Chapter 1

Linear Spaces

The basic background for solving equations is introduced here.

1.1 Linear Operators

Some mathematical operations have certain properties in common. These properties are given in the following definition.

Definition 1.1 An operator T which maps a linear space X into a linear space Y over the same scalar field S is said to be additive if

$$T(x+y) = T(x) + T(y), \qquad \text{for all } x, y \in X,$$

and homogeneous if

$$T(sx) = sT(x), \qquad \text{for all } x \in X, s \in S.$$

An operator that is additive and homogeneous is called a linear operator.

Many examples of linear operators exist.

Example 1.1 Define an operator T from a linear space X into it self by $T(x) = sx$, $s \in S$. Then T is a linear operator.

Example 1.2 The operator $D = \frac{d}{dt}$ mapping $X = C^1[0,1]$ into $Y = C[0,1]$ given by

$$D(x) = \frac{dx}{dt} = y(t), \ 0 \le t \le 1,$$

is linear.

If X and Y are linear spaces over the same scalar field S, then the set $L(X, Y)$ containing all linear operators from X into Y is a linear space over S if addition is defined by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \qquad \text{for all } x \in X,$$

and scalar multiplication by

$$(sT)(x) = s(T(x)), \qquad \text{for all } x \in X, \ s \in S.$$

We may also consider linear operators B mapping X into $L(X, Y)$. For an $x \in X$ we have

$$B(x) = T,$$

a linear operator from X into Y. Hence, we have

$$B(x_1, x_2) = (B(x_1))(x_2) = y \in Y.$$

B is called a bilinear operator from X into Y. The linear operators B from X into $L(X, Y)$ form a linear space $L(X, L(X, Y))$. This process can be repeated to generate j-linear operators ($j > 1$ an integer).

Definition 1.2 A linear operator mapping a linear space X into its scalar S is called a linear functional in X.

Definition 1.3 An operator Q mapping a linear space X into a linear space Y is said to be nonlinear if it is not a linear operator from X into Y.

1.2 Continuous Linear Operators

Some metric concepts of importance are introduced here.

Definition 1.4 An operator F from a Banach space X into a Banach space Y is continuous at $x = x^*$ if

$$\lim_{n\to\infty} \|x_n - x^*\|_X = 0 \implies \lim_{n\to\infty} \|F(x_n) - F(x^*)\|_Y = 0$$

Theorem 1.1 *If a linear operator T from a Banach space X into a Banach space Y is continuous at $x^* = 0$, then it is continuous at every point x of space X.*

Proof. We have $T(0) = 0$, and from $\lim_{n\to\infty} \|x_n\| = 0$ we get $\lim_{n\to\infty} \|T(x_n)\| = 0$. If sequence $\{x_n\}\,(n \geq 0)$ converges to x^* in X, by setting $y_n = x_n - x^*$ we obtain $\lim_{n\to\infty} \|y_n\| = 0$. By hypothesis this implies that

$$\lim_{n\to\infty} \|T(x_n)\| = \lim_{n\to\infty} \|T(x_n - x^*)\| = \lim_{n\to\infty} \|T(x_n) - T(x^*)\| = 0.$$ □

Definition 1.5 An operator F from a Banach space X into a Banach space Y is bounded on the set A in X if there exists a constant $c < \infty$ such that

$$\|F(x)\| \le c\|x\|, \qquad \text{for all } x \in A.$$

The greatest lower bound (infimum) of numbers c satisfying the above inequality is called the bound of F on A. An operator which is bounded on a ball (open) $U(z,r) = \{x \in X \mid \|x - z\| < r\}$ is continuous at z. It turns out that for linear operators the converse is also true.

Theorem 1.2 *A continuous linear operator T from a Banach space X into a Banach space Y is bounded on X.*

Proof. By the continuity of T there exists $\varepsilon > 0$ such that $\|T(z)\| < 1$, if $\|z\| < \varepsilon$. For $0 \ne z \in X$

$$\|T(z)\| \le \tfrac{1}{\varepsilon}\|z\|, \tag{1.1}$$

since $\|cz\| < \varepsilon$ for $|c| < \frac{\varepsilon}{\|z\|}$, and $\|T(cz)\| = |c| \cdot \|T(z)\| < 1$. Letting $c = \varepsilon^{-1}$ in (1.1), we conclude that operator T is bounded on X. □

The bound on X of a linear operator T denoted by $\|T\|_X$ or simply $\|T\|$ is called the norm of T. As in Theorem 1.2 we get

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|. \tag{1.2}$$

Hence, for any bounded linear operator T

$$\|T(x)\| \le \|T\| \cdot \|x\|, \qquad \text{for all } x \in X. \tag{1.3}$$

From now on, $L(X,Y)$ denotes the set of all bounded linear operators from a Banach space X into another Banach space Y. It also follows immediately that $L(X,Y)$ is a linear space if equipped with the rules of addition and scalar multiplication introduced in Section 1.1.

The proof of the following result is left as an exercise (see also [101], [124]).

Theorem 1.3 *The set $L(X,Y)$ is a Banach space for the norm* (1.2).

1.3 Equations

In a Banach space X solving a linear equation can be stated as follows: given a bounded linear operator T mapping X into itself and some $y \in X$, find an $x \in X$ such that

$$T(x) = y. \tag{1.4}$$

The point x (if it exists) is called a solution of Equation (1.4).

Definition 1.6 If T is a bounded linear operator in X and a bounded linear operator T_1 exists such that

$$T_1 T = T T_1 = I, \tag{1.5}$$

where I is the identity operator in X (i.e., $I(x) = x$ for all $x \in X$), then T_1 is called the inverse of T and we write $T_1 = T^{-1}$. That is,

$$T^{-1} T = T T^{-1} = I. \tag{1.6}$$

If T^{-1} exists, then Equation (1.4) has the unique solution

$$x = T^{-1}(y). \tag{1.7}$$

The proof of the following result is left as an exercise (see also [140], [185], [188]).

Theorem 1.4 *(Banach Lemma on Invertible Operators). If T is a bounded linear operator in X, T^{-1} exists if and only if there is a bounded linear operator P in X such that P^{-1} exists and*

$$\|I - PT\| < 1. \tag{1.8}$$

If T^{-1} exists, then

$$T^{-1} = \sum_{n=0}^{\infty} (I - PT)^n P \qquad \textit{(Neumann Series)} \tag{1.9}$$

and

$$\|T^{-1}\| \leq \frac{\|P\|}{1 - \|I - PT\|}. \tag{1.10}$$

Based on Theorem 1.4 we can immediately introduce a computational theory for Equation (1.4) composed by three factors:

(A) *Existence and Uniqueness.* Under the hypotheses of Theorem 1.4 Equation (1.4) has a unique solution x^*.

(B) *Approximation.* The iteration

$$x_{n+1} = P(y) + (I - PT)(x_n) \quad (n \geq 0) \tag{1.11}$$

gives a sequence $\{x_n\}$ $(n \geq 0)$ of successive approximations, which converges to x^* for any initial guess $x_0 \in X$.

(C) *Error Bounds.* Clearly the speed of convergence of iteration $\{x_n\}$ $(n \geq 0)$ to x^* is governed by the estimate:

$$\|x_n - x^*\| \leq \frac{\|I - PT\|^n}{1 - \|I - PT\|}\|P(y)\| + \|I - PT\|^n \|x_0\|. \tag{1.12}$$

1.4 Computing the Inverse of a Linear Operator

Let T be a bounded linear operator in X. One way to obtain an approximate inverse is to make use of an operator sufficiently close to T.

Theorem 1.5 *If T is a bounded linear operator in X, T^{-1} exists if and only if there is a bounded linear operator P_1 in X such that P_1^{-1} exists, and*

$$\|P_1 - T\| \leq \left\|P_1^{-1}\right\|^{-1}. \tag{1.13}$$

If T^{-1} exists, then

$$T^{-1} = \sum_{n=0}^{\infty} \left(I - P_1^{-1}T\right)^n P_1^{-1} \tag{1.14}$$

and

$$\left\|T^{-1}\right\| \leq \frac{\left\|P^{-1}\right\|}{1 - \left\|I - P_1^{-1}T\right\|} \leq \frac{\left\|P_1^{-1}\right\|}{1 - \left\|P_1^{-1}\right\| \|P_1 - T\|}. \tag{1.15}$$

Proof. Let $P = P_1^{-1}$ in Theorem 1.4 and note that by (1.13)

$$\left\|I - P_1^{-1}T\right\| = \left\|P_1^{-1}(P_1 - T)\right\| \leq \left\|P_1^{-1}\right\| \cdot \|P_1 - T\| < 1. \tag{1.16}$$

That is, (1.8) is satisfied. The bounds (1.15) follow from (1.10) and (1.16). That proves the sufficiency. The necessity is proved by setting $P_1 = T$, if T^{-1} exists. □

The following result is equivalent to Theorem 1.4.

Theorem 1.6 *A bounded linear operator T in a Banach space X has an inverse T^{-1} if and only if linear operators P, P^{-1} exist such that the series*

$$\sum_{n=0}^{\infty} (I - PT)^n P \tag{1.17}$$

converges. In this case we have

$$T^{-1} = \sum_{n=0}^{\infty} (I - PT)^n P.$$

Proof. If series (1.17) converges, then it converges to T^{-1} (see Theorem 1.4). The existence of P, P^{-1} and the convergence of series (1.17) is again established as in Theorem 1.4, by taking $P = T^{-1}$, when it exists. □

Definition 1.7 A linear operator N in a Banach space X is said to be nilpotent if

$$N^m = 0, \tag{1.18}$$

for some positive integer m.

Theorem 1.7 *A bounded linear operator T in a Banach space X has an inverse T^{-1} and only if there exist linear operators P, P^{-1} such that $I - PT$ is nilpotent.*

Proof. If P, P^{-1} exists and $I - PT$ is nilpotent, then series

$$\sum_{n=0}^{\infty} (I - PT)^n P = \sum_{n=0}^{m-1} (I - PT)^n P$$

converges to T^{-1} by Theorem 1.6. Moreover, if T^{-1} exists, then $P = T^{-1}$, $P^{-1} = T$ exists, and $I - PT = I - T^{-1}T = 0$ is nilpotent. □

1.5 Fréchet Derivatives

The computational techniques to be considered later make use of the derivative in the sense of Fréchet [185], [186], [229].

Definition 1.8 Let F be an operator mapping a Banach space X into a Banach space Y. If there exists a bounded linear operator L from X into

Y such that

$$\lim_{\|\Delta x\|\to 0} \frac{\|F(x_0+\Delta x)-F(x_0)-L(\Delta x)\|}{\|\Delta x\|}=0, \tag{1.19}$$

then P is said to be Fréchet differentiable at x_0, and the bounded linear operator

$$P'(x_0)=L \tag{1.20}$$

is called the first Fréchet-derivative of F at x_0. The limit in (1.19) is supposed to hold independently of the way that Δx approaches 0. Moreover, the Fréchet differential

$$\delta F(x_0,\Delta x)=F'(x_0)\,\Delta x \tag{1.21}$$

is an arbitrary close approximation to the difference $F(x_0+\Delta x)-F(x_0)$ relative to $\|\Delta x\|$, for $\|\Delta x\|$ small.

If F_1 and F_2 are differentiable at x_0, then

$$(F_1+F_2)'(x_0)=F_1'(x_0)+F_2'(x_0). \tag{1.22}$$

Moreover, if F_2 is an operator from a Banach space X into a Banach space Z, and F_1 is an operator from Z into a Banach space Y, their composition $F_1\circ F_2$ is defined by

$$(F_1\circ F_2)(x)=F_1(F_2(x)), \qquad \text{for all } x\in X. \tag{1.23}$$

It follows from Definition 1.8 that $F_1\circ F_2$ is differentiable at x_0 if F_2 is differentiable at x_0 and F_1 is differentiable at $F_2(x_0)$ of Z, with (chain rule):

$$(F_1\circ F_2)'(x_0)=F_1'(F_2(x_0))F_2'(x_0). \tag{1.24}$$

In order to differentiate an operator F we write:

$$F(x_0+\Delta x)-F(x_0)=L(x_0,\Delta x)\Delta x+\eta(x_0,\Delta x), \tag{1.25}$$

where $L(x_0,\Delta x)$ is a bounded linear operator for given $x_0,\Delta x$ with

$$\lim_{\|\Delta x\|\to 0} L(x_0,\Delta x)=L, \tag{1.26}$$

and

$$\lim_{\|\Delta x\|\to 0} \frac{\|\eta(x_0,\Delta x)\|}{\|\Delta x\|}=0. \tag{1.27}$$

Estimates (1.26) and (1.27) give

$$\lim_{\|\Delta x\|\to 0} L(x_0, \Delta x) = F'(x_0). \tag{1.28}$$

If $L(x_0, \Delta x)$ is a continuous function of Δx in some ball $U(0, R)$ $(R > 0)$, then

$$L(x_0, 0) = F'(x_0). \tag{1.29}$$

We need the definition of a mosaic:

Higher-order derivatives can be defined by induction:

Definition 1.9 If F is $(m-1)$-times Fréchet-differentiable ($m \geq 2$ an integer), and an m-linear operator A from X into Y exists such that

$$\lim_{\|\Delta x\|\to 0} \frac{\left\|F^{(m-1)}(x_0 + \Delta x) - F^{(m-1)}(x_0) - A(\Delta x)\right\|}{\|\Delta x\|} = 0, \tag{1.30}$$

then A is called the m-Fréchet-derivative of F at x_0, and

$$A = F^{(m)}(x_0) \tag{1.31}$$

Higher partial derivatives in product spaces can be defined as follows: Define

$$X_{ij} = L(X_j, X_i), \tag{1.32}$$

where $X_1, X_2, \ldots$ are Banach spaces and $L(X_j, X_i)$ is the space of bounded linear operators from X_j into X_i. The elements of X_{ij} are denoted by L_{ij}, etc. Similarly,

$$X_{ijm} = L(X_m, X_{ij}) = L(X_m, L(X_j, X_i)) \tag{1.33}$$

denotes the space of bounded bilinear operators from X_k into X_{ij}. Finally, we write

$$X_{ij_1j_2\cdots j_m} = L\left(X_{jk}, X_{ij_1j_2\cdots j_{m-1}}\right), \tag{1.34}$$

which denotes the space of bounded linear operators from X_{jm} into $X_{ij_1j_2\cdots j_{m-1}}$. The elements $A = A_{ij_1j_2\cdots j_m}$ of $X_{ij_1j_2\cdots j_m}$ are a generalization of m-linear operators [10], [54].

Consider an operator F_i from space

$$X = \prod_{p=1}^{n} X_{j_p} \tag{1.35}$$

into X_i, and that F_i has partial derivatives of orders $1, 2, \ldots, m-1$ in some ball $U(x_0, R)$, where $R > 0$ and

$$x_0 = \left(x_{j_1}^{(0)}, x_{j_2}^{(0)}, \ldots, x_{j_n}^{(0)}\right) \in X. \tag{1.36}$$

For simplicity and without loss of generality we renumber the original spaces so that

$$j_1 = 1, j_2 = 2, \ldots, j_n = n. \tag{1.37}$$

Hence, we write

$$x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}). \tag{1.38}$$

A partial derivative of order $(m-1)$ of F_i at x_0 is an operator

$$A_{iq_1q_2\cdots q_{m-1}} = \frac{\partial^{(m-1)} F_i(x_0)}{\partial x_{q_1} \partial x_{q_2} \cdots \partial x_{q_{m-1}}} \tag{1.39}$$

(in $X_{iq_1q_2\cdots q_{m-1}}$) where

$$1 \leq q_1, q_2, \ldots, q_{m-1} \leq n. \tag{1.40}$$

Let $P(X_{q_m})$ denote the operator from X_{q_m} into $X_{iq_1q_2\cdots q_{m-1}}$ obtained from (1.39) by letting

$$x_j = x_j^{(0)}, \quad j \neq q_m, \tag{1.41}$$

for some q_m, $1 \leq q_m \leq n$. Moreover, if

$$P'(x_{q_m}^{(0)}) = \frac{\partial}{\partial x_{q_m}} \cdot \frac{\partial^{m-1} F_i(x_0)}{\partial x_{q_1} \partial x_{q_2} \cdots \partial x_{q_{m-1}}} = \frac{\partial^m F_i(x_0)}{\partial x_{q_1} \cdots \partial x_{q_m}}, \tag{1.42}$$

exists, it will be called the partial Fréchet-derivative of order m of F_i with respect to $x_{q_1}, \ldots, x_{q_m}$ at x_0.

Furthermore, if F_i is Fréchet-differentiable m times at x_0, then

$$\frac{\partial^m F_i(x_0)}{\partial x_{q_1} \cdots \partial x_{q_m}} x_{q_1} \cdots x_{q_m} = \frac{\partial^m F_i(x_0)}{\partial x_{s_1} \partial x_{s_2} \cdots \partial x_{s_m}} x_{s_1} \cdots x_{s_m} \tag{1.43}$$

for any permutation $s_1, s_2, \ldots, s_m$ of integers $q_1, q_2, \ldots, q_m$ and any choice of points $x_{q_1}, \ldots, x_{q_m}$, from $X_{q_1}, \ldots, X_{q_m}$ respectively. Hence, if $F = (F_1, \ldots, F_t)$ is an operator from $X = X_1 \times X_2 \times \cdots \times X_n$ into $Y = Y_1 \times Y_2 \times \cdots \times Y_t$, then

$$F^{(m)}(x_0) = \left(\frac{\partial^m F_i}{\partial x_{j_1} \cdots \partial x_{j_m}}\right)_{x=x_0} \tag{1.44}$$

$i = 1, 2, \ldots, t$, $j_1, j_2, \ldots, j_m = 1, 2, \ldots, n$, is called the m-Fréchet derivative of F at $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$.

1.6 Integration

In this section we state results concerning the mean value theorem, Taylor's theorem, and Riemannian integration. The proofs are left out as exercises.

The mean value theorem for differentiable real functions f:

$$f(b) - f(a) = f'(c)(b - a), \tag{1.45}$$

where $c \in (a, b)$, does not hold in a Banach space setting. However, if F is a differentiable operator between two Banach spaces X and Y, then

$$\|F(x) - F(y)\| \leq \sup_{\bar{x} \in L(x,y)} \|F'(\bar{x})\| \cdot \|x - y\|, \tag{1.46}$$

where

$$L(x, y) = \{z : z = \lambda y + (1 - \lambda)x, \ 0 \leq \lambda \leq 1\}. \tag{1.47}$$

Set

$$z(\lambda) = \lambda y + (1 - \lambda)x, \quad 0 \leq \lambda \leq 1, \tag{1.48}$$

and

$$F(\lambda) = F(z(\lambda)) = F(\lambda y + (1 - \lambda)x). \tag{1.49}$$

Divide the interval $0 \leq \lambda \leq 1$ into n subintervals of lengths $\Delta\lambda_i$, $i = 1, 2, \ldots, n$, choose points λ_i inside corresponding subintervals and as in the real Riemann integral consider sums

$$\sum_{\sigma} F(\lambda_i)\Delta\lambda_i = \sum_{i=1}^{n} F(\lambda_i)\Delta\lambda_i, \tag{1.50}$$

where σ is the partition of the interval, and set

$$|\sigma| = \max_{(i)} \Delta\lambda_i. \tag{1.51}$$

Definition 1.10 If

$$S = \lim_{|\sigma| \to 0} \sum_{\sigma} F(\lambda_i)\, \Delta\lambda_i \tag{1.52}$$

exists, then it is called the Riemann integral from $F(\lambda)$ from 0 and 1, denoted by

$$S = \int_0^1 F(\lambda)\, d\lambda = \int_x^y F(\lambda)\, d\lambda. \tag{1.53}$$

Definition 1.11 A bounded operator $P(\lambda)$ on $[0,1]$ such that the set of points of discontinuity is of measure zero is said to be integrable on $[0,1]$.

We now state the famous Taylor theorem [161].

Theorem 1.8 *If F is m-times Fréchet-differentiable in $U(x_0,R)$, $R>0$, and $F^{(m)}(x)$ is integrable from x to any $y \in U(x_0,R)$, then*

$$F(y) = F(x) + \sum_{n=1}^{m-1} \tfrac{1}{n!} F^{(n)}(x)(y-x)^n + R_m(x,y), \tag{1.54}$$

$$\left\| F(y) - \sum_{n=0}^{m-1} \tfrac{1}{n!} F^{(n)}(x)(y-x)^n \right\| \le \sup_{\bar{x}\in L(x,y)} \left\| F^{(m)}(\bar{x}) \right\| \frac{\|y-x\|^m}{m!}, \tag{1.55}$$

where

$$R_m(x,y) = \int_0^1 F^{(m)}\big(\lambda y + (1-\lambda)x\big)\,(y-x)^m \tfrac{(1-\lambda)^{m-1}}{(m-1)!} d\lambda. \tag{1.56}$$

1.7 Exercises

1.1 Show that the operators introduced in Examples 1.1 and 1.2 are indeed linear.

1.2. Show that the Laplace transform

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is a linear operator mapping the space of real functions $x = x(x_1,x_2,x_3)$ with continuous second derivatives on some subset D of R^3 into the space of continuous real functions on D.

1.3. Define $T : C''[0,1] \times C'[0,1] \to C[0,1]$ by

$$T(x,y) = \left(\alpha \frac{d^2}{dt^2}\ \beta \frac{d}{dt}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \frac{d^2 x}{dt^2} + \beta \frac{dy}{dt},\ 0 \le t \le 1.$$

Show that T is a linear operator.

1.4. In an inner product $\langle \cdot, \cdot \rangle$ space show that for any fixed z in the space

$$T(x) = \langle x, z \rangle$$

is a linear functional.

1.5. Show that an additive operator T from a real Banach space X into a real Banach space Y is homogeneous if it is continuous.

1.6. Show that matrix $A = \{a_{ij}\}$, $i, j = 1, 2, \ldots, n$ has an inverse if

$$|a_{ii}| > \tfrac{1}{2} \sum_{j=1}^{n} |a_{ij}| > 0, \quad i = 1, 2, \ldots, n.$$

1.7. Show that the linear integral equation of second Fredholm kind in $C[0,1]$

$$x(s) - \lambda \int_0^1 K(s,t)x(t)dt = y(s), \quad 0 \le \lambda \le 1,$$

where $K(s,t)$ is continuous on $0 \le s, t \le 1$, has a unique solution $x(s)$ for $y(s) \in C[0,1]$ if

$$|\lambda| < \left[\max_{[0,1]} \int_0^1 |K(s,t)|dt \right]^{-1}.$$

1.8. Prove Theorem 1.3.

1.9. Prove Theorem 1.4.

1.10. Show that the operators defined below are all linear.

(a) Identity operator. The identity operator $I_X : X \to X$ given by $I_X(x) = x$, for all $x \in X$.

(b) Zero operator. The zero operator $O : X \to Y$ given by $O(x) = 0$, for all $x \in X$.

(c) Integration. $T : C[a,b] \to C[a,b]$ given by $T(x(t)) = \int_0^1 x(s)ds$, $t \in [a,b]$.

(d) Differentiation. Let X be the vector space of all polynomials on $[a,b]$. Define T on X by $T(x(t)) = x'(t)$.

(e) Vector algebra. The cross product with one factor kept fixed. Define $T_1 : \mathbb{R}^3 \to \mathbb{R}^5$. Similarly, the dot product with one fixed factor. Define $T_2 : \mathbb{R}^3 \to \mathbb{R}$.

(f) Matrices. A real matrix $A = \{a_{ij}\}$ with m rows and n columns. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $y = Ax$.

1.11. Let T be a linear operator. Show:

(a) the $R(T)$ (range of T) is a vector space;
(b) if $\dim(T) = n < \infty$, then $\dim R(T) \leq n$;
(c) the null/space $N(T)$ is a vector space.

1.12. Let X, Y be vector spaces, both real or both complex. Let $T : D(T) \to Y$ (domain of T) be a linear operator with $D(T) \subseteq X$ and $R(T) \subseteq Y$. Then, show:

(a) the inverse $T^{-1} : R(T) \to D(T)$ exists if and only if

$$T(x) = 0 \Rightarrow x = 0;$$

(b) if T^{-1} exists, it is a linear operator;
(c) if $\dim D(T) = n < \infty$ and T^{-1} exists, then $\dim R(T) = \dim D(T)$.

1.13. Let $T : X \to Y$, $P : Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then, show: the inverse $(ST)^{-1} : Z \to X$ of the product ST exists, and

$$(ST)^{-1} = T^{-1}S^{-1}.$$

1.14. If the product (composite) of two linear operators exists, show that it is linear.

1.15. Let X be the vector space of all complex 2×2 matrices and define $T : X \to X$ by $T(x) = cx$, where $c \in X$ is fixed and cx denotes the usual product of matrices. Show that T is linear. Under what conditions does T^{-1} exist?

1.16. Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $R(T) = Y$ if and only if T^{-1} exists.

1.17. Define the integral operator $T : C[0,1] \to C[0,1]$ by $y = T(x)$, where $y(t) = \int_0^1 k(x,s)x(s)ds$ and k is continuous on $[0,1] \times [0,1]$. Show that T is linear and bounded.

1.18. Show that the operator T defined in 10(f) is bounded.

1.19. If a normed space X is finite dimensional then show that every linear functional on X is bounded.

1.20. Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subseteq X$ and X, Y are normed spaces. Show:

(a) T is continuous if and only if it is bounded;
(b) if T is continuous at a single point, it is continuous.

1.21. Let T be a bounded linear operator. Show:

(a) $x_n \to x$ (where x_n, $x \in D(T)$) $\Rightarrow T(x_n) \to T(x)$;
(b) the null space $N(T)$ is closed.

1.22. If $T \neq 0$ is a bounded linear operator, show that for any $x \in D(T)$ such that $\|x\| < 1$, we have $\|T(x)\| < \|T\|$.

1.23. Show that the operator $T : \ell^\infty \to \ell^\infty$ defined by $y = (y_i) = T(x)$, $y_i = \frac{x_i}{i}$, $x = (x_i)$, is linear and bounded.

1.24. Let $T : C[0,1] \to C[0,1]$ be defined by

$$y(t) = \int_0^t x(s)ds.$$

Find $R(T)$ and $T^{-1} : R(T) \to C[0,1]$. Is T^{-1} linear and bounded?

1.25. Show that the functionals defined on $C[a,b]$ by

$$f_1(x) = \int_a^b x(t)y_0(t)dt \quad (y_0 \in C[a,b])$$

$$f_2(x) = c_1 x(a) + c_2 x(b) \quad (c_1, c_2 \text{ fixed})$$

are linear and bounded.

1.26. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^0 x(t)dt - \int_0^1 x(t)dt.$$

1.27. Show that

$$f_1(x) = \max_{t \in J} x(t), \quad f_2(x) = \min_{t \in J} x(t), \quad J = [a,b]$$

define functionals on $C[a,b]$. Are they linear? Bounded?

1.28. Show that a function can be additive and not homogeneous. For example, let $z = x + iy$ denote a complex number, and let $T : \mathbb{C} \to \mathbb{C}$ be given by

$$T(z) = \bar{z} = x - iy.$$

1.29. Show that a function can be homogeneous and not additive. For example, consider the operator $T : \mathbb{R}^2 \to \mathbb{R}$ given by

$$T((x_1, x_2)) = \frac{x_1^2}{x_2}.$$

1.30. Let F be an operator in $C[0,1]$ defined by

$$F(x)(s) = x(s)\int_0^1 \frac{s}{s+t}x(t)dt, \quad 0 \le \lambda \le 1.$$

Show that for $x_0, z \in C[0,1]$

$$F'(x_0)z = x_0(s)\int_0^1 \frac{s}{s+t}z(t)dt + z(s)\int_0^1 \frac{s}{s+t}x_0(t)dt.$$

1.31. Find the Fréchet-derivative of the operator F in $\mathbb{R}^2_\infty$ given by

$$F\binom{x}{y} = \begin{pmatrix} x^2 + 7x + 2xy - 3 \\ x + y^3 \end{pmatrix}.$$

1.32. Find the first and second Fréchet-derivatives of the Uryson operator

$$U(x) = \int_0^1 k(s,t,x(t))dt$$

in $C[0,1]$ at $x_0 = x_0(s)$.

1.33. Find the Fréchet-derivative of the Riccati differential operator

$$R(z) = \frac{dz}{dt} + p(t)z^2 + q(t)z + r(t),$$

from $C'[0,s]$ into $C[0,s]$ at $z_0 = z_0(t)$ in $C'[0,s]$.

1.34. Find the first two Fréchet-derivatives of the operator

$$F\binom{x}{y} = \begin{pmatrix} x^2 + y^2 - 3 \\ x \sin y \end{pmatrix} \quad \text{in } \mathbb{R}^2.$$

1.35. Consider the partial differential operator

$$F(x) = \Delta x - x^2$$

from $C^2(I)$ into $C(I)$, the space of all continuous function on the square $0 \le \alpha, \beta \le 1$. Show that

$$F'(x_0)z = \Delta z(\alpha,\beta) - 2x_0(\alpha,\beta)z(\alpha,\beta),$$

where Δ is the usual Laplace operator.

1.36. Let $F(L) = L^3$, in $L(x)$. Show:

$$F'(L_0) = L_0[\ ]L_0 + L_0^2[\ ] + [\ ]L_0.$$

1.37. Let $F(L) = L^{-1}$, in $L(x)$. Show:

$$F'(L_0) = -L_0^{-1}[\ ]L_0^{-1},$$

provided that L_0^{-1} exists.

1.38. Show estimates (1.45) and (1.46).

1.39. Show Taylor's Theorem 1.8.

1.40. Integrate the operator

$$F(L) = L^{-1} \text{ in } L(X)$$

from $L_0 = I$ to $L_1 = A$, where $\|I - A\| < 1$.