

Preface

Researchers in computational sciences are faced with the problem of solving a variety of equations. A large number of problems are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modelled by difference or differential equations, and their solutions represent usually the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $x' = f(x)$, where x is the state, then the equilibrium states are determined by solving the equations $f(x) = 0$. Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except special cases, the most commonly used solutions methods are iterative, when starting from one or several initial approximations a sequence is constructed, which converges to a solution of the equation. Iteration methods are applied also for solving optimization problems. In such cases the iteration sequences converge to an optimal solution of the problem in hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

To complicate the matter further, many of these equations are nonlinear. However, all may be formulated in terms of operators mapping a linear space into another, the solutions being sought as points in the corresponding space. Consequently, computational methods that work in this general setting for the solution of equations apply to a large number of problems, and lead directly to the development of suitable computer programs to obtain accurate approximate solutions to equations in the appropriate space.

This book is intended for researchers, practitioners and students in com-

putational sciences. The goal is to introduce these powerful concepts and techniques at the earliest possible stage. The reader is assumed to have had basic courses in numerical analysis, computer programming, computational linear algebra, and an introduction to real, complex, and functional analysis.

We have divided the material into several chapters. Each section of a chapter is as independent from another as possible, so the reader interested in a particular method/result can access directly the information without studying previous or following sections. Each chapter contains several new theoretical results and important applications in engineering, in dynamic economic systems, in input-output systems, in the solution of nonlinear and linear differential equations, and optimization problems. Sections have been written as independent of each other as possible. Hence the interested reader can go directly to a certain section and understand the material without having to go back and forth in the whole textbook to find related material.

There are three basic problems connected with iterative methods.

Problem 1 *Show that the iterates are well defined. For example, if the algorithm requires the evaluation of F at each x_n , it has to be guaranteed that the iterates remain in the domain of F . It is, in general, impossible to find the exact set of all initial data for which a given process is well defined, and we restrict ourselves to giving conditions which guarantee that an iteration sequence is well defined for certain specific initial guesses.*

Problem 2 *Concerns the convergence of the sequences generated by a process and the question of whether their limit points are, in fact, solutions of the equation. There are several types of such convergence results. The first, which we call a local convergence theorem, begins with the assumption that a particular solution x^* exists, and then asserts that there is a neighborhood U of x^* such that for all initial vectors in U the iterates generated by the process are well defined and converge to x^* . The second type of convergence theorem, which we call semilocal, does not require knowledge of the existence of a solution, but states that, starting from initial vectors for which certain—usually stringent—conditions are satisfied, convergence to some (generally nearby) solutions x^* is guaranteed. Moreover, theorems of this type usually include computable (at least in principle) estimates for the error $x_n - x^*$, a possibility not afforded by the local convergence theorems. Finally, the third and most elegant type of convergence result, the global theorem, asserts that starting anywhere in a linear space, or at least in a large part of it, convergence to a solution is assured.*

Problem 3 *Concerns the economy of the entire operations, and, in particular, the question of how fast a given sequence will converge. Here, there are two approaches, which correspond to the local and semilocal convergence theorems. As mentioned above, the analysis which leads to the semilocal type of theorem frequently produces error estimates, and these, in turn, may sometimes be reinterpreted as estimates of the rate of convergence of the sequence. Unfortunately, however, these are usually overly pessimistic. The second approach deals with the behavior of the sequence $\{x_n\}$ when n is large, and hence when x_n is near the solutions x^* . This behavior may then be determined, to a first approximation, by the properties of the iteration function near x^* and leads to so-called asymptotic rates of convergence.*

We have included a variety of new results dealing with problems 1–3.

This textbook is an outgrowth of research work undertaken by us and complements/updates earlier works of ours focusing on in depth treatment of convergence theory for iterative methods [68]–[99], and the references there. Such a comprehensive study of optimal iterative procedures appears to be needed and should benefit not only those working in the field but also those interest in, or in need of, information about specific results or techniques. We have endeavored to make the main text as self contained as possible, to prove all results in full detail and to include a number of exercises throughout the textbook. In order to make the study useful as a reference source, we have complemented each section with a set of “Remarks” in which literature citations are given, other related results are discussed, and various possible extensions of the results of the text are indicated. For completion, the book ends with a comprehensive list of references. Because we believe our readers come from diverse backgrounds and have varied interests, we provide “recommended reading” throughout the textbook. Often a long textbook summarized knowledge in a field. This textbook, however, may be viewed as a report on work in progress. We provide a foundation for a scientific field that is rapidly changing. Therefore we list numerous conjectures and open problems as well as alternative models which need to be explored.

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