V U(\mathbf{r}')F(\mathbf{r}')G(\mathbf{r}_>,\mathbf{r}')d^3r', \tag{1.54}$$

for $\mathbf{r}$ outside V and $\mathbf{r}'$ in V.

$\sum(\mathbf{r})$ denotes the surface integral:

$$\sum(\mathbf{r}) = \int_S \left(U\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial U}{\partial \mathbf{n}}\right) dS. \tag{1.55}$$

Equation (1.51) constitutes the *scalar* form of the *extinction theorem*. It was derived in the context of quantum mechanics by Lax [1.26]. Once the field on surface S is obtained from Eq. (1.51), the exterior field can be calculated from Eq. (1.52). On the other hand, Eqs. (1.53) and (1.54) are well known in potential scattering, giving the wavefield, both inside and outside V, in terms of the scattering potential.

Equations (1.51) and (1.53) also imply the equation:

$$U(\mathbf{r}_<) = -\frac{1}{4\pi}\int_V U(\mathbf{r}')F(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r' - \frac{1}{4\pi}\sum(\mathbf{r}_<). \tag{1.56}$$

Equation (1.56) expresses the wavefield $U(\mathbf{r})$ inside a volume V in terms of its boundary value at the surface S that limits V.

When $F(\mathbf{r}) = 0$, namely, when volume V is source-free, Eq. (1.56) becomes identical to the scalar form of the *Helmholtz–Kirchhoff integral theorem* of the scalar theory of diffraction.

1.6.4. *Natural modes*

The natural modes of the wave equations are well behaved outgoing solutions that cannot be generated by any incident field. These modes may be either *radiative* (complex eigenfrequencies ω), or *nonradiative* (real eigenfrequencies ω), or *surface* or *bulk* modes. Wolf [1.18] wrote the equations for these modes in terms of the expressions discussed in Secs. 1.6.2 and 1.6.3. According to this analysis, since these modes are not generated by any incident field, one has, from Eqs. (1.35) and (1.36):

$$\mathbf{S}_e(\mathbf{r}_<) = 0, \tag{1.57}$$

$$\mathbf{E}(\mathbf{r}_>) = \frac{1}{4\pi}\mathbf{S}_e(\mathbf{r}_>). \tag{1.58}$$

There are similar equations for the magnetic vector.

Likewise, in the scalar case, the equation for the modes obtained from Eq. (1.51), is:

$$\sum(\mathbf{r}_<) = 0. \tag{1.59}$$

Equations (1.57)–(1.59) define the set of eigenvalues k (or equivalently, of eigenfrequencies ω) of the modes that constitute solutions to these equations.

Equation (1.59) has been applied by Pattanayak and Wolf [1.25] to a central potential in order to characterize the nonradiative modes (*bound states*); this provided an alternative method to the one previously used by Humblet and Rosenfeld [1.27] for *resonance states* of the Schrödinger equation, of which the dispersion formula for nuclear reactions of Kapur and Peierls [1.28] is a particular case. Further references in connection with some problems of condensed matter physics can be found in Ref. 1.25. Also Agarwal [1.29] has derived the dispersion relation for *surface polaritons* and *Brewster modes* by applying Eqs. (1.57) and (1.58) to a plane interface separating two media.

1.6.5. *Other extensions and uses of the extinction theorem*

The extinction theorem has been generalized to spatially dispersive media by Agarwal *et al.* [1.30]; to bounded media by Birman and Sein [1.31], and to bounded gyrotropic media by Puri and Birman [1.32]. In Chapters 6 and 7 we discuss its applications to diffraction problems and scattering from rough surfaces. A mapping in the complex plane for two-dimensional scalar wavefields has been shown [1.33] to lead to the extinction theorem, as well as to the Helmholtz–Kirchhoff theorem, when the Cauchy integral is applied to a certain class of generalized analytic functions.

Problems

1.1. Prove that the dyadic Green function $\mathscr{G}(\mathbf{r},\mathbf{r}')$ can be written as:

$$\mathscr{G}(\mathbf{r},\mathbf{r}') = \frac{1}{k^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times G(\mathbf{r},\mathbf{r}')\mathscr{I} - 4\pi\delta(\mathbf{r}-\mathbf{r}')\mathscr{I};$$

$G(\mathbf{r},\mathbf{r}')$ being the scalar Green function and $\mathscr{I}$ denoting the unit dyadic.

1.2. Using the result of Problem 1.1, and the vector Green theorem for two vectors $\mathbf{P}$ and $\mathbf{Q}$ (cf. e.g. Ref. 1.22, Eq. (13.1.7)):

$$\begin{aligned}\int_V [\mathbf{P}\cdot\boldsymbol{\nabla}^2\mathbf{Q} - \mathbf{Q}\cdot\boldsymbol{\nabla}^2\mathbf{P}]d^3r \\ = \int_S [(\mathbf{P}\boldsymbol{\nabla}\cdot\mathbf{Q} - \mathbf{Q}\boldsymbol{\nabla}\cdot\mathbf{P})\cdot\mathbf{n} - (\mathbf{P}\cdot(\mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{Q})) \\ + \boldsymbol{\nabla}\times\mathbf{P}\cdot(\mathbf{n}\times\mathbf{Q}))]dS.\end{aligned}$$

Prove that Eq. (1.38) can be expressed in the form given by Eq. (1.40).

1.3. By using the result of Problem 1.1 and the vector Green theorem prove that $\sum_e^{(-)}(\mathbf{r}_<)$ given by Eq. (1.39) may be written as:

$$\begin{aligned}\sum\nolimits_e^{(-)}(\mathbf{r}_<) = \int_S [(\mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r}')))G(\mathbf{r}_<,\mathbf{r}') + (\mathbf{n}\times\mathbf{E}(\mathbf{r}')) \\ \times\nabla G(\mathbf{r}_<,\mathbf{r}') + (\mathbf{n}\cdot\mathbf{E}(\mathbf{r}'))\boldsymbol{\nabla}G(\mathbf{r}_<,\mathbf{r}')]dS.\end{aligned}$$

This expression of $\sum_e^{(-)}(\mathbf{r})$, when substituted into Eq. (1.34), yields the field at a point $\mathbf{r}_<$, inside the volume V, in terms of the boundary values on a surface S. When $\mathbf{F}_e(\mathbf{r}) = 0$ the resulting equation is known as the Stratton–Chu formula (see Sec. 4.15 of Ref. 1.1 or Sec. 9.9 of Ref. 1.34).

1.4. Prove that Eq. (1.34) may be expressed in the form of Eq. (1.44) with the Hertz vectors $\boldsymbol{\Pi}_e(\mathbf{r})$ and $\boldsymbol{\Pi}_m(\mathbf{r})$ given by Eqs. (1.42) and (1.43), respectively.

1.5. Prove that the Hertz vectors $\mathbf{\Pi}_e(\mathbf{r})$ and $\mathbf{\Pi}_m(\mathbf{r})$ satisfy the Helmholtz equations:

$$(\nabla^2 + k^2)\mathbf{\Pi}_e(\mathbf{r}) = -4\pi\mathbf{P}(\mathbf{r}),$$
$$(\nabla^2 + k^2)\mathbf{\Pi}_m(\mathbf{r}) = -4\pi\mathbf{M}(\mathbf{r}).$$

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Appendix 1.1

In this appendix, the paper by E. Wolf entitled "A Generalized Extinction Theorem and its Role in Scattering Theory" is reproduced. This paper was published in: *Coherence and Quantum Optics*, pp. 339–357, eds. L. Mandel and E. Wolf, Plenum Press, New York, 1973. Permission to reproduce the paper was granted by Prof. E. Wolf and Plenum Publishing Corporation.

A GENERALIZED EXTINCTION THEOREM AND ITS ROLE IN SCATTERING THEORY*

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When an electromagnetic wave is incident on a homogeneous medium with a sharp boundary, it is extinguished inside the medium in the process of interaction and is replaced by a wave propagated in the medium with a velocity different from that of the incident wave. A classic theorem of molecular optics due to P.P. Ewald (1912) and C.W. Oseen (1915) expresses the extinction of the incident wave in terms of an integral relation, that involves the induced field on the boundary of the medium. Various generalizations of this theorem have recently been proposed and it was also shown that the customary physical interpretation of the theorem is incorrect.

*Research supported by the Air Force Office of Scientific Research and the Army Research Office (Durham).

In this paper results of a recent investigation carried out in collaboration with D.N. Pattanayak are presented, which provide a generalization of the extinction theorem to any medium. Like the recent generalization due to J.J. Sein our derivation is based entirely on Maxwell's theory and not on molecular optics. A hypothesis is put forward as to the true physical significance of the extinction theorem and it is shown how the theorem may be used to solve scattering problems in a novel way. An analogous extinction theorem for non-relativistic quantum mechanics is also presented.

One of the most poorly understood theorems of classical electrodynamics is undoubtedly the so-called *extinction theorem* first formulated by P.P. Ewald [1] in 1912 in his basic investigations on the foundations of crystal optics and later by C.W. Oseen [2] in 1915 in his studies of dispersion of light in material media. Let me first say a few words about the usual formulation of the theorem.

Suppose that a plane electromagnetic wave is incident from vacuo on a material medium with a sharp boundary. The medium will for the moment be assumed to be of the simplest kind — a linear, homogeneous, isotropic, non-magnetic dielectric.

We know that under the influence of the incident electromagnetic field another field will be generated inside the dielectric, which will have a different wave number and hence a different phase velocity. We may, therefore, say that inside the medium, the incident wave, propagated with the vacuum velocity of light c, is somehow *extinguished* by the interaction with the medium and is replaced by a new wave propagated with the velocity c/n, where n is the refractive index of the medium. The question then arises: how does the extinguishing of the incident wave come about? The Ewald–Oseen theorem provides an answer to this question. In mathematical terms the theorem may be expressed in the form: [3]

$$\underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi k^2}\nabla\times\nabla\times\int_S \left\{\underline{E}(\underline{r}')\frac{\partial}{\partial \mathrm{n}}G_0(\underline{r}_<,\underline{r}') - G_0(\underline{r}_<,\underline{r}')\frac{\partial}{\partial \mathrm{n}}\underline{E}(\underline{r}')\right\}dS = 0, \tag{1}$$

valid at every point $\underline{r}_<$ inside the volume V bounded by a surface S (see Fig. 1) occupied by the medium. Here $\underline{E}^{(\mathrm{i})}$ and $\underline{E}$ represent the Fourier transforms (for frequency components $\omega = kc$, c being the vacuum velocity of light) of the incident electric field and of the total electric field generated inside the medium respectively (and taken in the integral in (1) in the limit as the surface is approached from inside V),

$$G_0(\underline{r},\underline{r}') = \exp\{ik|\underline{r}-\underline{r}'|\}/|\underline{r}-\underline{r}'| \tag{2}$$

is the outgoing free-space Green's function of the Helmholtz equation and $\partial/\partial\mathrm{n}$ denotes differentiation along the outward normal to the boundary surface.

The relation (1), which is essentially in the form as formulated by Oseen, was originally derived not from the macroscopic Maxwell theory, but rather from molecular optics, which is a microscopic theory. In this later theory the response of the medium to the incident field is expressed in terms of elementary dipole fields, generated by the interaction of the incident wave with the individual molecules of the medium. We note that in (1), the second term formally cancels the incident electric field at every point inside the medium. Since the second term involves the values of the total field $\underline{E}$ on the boundary surface S only it has been generally asserted that Eq. (1) implies that the incident field is extinguished entirely by those molecular dipoles that are situated on the boundary S of the dielectric. This is the original formulation of the Ewald–Oseen extinction theorem.

In the last few years the Ewald–Oseen extinction theorem has attracted a good deal of attention and various modifications and generalizations of it for more complicated media have been proposed and applied to numerous problems of current research interest. Here is a partial listing of the relevant publications, indicating the authors, year of publication and topics. The complete references are given in footnote 5.

(a)	A. Wierzbicki	(1961, 1962)	Quadrupole radiation, reflection, refraction
(b)	N. Bloembergen and P.S. Pershan	(1962)	Non-linear optics
(c)	B.A. Sotskii	(1963)	Metals, optically active media
(d)	R.K. Bullough	(1968)	Many-body optics
(e)	J.J. Sein	(1969, 1970)	Spatial dispersion, excitons
(f)	É. Lalor	(1969)	New formulation
(g)	É. Lalor and E. Wolf	(1971)	Interaction of charged particle with a dielectric
(h)	T. Suzuki	(1971)	Diffraction
(i)	G.S. Agarwal, D. Pattanayak and E. Wolf	(1971)	Spatial Dispersion
(j)	É. Lalor and E. Wolf	(1972)	Refraction and reflection
(k)	J.R. Birman and J.J. Sein	(1972)	Polaritons in bounded media

It is thus clear that the extinction theorem is playing an increasingly greater role in widely different areas. Of the numerous investigations those of J.J. Sein [4, 5e] are of particular relevance to the subject matter of this talk. Sein showed that the extinction theorem which, as I already noted, was derived originally from molecular optics may also be derived from Maxwell's theory and he also showed that the traditional interpretation of the theorem is incorrect [4].

In this talk I will present results of a recent investigation that I carried out in collaboration with D. Pattanayak [6], (see also [7]), in which we have attempted to answer the following two questions:

(1) Can the extinction theorem be generalized within the framework of Maxwell's theory to a medium of any kind, i.e. with arbitrary response?

(2) What is the true meaning of the theorem?

The first question has been partially answered already by Sein, but we will present quite a general and rigorous answer to it. Let me add that Sein's recognition that the extinction theorem follows also from Maxwell's theory represents an important contribution, since attempts to generalize it within the framework of molecular optics encounters formidable difficulties because of the local field corrections (associated with the Lorentz internal field).

As regards the second question — namely what is the true meaning of the theorem — we will put forward a hypothesis, supported by a few explicit solutions that we obtained with the help of the theorem.

We will also show that the extinction theorem has a strict analogue in potential scattering in non-relativistic quantum mechanics.

The full derivation of our main results is rather lengthy and I will only indicate the main steps.

Let us then consider the scattering of monochromatic electro-magnetic wave incident from vacuo on a body with a sharp boundary. We assume the body to be of arbitrary kind; its response could be, for example, non-linear or non-local as in the case of spatial dispersion.

From Maxwell's equations for monochromatic fields, on eliminating, the electric displacement vector $\underline{D}$ and the magnetic induction vector $\underline{B}$ via the relations

$$\underline{D} = \underline{E} + 4\pi\underline{P}, \quad \underline{B} = \underline{H} + 4\pi\underline{M}, \tag{3}$$

where $\underline{P}$ and $\underline{M}$ denote the polarization and magnetization vectors respectively, we obtain the four equations

$$\hat{\mathrm{L}}\,\underline{E} = \underline{F}_{\mathrm{e}}, \tag{4a}$$

$$\hat{\mathrm{L}}\,\underline{H} = \underline{F}_{\mathrm{h}}, \tag{4b}$$

$$\nabla\cdot\underline{E} = 4\pi(\rho - \nabla\cdot\underline{P}) \tag{4c}$$

$$\nabla\cdot\underline{H} = -4\pi\nabla\cdot\underline{M}, \tag{4d}$$

where $\hat{\mathrm{L}}$ is the operator

$$\hat{\mathrm{L}} = -k^2 + \nabla\times\nabla\times \tag{5}$$

and the source terms $\underline{F}_{\mathrm{e}}$ and $\underline{F}_{\mathrm{h}}$ are given by

$$\underline{F}_{\mathrm{e}} = 4\pi\left[\frac{\mathrm{ik}}{\mathrm{c}}\underline{j} + \mathrm{k}^2\underline{P} + \mathrm{ik}\nabla\times\underline{M}\right], \tag{6a}$$

$$\underline{F}_{\mathrm{h}} = 4\pi\left[\frac{1}{\mathrm{c}}\nabla\times\underline{j} - \mathrm{ik}\nabla\times\underline{P} + \mathrm{k}^2\underline{M}\right]. \tag{6b}$$

The vectors $\underline{E}$, $\underline{H}$, $\underline{j}$, $\underline{P}$ and $\underline{M}$ and the scalar ρ are, of course, functions of position ($\underline{r}$) and are taken at a fixed frequency ω.

The sources of the incident field are assumed to be in the domain V exterior to V (see Fig. 1) and we take them to be at a finite distance [8] from V. It is clear then, since the exterior $\tilde{V}$ of V is vacuo, that

$$\underline{F}_{\mathrm{e}}(\underline{r}) = 4\pi\left[\frac{ik}{c}\underline{j}_{\mathrm{c}} + k^2\underline{P} + ik\nabla\times\underline{M}\right] \qquad \text{if } \underline{r}\in V \tag{7a}$$

$$= \frac{4\pi ik}{c}\underline{j}_{\mathrm{ext}} \qquad \text{if } \underline{r}\in\tilde{V}, \tag{7b}$$

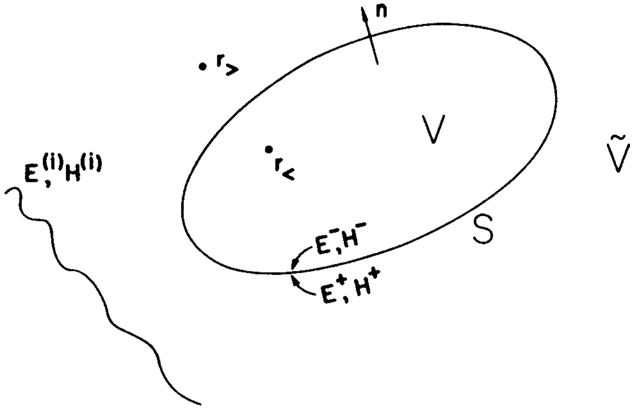

Fig. 1. Notation relating to scattering of an electromagnetic wave on a material medium.

$$\underline{F}_{\rm h}(\underline{r}) = 4\pi \left[\frac{1}{c}\nabla \times \underline{j}_{\rm c} - ik\nabla \times \underline{P} + k^2 \underline{M}\right] \quad \text{if } \underline{r} \in V \tag{8a}$$

$$= \frac{4\pi}{c}\nabla \times \underline{j}_{\rm ext} \quad \text{if } \underline{r} \in \underline{\tilde{V}}. \tag{8b}$$

In Eqs. (7) and (8), $\underline{j}_{\rm c}$ denotes the conduction current density and $\underline{j}_{\rm ext}$ denotes the external current density (representing the source). The corresponding charge densities are, of course, related to the current densities by the continuity equation. Further the polarization vector $\underline{P}$ and the magnetization vector $\underline{M}$ in (7a) and (8a) are assumed to be given functions of the electromagnetic field vectors $\underline{E}$ and $\underline{H}$, whose exact from depends on the nature of the medium. It is to be noted that because of this fact, the equations (4a) and (4b) are in general *coupled* to each other.

To complete the formulation we must specify the behavior of the fields at the boundary. As is well known Maxwell's equations imply that across the boundary

$$\underline{n} \times (\underline{E}^{+} - \underline{E}^{-}) = 0, \quad \underline{n} \times (\underline{H}^{+} - \underline{H}^{-}) = \frac{4\pi}{c}\underline{K}, \tag{9}$$

where the superscripts plus and minus denote limiting values as the boundary surface S is approached from outside and inside of the medium respectively (see Fig. 1), $\underline{n}$ is the unit outward normal to the boundary surface S and $\underline{K}$ represents the surface current density which will be non-zero only for a perfect conductor. Although these conditions are generally referred to as *boundary conditions*, they are, actually jump conditions — or *saltus* conditions as they are called in the older literature. A clear appreciation of the difference between true boundary conditions and the saltus conditions is, as we shall see later, at the heart of the proper interpretation of the extinction theorem.

For later purpose we also note that the corresponding problem in non-relativistic quantum mechanics, namely the scattering from a finite step potential is mathematically much simpler since in place of several coupled equations involving the vector fields $\underline{E}$ and $\underline{H}$, the quantum mechanical problem involves only a single equation for the Schrödinger scalar wave function $\psi(\underline{r})$. Moreover, since ψ and its normal $\partial\psi/\partial \mathrm{n}$ must, according to basic postulates of quantum mechanics be *continuous* across a finite potential step we now have in place of the conditions (9) the conditions

$$\psi^{+} - \psi^{-} = 0, \quad \left(\frac{\partial\psi}{\partial \mathrm{n}}\right)^{+} - \left(\frac{\partial\psi}{\partial \mathrm{n}}\right)^{-} = 0. \tag{10}$$

These are *continuity conditions* and not true boundary conditions either.

Returning to the electromagnetic problem we introduce a dyadic Green's function associated with the $\hat{\mathrm{L}}$-operator,

$$\hat{\mathrm{L}}\,\underline{\underline{G}} = 4\pi\delta(\underline{r} - \underline{r}')\,\underline{\underline{U}}, \tag{11}$$

($\underline{\underline{U}}$ = unit dyadic), which obeys the vectorial form of the Sommerfeld radiation condition at infinity:

$$\lim_{r\to\infty} r[\nabla \times \underline{\underline{G}} - ik\,\hat{\underline{r}} \times \underline{\underline{G}}] = 0, \tag{12}$$

where $\hat{\underline{r}}$ is the unit vector in the direction of $\underline{r}$ and $\mathrm{r} = |\underline{r}|$. It is known that [9]

$$\underline{\underline{G}}(\underline{r}, \underline{r}') = \left(\underline{\underline{U}} + \frac{1}{k^2}\nabla\nabla\right) G_0(\underline{r}, \underline{r}'), \tag{13a}$$

where G_0 is the outgoing free-space Green's function of the Helmholtz equation, viz.

$$G_0(\underline{r}, \underline{r}') = \frac{\mathrm{e}^{ik}|\underline{r} - \underline{r}'|}{|\underline{r} - \underline{r}'|}. \tag{13b}$$

Now our equation involving the electric field is of the form

$$\hat{\mathrm{L}}\,\underline{E} = \underline{F}_{\mathrm{e}}, \tag{14}$$

with a similar equation involving $\underline{H}$. From (11) and (14) one obtains, if one also uses the vectorial form of Green's theorem, the following identity valid for integration through any domain V' bounded by a closed surface S':

$$\int_{V'} E(\underline{r}')\delta(\underline{r} - \underline{r}')\mathrm{d}^3\underline{r}' = \frac{1}{4\pi}\int_{\mathrm{V}'} \underline{F}_{\mathrm{e}}(\underline{r}') \cdot \underline{\underline{G}}(\underline{r}, \underline{r}')\mathrm{d}^3\underline{r}' - \frac{1}{4\pi}\sum\nolimits_{\mathrm{e}}(\underline{r}), \tag{15}$$

where

$$\sum\nolimits_{\mathrm{e}}(\underline{r}) = \int_{S'} \{[\underline{n} \times \nabla \times \underline{E}(\underline{r}')] \cdot \underline{\underline{G}}(\underline{r}, \underline{r}') + [\underline{n} \times \underline{E}(\underline{r}')] \cdot \nabla \times \underline{\underline{G}}\} dS' \tag{16}$$

and n is the unit normal to S' pointing outward from the volume V'.

Let us now take the volume V' to coincide either with the scattering volume V or with the exterior $\tilde{V}$ of it. Also we can take the field point $\underline{r}$ to be either in V or in $\tilde{V}$. Applying the theorem (15) separately to each of these four cases we obtain the following relations:

(a) $\underline{\underline{r} \in V,\ \underline{r}' \in V}$:

$$\underline{E}(\underline{r}_<) = \frac{1}{4\pi} \int_{\mathrm{V}} \underline{F}_{\mathrm{e}} \cdot \underline{\underline{G}}\, d^3\underline{r}' - \frac{1}{4\pi} \sum\nolimits_{\mathrm{e}}^{(-)} (\underline{r}_<), \tag{17a}$$

(b) $\underline{\underline{r} \in V,\ \underline{r}' \in \tilde{V}}$:

$$0 = \frac{ik}{c} \int_{\tilde{V}} \underline{j}_{\mathrm{ext}} \cdot \underline{\underline{G}}\, d^3\underline{r}' + \frac{1}{4\pi} \cdot \sum\nolimits_{\mathrm{e}}^{(+)} (\underline{r}_<), \tag{17b}$$

(c) $\underline{\underline{r} \in \tilde{V},\ \underline{r}' \in \tilde{V}}$:

$$\underline{E}(\underline{r}_>) = \frac{ik}{c} \int_{\tilde{V}} \underline{j}_{\mathrm{ext}} \cdot \underline{\underline{G}}\, d^3\underline{r}' + \frac{1}{4\pi} \sum\nolimits_{\mathrm{e}}^{(+)} (\underline{r}_>), \tag{17c}$$

(d) $\underline{\underline{r} \in \tilde{V},\ \underline{r}' \in V}$:

$$0 = \frac{1}{4\pi} \int_{V} \underline{F}_{\mathrm{e}} \cdot \underline{\underline{G}}\, d^3\underline{r}' - \frac{1}{4\pi} \sum\nolimits_{\mathrm{e}}^{(-)} (\underline{r}_>). \tag{17d}$$

Here

$$\sum\nolimits_{\mathrm{e}}^{(+)} = \int_{S^{\pm}} \{[\mathrm{n} \times \nabla \times \underline{E}] \cdot \underline{\underline{G}} + [\underline{n} \times \underline{E}] \cdot \nabla \times \underline{\underline{G}}\} dS, \tag{18}$$

and the upper or lower signs are taken on $\sum_{\mathrm{e}}^{\pm}$ and on $S^{\pm}$ according as the limiting values are taken from outside (S^+) or inside (S^-) of the scattering volume. In deriving (17b) and (17c) we also used the radiation condition (12) which ensures that there is no contribution from a sphere of infinitely large radius (the outer boundary of the volume V).

Now the integral containing the external current, taken over the exterior $\tilde{V}$ of our scattering volume has a clear physical meaning. From the significance of $\underline{\underline{G}}$ as the outgoing free-space Green's function of the $\tilde{\mathrm{L}}$-operator, this integral must evidently represent the unperturbed incident field, i.e.

$$\frac{ik}{c} \int_{\tilde{V}} \underline{j}_{\mathrm{ext}}(\underline{r}') \cdot \underline{\underline{G}}(\underline{r}, \underline{r}') d^3\underline{r}' = \underline{E}^{(\mathrm{i})}(\underline{r}), \tag{19}$$

irrespective whether the point $\underline{r}$ is situated inside or outside the scattering volume. Hence the equations (17b) and (17c) may be expressed in the compact form

$$\underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi}\sum\nolimits_{\mathrm{e}}^{(+)}(\underline{r}_<) = 0, \tag{20a}$$

$$\underline{E}(\underline{r}_>) = \underline{E}^{(\mathrm{i})}(\underline{r}_>) + \frac{1}{4\pi}\sum\nolimits_{\mathrm{e}}^{(+)}(\underline{r}_>). \tag{20b}$$

We note that (20a) has some resemblance to the Ewald–Oseen extinction theorem, since it expressed the cancellation of the incident field at every point $\underline{r}$ inside the scattering medium V in terms of an integral involving the field on the boundary of the medium only. However because $\sum_{\mathrm{e}}^{(+)}$ rather than $\sum_{\mathrm{e}}^{(-)}$ appears in this surface integral, the integral involves the limiting values of the field taken from the outside, rather than from the inside of the scattering volume; but one can easily transform (20a) and also (20b) so as to involve the limiting values from the inside, since from the definition (18) of $\sum_{\mathrm{e}}^{(-)}$ and $\sum_{\mathrm{e}}^{(+)}$ and from the saltus conditions (9) one easily finds that

$$\frac{1}{4\pi}\left[\sum\nolimits_{\mathrm{e}}^{(+)}(\underline{r}) - \sum\nolimits_{\mathrm{e}}^{(-)}(\underline{r})\right] = -ik\int_{S-}\left(\underline{n}\times\underline{M} - \frac{1}{c}\underline{K}\right)\cdot \underline{\underline{G}}\, ds \tag{21}$$

Using this result in (20a) and (20b) and the expressions for $\sum_{\mathrm{e}}^{(+)}$ one then obtains the following two relations:

$$\boxed{\begin{aligned} \underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi}\underline{S}_{\mathrm{e}}(\underline{r}_<) &= 0, && (22)\\ \underline{E}(\underline{r}_>) = \underline{E}^{(\mathrm{i})}(\underline{r}_>) &+ \frac{1}{4\pi}\underline{S}_{\mathrm{e}}(\underline{r}_>), && (23)\end{aligned}}$$

where

$$\underline{S}_{\mathrm{e}}(\underline{r}) = \int_{S-}\left\{\left[\underline{n}\times(\nabla\times\underline{E} - 4\pi ik\underline{M}) + \frac{4\pi ik}{c}\underline{K}\right]\right.$$
$$\left.\cdot\,\underline{\underline{G}}(\underline{r}\cdot\underline{r}') + [\underline{n}\times\underline{E}]\cdot\nabla\times\underline{\underline{G}}(\underline{r},\underline{r}')\right\}dS. \tag{24}$$

The relation (22) must be satisfied at each point $\underline{r}_<$ inside the scattering volume V. It is one form of our *generalized* Ewald–Oseen theorem, valid for scattering by *any* medium, irrespective of the nature of the constitutive relations. I will indicate shortly how it reduces to the usual form of the Ewald–Oseen theorem when the medium is of the simplest kind. But first I want to say a little about what I believe is the true meaning of the theorem and also discuss briefly the significance of the complementary relation (23). For this purpose we also must note there is an analogous set of relations to (22) and (23), involving the *magnetic* rather than the electric field. They can be derived in a similar way and are

$$\underline{H}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi}\underline{S}_{\mathrm{h}}(\underline{r}_<) = 0, \tag{25}$$

$$\underline{H}(\underline{r}_>) = \underline{H}^{(\mathrm{i})}(\underline{r}_>) + \frac{1}{4\pi}\underline{S}_{\mathrm{h}}(\underline{r}_>), \tag{26}$$

where

$$\underline{S}_{\mathrm{h}}(r) = \int_{S^-} \{[\underline{n} \times (\nabla \times \underline{H} + 4\pi ik\underline{P} - (4\pi/\mathrm{c})\underline{j})] \cdot \underline{\underline{G}} + [\underline{n} \times \underline{H} + (4\pi/\mathrm{c})\underline{K}] \cdot \nabla \times \underline{\underline{G}}\} dS. \tag{27}$$

We know that inside the medium, the $\underline{E}$ and $\underline{H}$ fields obey the equations (4a) and (4b),

$$\hat{\mathrm{L}}\,\underline{E}(\underline{r}_<) = \underline{F}_{\mathrm{e}}(\underline{r}_<), \quad \hat{\mathrm{L}}\,\underline{H}(\underline{r}_<) = \underline{F}_{\mathrm{h}}(\underline{r}_<).$$

However these are *general field equations* valid inside the medium. They do not completely specify the scattered field since they involve neither the incident field, nor any boundary conditions. Our hypothesis is that the *two extinction theorems* (22) *and* (25) *represent boundary conditions subject to which the* (*generally coupled*) *field equations* (4) *provide unique solution for the fields* $\underline{E}$, $\underline{H}$ *inside the scattering medium* (*i.e. inside the volume* V), *when an electromagnetic field* $\underline{E}^{(\mathrm{i})}$, $\underline{H}^{(\mathrm{i})}$ *is incident on the medium.* Thus, according to this hypothesis, the two extinction theorems allow us to replace the original saltus problem — involving the solution both inside and outside the medium — by a *boundary value problem* for determining the field inside the scattering medium. The boundary conditions for this later problem are of a somewhat unusual kind, having the form of *non-local* relations. Once the solution inside the scattering medium has been obtained, the solution outside it may be determined from Eqs. (23) and (26) by substituting the boundary values into the surface integrals occuring in these formulae. Our new interpretation of the extinction theorems is supported by explicit solutions that were obtained for several special cases [10].

Up to this point we have considered only two of the four relations (17), namely (17b) and (17c). If the other two relations, viz. (17a) and (17d) are also used, as well as the relations (19) and (21) and the corresponding formulae involving the magnetic fields one obtains an alternative set of equations in the form of *integro-differential equations* valid both inside and outside the medium — for the unknown electromagnetic fields $\underline{E}$ and $\underline{H}$. Lack of time prevents a discussion of this point here but we will later consider briefly the analogous situation for the case of quantum mechanical potential scattering.

With the help of various vector identities and Maxwell's equations, our general extinction theorems may be expressed in many alternative but equivalent forms. The extinction theorem for the electric field may, for example, be transformed into the form

$$\underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{k^2}\nabla \times \nabla \times [\underline{I}^{(\mathrm{E})}(\underline{r}_<) + \underline{I}^{(\mathrm{P})}(\underline{r}_<) + \underline{I}^{(\mathrm{M})}(\underline{r}_<) + \underline{I}^{(\mathrm{J})}(\underline{r}_<)] = 0, \tag{28}$$

where

$$\underline{I}^{(\mathrm{E})}(\underline{r}_<) = \frac{1}{4\pi}\int_{S^-}\left\{\underline{E}(\underline{r}')\frac{\partial G_0(\underline{r}_<,\underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}_<,\underline{r}')\frac{\partial \underline{E}(\underline{r}')}{\partial \mathrm{n}}\right\}dS, \tag{29}$$

$$\underline{I}^{(\mathrm{P})}(\underline{r}_<) = -\int_{S^-}[\underline{n}\nabla\cdot\underline{P}(\underline{r}')]G_0(\underline{r}_<,\underline{r}')dS, \tag{30}$$

$$\underline{I}^{(\mathrm{M})}(\underline{r}_<) = -ik\int_{S^-}[\underline{n}\times\underline{M}(\underline{r}')]G_0(\underline{r}_<,\underline{r}')dS, \tag{31}$$

$$\underline{I}^{(\mathrm{J})}(\underline{r}_<) = -\frac{i}{kc}\int_{S^-}[\underline{n}\nabla\cdot\underline{j}(\underline{r}') - \underline{K}(\underline{r}')]G_0(\underline{r}_<,\underline{r}')dS. \tag{32}$$

Clearly if the medium is non-magnetic, $\underline{I}^{(\mathrm{M})} \equiv 0$, if it is a nonconductor $\underline{I}^{(\mathrm{j})} \equiv 0$. For a linear, homogeneous, spatially non-dispersive non-magnetic dielectric, not only do these two terms vanish, but so does also the term $\underline{I}^{(\mathrm{P})}$, unless the frequency of the incident field coincides with a frequency at which the dielectric constant vanishes; for except in this case the polarization field is necessarily transverse [11] (i.e. $\nabla\cdot\underline{P} = 0$). If we exclude this exceptional case, (28) reduces to

$$\underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi k^2}\nabla\times\nabla\times\int_{S^-}\left\{\underline{E}(\underline{r}')\frac{\partial G_0(\underline{r}_<,\underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}_<,\underline{r}')\frac{\partial \underline{E}(\underline{r}')}{\partial \mathrm{n}}\right\}dS = 0, \tag{33}$$

which is seen to be identical with the Oseen formulaiton (1) of the extinction theorem. In this case ($\underline{M} = \nabla\cdot\underline{P} = \underline{j} = \underline{K} = 0$, $\underline{P} = \chi\underline{E}$, χ being a constant), the equation of motion (4a) is not coupled to (4b) (since no magnetic term now occurs on the r.h.s. of (6a)) and reduces to

$$\nabla^2\underline{E}(\underline{r}_<) + \mathrm{n}^2k^2\underline{E}(\underline{r}_<) = 0, \tag{34}$$

where

$$\mathrm{n}^2 = 1 + 4\pi\chi. \tag{35}$$

According to our hypothesis the electric field $\underline{E}$ inside the medium is that solution of (34), which obeys the condition (33) at every point $\underline{r}$ inside the medium. Once this solution has been found, the field outside the medium is obtained from the formula

$$\underline{E}(\underline{r}_>) = \underline{E}^{(\mathrm{i})}(\underline{r}_>) + \frac{1}{4\pi k^2} \times\nabla\times\nabla\times\int_{S^-}\left\{\underline{E}(\underline{r}')\frac{\partial G_0(\underline{r}_>,\underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}_>,\underline{r}')\frac{\partial \underline{E}(\underline{r}')}{\partial \mathrm{n}}\right\}dS, \tag{36}$$

to which Eq. (23) may be shown to reduce in the present case.

One can also show that in some other special cases, our general extinction theorem (22) for the electric field reduces to various extinction theorems derived in recent years by other authors. Moreover, one can readily show that in the special case when no scattering medium is present at all our general extinction theorem for the electric field reduces to

$$\underline{E}^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi}\int_{S^-}\left\{\underline{E}^{(\mathrm{i})}(\underline{r}')\frac{\partial G_0(\underline{r}_<,\underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}_<,\underline{r}')\frac{\partial \underline{E}^{(\mathrm{i})}(\underline{r}')}{\partial \mathrm{n}}\right\}dS = 0, \qquad (37)$$

which will be recognized as the classic *integral theorem of Helmholtz and Kirchhoff* (cf. for example, Ref. 3a, p. 377).

Returning to the general forms (22) and (25) of the extinction theorems it seems quite remarkable that for a completely arbitrary medium — e.g. an inhomogeneous, anisotropic, non-linear or spatially dispersive medium — the cancelation of the incident field inside the medium is expressible entirely by the values that the field takes at the boundary of the medium.

Finally I will show that the main results that we obtained for electromagnetic scattering have a strict analogue in non-relativistic quantum-mechanical potential scattering. Consider scattering of a free particle of momentum $\underline{p}$ on a three-dimensional potential barrier or potential well, characterized by a potential $\mathcal{V}(\underline{r})$ that vanishes outside a finite volume V, bounded by a surface S (Fig. 2). For simplicity we assume that the potential $\mathcal{V}(\underline{r})$ has at most a finite discontinuity on S.

The Schrödinger equation for this problem may be written in the form

$$(\nabla^2 + k^2)\psi(\underline{r}) = U(\underline{r})\psi(\underline{r}), \qquad (38)$$

where

$$k^2 = \frac{2\mathrm{m}}{\hbar^2}\mathrm{E}, \quad U(\underline{r}) = \frac{2\mathrm{m}}{k^2}\mathcal{V}(\underline{r}) \quad \text{if r} \in V$$
$$= 0 \quad \text{if r} \in \tilde{V}. \qquad (39)$$

Here m denotes the mass of the particle and $\mathrm{E} = \mathrm{p}^2/2\mathrm{m}$ its energy, $\hbar$ is the Planck constant divided by 2π and $\tilde{V}$ denotes the (infinite) domain outside V.

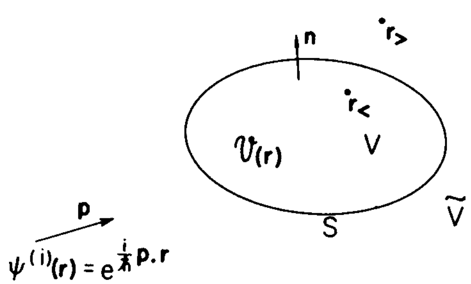

Fig. 2. Notation relating to quantum mechanical potential scattering.

The equation for the associated Green's function is

$$(\nabla^2 + k^2)G_0(\underline{r}, \underline{r}') = -4\pi\delta(\underline{r} - \underline{r}'), \tag{40}$$

the Green's function being, of course, the outgoing spherical wave [Eq. (13b) above].

From (38) and (39) we obtain, if we also use Green's theorem, the following identity valid for integration throughout any domain V′ bounded by a closed surface S′:

$$\int_{V'} \psi(\underline{r}')\delta(\underline{r} - \underline{r}')d^3\underline{r}' = -\frac{1}{4\pi}\int_{V'} U(\underline{r}')\psi(\underline{r}')G_0(\underline{r}, \underline{r}')d^3\underline{r}' - \frac{1}{4\pi}\sum(\underline{r}), \tag{41}$$

where

$$\sum(r) = \int_{S'}\left\{\psi(\mathrm{r}')\frac{\partial G_0(\underline{r}, \underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}, \underline{r}')\frac{\partial\psi(\underline{r}')}{\partial \mathrm{n}}\right\}dS \tag{42}$$

and ∂/∂n denotes differentiation along the outward normal to S'.

Let us now take the volume V' to coincide either with the scattering volume V or with the exterior $\tilde{V}$ of it. Again the field point $\underline{r}$ may be taken to be either in V or in $\tilde{V}$. Thus, in analogy with the electromagnetic case, we obtain four formulae:

(a) $\underline{\underline{r} \in V,\ \underline{r}' \in V}$

$$\psi(\underline{r}_<) = -\frac{1}{4\pi}\int_V \psi(r')U(\underline{r}')G_0(\underline{r}_<, \underline{r}')\mathrm{d}^3\underline{r}' - \frac{1}{4\pi}\sum(\underline{r}_<), \tag{43a}$$

(b) $\underline{\underline{r} \in V,\ \underline{r}' \in \tilde{V}}$

$$0 = -\frac{1}{4\pi}\int_{\tilde{V}} \psi(\underline{r}')U(\underline{r}')G_0(\underline{r}_<, \underline{r}')d^3\mathrm{r}' + \frac{1}{4\pi}\sum(\underline{r}_<) - \frac{1}{4\pi}\sum\nolimits^{(\infty)}(r_<), \tag{43b}$$

(c) $\underline{\underline{r} \in \tilde{V},\ \underline{r}' \in \tilde{V}}$

$$\psi(\underline{r}_>) = -\frac{1}{4\pi}\int_{\tilde{V}} \psi(\underline{r}')U(\underline{r}')G_0(\underline{r}_>, \underline{r}')d^3\underline{r}' + \frac{1}{4\pi}\sum(\underline{r}_>) - \frac{1}{4\pi}\sum\nolimits^{(\infty)}(r_>), \tag{43c}$$

(d) $\underline{\underline{r} \in \tilde{V},\ \underline{r}' \in V}$

$$0 = -\frac{1}{4\pi}\int_V \psi(\underline{r}')U(\underline{r}')G_0(\underline{r}_>, \underline{r}')d^3\mathrm{r}' - \frac{1}{4\pi}\sum(\underline{r}_>), \tag{43d}$$

where

$$\sum(\underline{r}) = \int_S\left\{\psi(\underline{r}')\frac{\partial G_0(\underline{r}, \underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}, \underline{r}')\frac{\partial\psi(\underline{r}')}{\partial \mathrm{n}}\right\}dS, \tag{44a}$$

$$\sum^{(\infty)}(r) = \lim_{R\to\infty} \int_{S_\mathrm{R}} \left\{ \psi(\underline{r}') \frac{\partial G_0(\underline{r}, \underline{r}')}{\partial \mathrm{n}} - G_0(\underline{r}, \underline{r}') \frac{\partial \psi(\underline{r}')}{\partial \mathrm{n}} \right\} dS, \qquad (44\mathrm{b})$$

$\sum^{(\infty)}$ being a contribution from a sphere of limitingly large radius $R \to \infty$ and $\partial/\partial\mathrm{n}$ denotes differentiation along the outward normals to the respective volume regions. In the integral (44a) for $\sum(\underline{r})$ we need not distinguish between limits from inside and outside of V since ψ and $\partial\psi/\partial\mathrm{n}$ must be continuous at the surface S [cf. Eq. (10) above].

The integrals over V in (43b) and (43c) vanish since $U(\underline{r}) = 0$ in $\tilde{V}$. Also the contribution from the surface at infinity must evidently represent the incident wave, i.e.

$$-\frac{1}{4\pi} \sum^{(\infty)} (\underline{r}) = \psi^{(\mathrm{i})}(\underline{r}). \qquad (45)$$

Hence (40b) and (40c) reduce to

$$\psi^{(\mathrm{i})}(\underline{r}_<) + \frac{1}{4\pi} \sum(\underline{r}_<) = 0, \qquad (46)$$

$$\psi(\underline{r}_>) = \psi^{(\mathrm{i})}(\underline{r}_>) + \frac{1}{4\pi} \sum(\underline{r}_>). \qquad (47)$$

Further from (43a) we obtain if we also use (46)

$$\psi(\underline{r}_<) = \psi^{(\mathrm{i})}(\underline{r}_<) - \frac{1}{4\pi} \int_V \psi(\underline{r}')U(\underline{r}')G_0(\underline{r}_<, \underline{r}')d^3\underline{r}', \qquad (48)$$

and from (43d) if we use (47)

$$\psi(\underline{r}_>) = \psi^{(\mathrm{i})}(\underline{r}_>) - \frac{1}{4\pi} \int_V \psi(\underline{r}')U(\underline{r}')G_0(\underline{r}_>, \underline{r}')d^3\underline{r}'. \qquad (49)$$

The relations (46) and (47) are strictly analogous to those that we found for the electromagnetic case. In particular (46) represents *an extinction theorem* expressing the cancellation of the incident wave function at every point inside the potential barrier or potential well in terms of the values of the total wave function ψ and its normal derivative $\partial\psi/\partial\mathrm{n}$ at all points on the boundary of the potential barrier or potential well. We assert that this theorem has the same kind of significance as we postulated for the electromagnetic extinction theorem: It is a (non-local) boundary condition subject to which the Schrödinger equation (38) has to be solved inside the scattering volume V. Once this solution is known the wave function outside V can be determined from (47) by substitution. Note that (47) involves only a surface integral $\sum(\underline{r}_>)$, not a volume integral as it does in the usual formulation. The other two equations (47) and (48), which are seen to be of the same form irrespective whether the field point is inside or outside V will be recognized as the *usual integral equations for potential scattering.* Mathematically they are equivalent to the Schrödinger equation (38) together with the two equations (46) and (47).

We made use of the quantum mechanical extinction theorem (46) and the associated formula (47) to solve simple scattering problems, in order to verify the correctness of this new formulation of scattering. The results agree with those obtained by conventional methods based on the integral equations (48) and (49). I might add that the extinction theorem also leads correctly to *bound states* under appropriate conditions. One finds in these cases that the Schrödinger equation can then be solved subject to our non-local boundary condition (expressed by the extinction theorem (46)) only when $\psi^{(1)}(\underline{r}) \equiv 0$.

Let me remark here that a quantum-mechanical extinction theorem, was also derived, from a different approach by Melvin Lax [12] in 1952 in his treatment of multiple scattering.

I will end by summarizing our main conclusions:

(1) We obtained on the basis of Maxwell's theory generalization of the classic Ewald–Oseen extinction theorem, valid rigorously for scattering from medium of any prescribed macroscopic response.

(2) We have put forward a hypothesis as to the true meaning of the theorem: it is a non-local boundary condition for the solution of the equation of motion for the *interior* scattering problem.

(3) We have shown that once the boundary values have been determined the *exterior* scattering problem can be solved in a novel way, involving only surface integrations.

(4) We have shown that these results have strict analogues in non-relativistic quantum-mechanical potential scattering and provide a new approach to solving such problems.

Finally let me say that scattering problems involving sharp boundaries, such as we have considered here are, in general, hard to solve even approximately, since the Born approximation cannot be used in such cases. It is possible that our new formulation might provide a basis for the development of useful approximate techniques for solving problems of this kind. For this new formulation takes very explicitly into account the sharp boundary, the very presence of which makes the usual perturbation methods inapplicable. We plan to discuss this and related problems in other publications.

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7. While this manuscript was being prepared for publication a paper reporting some closely related results was published by J. de Goede and P. Mazur, Physica *58*, 568 (1972). This paper also contains some additional references to publications concerning the extinction theorem.
8. A slightly different argument to that given below is needed if the incident field is a plane wave (i.e. if the source is at infinity), but the final formulae remain the same. The case of plane wave incidence is discussed explicitly in connection with the quantum mechanical extinction theorem, in the last part of this paper.
9. See, for example, Chen-To Tai, *Dyadic Green's Functions in Electromagnetic Theory* (In-text Educational Publishers, Scranton and San Francisco, 1971).
10. One of them was presented in reference 5j.
11. That this is so follows at once from the Maxwell equation $\nabla \cdot \underline{D} = 0$. For this implies that $0 = \nabla \cdot (\varepsilon \underline{E}) = \varepsilon \nabla \cdot \underline{E}$ (ε = dielectric constant), so that $\nabla \cdot \mathrm{E} = 0$ and hence also $\nabla \cdot \underline{P} = 0$ (because of the linearity of the medium), unless $\varepsilon = 0$.
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