Preface

Since Richard Feynman's death in 1988 it has become increasingly evident that he was one of the most brilliant and original theoretical physicists of the twentieth century.[1] The Nobel Prize in Physics for 1965, shared with Julian Schwinger and Sin-itiro Tomonaga, rewarded their independent path-breaking work on the renormalization theory of quantum electrodynamics (QED). Feynman based his own formulation of a consistent QED, free of meaningless infinities, upon the work in his doctoral thesis of 1942 at Princeton University, which is published here for the first time. His new approach to quantum theory made use of the Principle of Least Action and led to methods for the very accurate calculation of quantum electromagnetic processes, as amply confirmed by experiment. These methods rely on the famous "Feynman diagrams," derived originally from the path integrals, which fill the pages of many articles and textbooks. Applied first to QED, the diagrams and the renormalization procedure based upon them also play a major role in other quantum field theories, including quantum gravity and the current "Standard Model" of elementary particle physics. The latter theory involves quarks and leptons interacting through the exchange of renormalizable Yang–Mills non-Abelian gauge fields (the electroweak and color gluon fields).

The path-integral and diagrammatic methods of Feynman are important general techniques of mathematical physics that have many applications other than quantum field theories: atomic and molecular scattering, condensed matter physics, statistical mechanics, quantum liquids and solids, Brownian motion, noise, etc.[2] In addition to

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[1] Hans Bethe's obituary of Feynman [*Nature* **332** (1988), p. 588] begins: "Richard P. Feynman was one of the greatest physicists since the Second World War and, I believe, the most original."

[2] Some of these topics are treated in R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, Massachusetts, 1965). Also see M. C. Gutzwiller, "Resource Letter ICQM-1: The Interplay Between Classical and Quantum Mechanics," *Am. J. Phys.* **66** (1998), pp. 304–24; items 71–73 and 158–168 deal with path integrals.

its usefulness in these diverse fields of physics, the path-integral approach brings a new fundamental understanding of quantum theory. Dirac, in his transformation theory, demonstrated the complementarity of two seemingly different formulations: the matrix mechanics of Heisenberg, Born, and Jordan and the wave mechanics of de Broglie and Schrödinger. Feynman's independent path-integral theory sheds new light on Dirac's operators and Schrödinger's wave functions, and inspires some novel approaches to the still somewhat mysterious interpretation of quantum theory. Feynman liked to emphasize the value of approaching old problems in a new way, even if there were to be no immediate practical benefit.

Early Ideas on Electromagnetic Fields

Growing up and educated in New York City, where he was born on 11 May 1918, Feynman did his undergraduate studies at the Massachusetts Institute of Technology (MIT), graduating in 1939. Although an exceptional student with recognized mathematical prowess, he was not a prodigy like Julian Schwinger, his fellow New Yorker born the same year, who received his PhD in Physics from Columbia University in 1939 and had already published fifteen articles. Feynman had two publications at MIT, including his undergraduate thesis with John C. Slater on "Forces and Stresses in Molecules." In that work he proved a very important theorem in molecular and solid-state physics, which is now known as the Hellmann–Feynman theorem.[3]

While still an undergraduate at MIT, as he related in his Nobel address, Feynman devoted much thought to electromagnetic interactions, especially the self-interaction of a charge with its own field, which predicted that a pointlike electron would have an infinite mass. This unfortunate result could be avoided in classical physics, either by not calculating the mass, or by giving the theoretical electron an

[3] L. M. Brown (ed.), *Selected Papers of Richard Feynman, with Commentary* (World Scientific, Singapore, 2000), p. 3. This volume (hereafter referred to as *SP*) includes a complete bibliography of Feynman's work.

extended structure; the latter choice makes for some difficulties in relativistic physics.

Neither of these solutions are possible in QED, however, because the extended electron gives rise to non-local interaction and the infinite pointlike mass inevitably contaminates other effects, such as atomic energy level differences, when calculated to high accuracy. While at MIT, Feynman thought that he had found a simple solution to this problem: Why not assume that the electron does not experience any interaction with its own electromagnetic field? When he began his graduate study at Princeton University, he carried this idea with him. He explained why in his Nobel Address:[4]

> *Well, it seemed to me quite evident that the idea that a particle acts on itself is not a necessary one — it is a sort of silly one, as a matter of fact. And so I suggested to myself that electrons cannot act on themselves; they can only act on other electrons. That means there is no field at all. There was a direct interaction between charges, albeit with a delay.*

A new classical electromagnetic field theory of that type would avoid such difficulties as the infinite self-energy of the point electron. The very useful notion of a field could be retained as an auxiliary concept, even if not thought to be a fundamental one. There was a chance also that if the new theory were quantized, it might eliminate the fatal problems of the then current QED. However, Feynman soon learned that there was a great obstacle to this delayed action-at-a-distance theory: namely, if a radiating electron, say in an atom or an antenna, were not acted upon at all by the field that it radiated, then it would not recoil, which would violate the conservation of energy. For that reason, some form of radiative reaction is necessary.

[4] *SP*, pp. 9–32, especially p. 10.

The Wheeler–Feynman Theory

Trying to work through this problem at Princeton, Feynman asked his future thesis adviser, the young Assistant Professor John Wheeler, for help. In particular, he asked whether it was possible to consider that two charges interact in such a way that the second charge, accelerated by absorbing the radiation emitted by the first charge, itself emits radiation that reacts upon the first. Wheeler pointed out that there would be such an effect but, delayed by the time required for light to pass between the two particles, it could not be the force of radiation reaction, which is instantaneous; also the force would be much too weak. What Feynman had suggested was not radiation reaction, but the reflection of light!

However, Wheeler did offer a possible way out of the difficulty. First, one could assume that radiation always takes place in a totally absorbing universe, like a room with the blinds drawn. Second, although the principle of causality states that all observable effects take place at a time later than the cause, Maxwell's equations for the electromagnetic field possess a radiative solution other than that normally adopted, which is delayed in time by the finite velocity of light. In addition, there is a solution whose effects are advanced in time by the same amount. A linear combination of retarded and advanced solutions can also be used, and Wheeler asked Feynman to investigate whether some suitable combination in an absorbing universe would provide the required observed instantaneous radiative reaction?

Feynman worked out Wheeler's suggestion and found that, indeed, a mixture of one-half advanced and one-half retarded interaction in an absorbing universe would exactly mimic the result of a radiative reaction due to the electron's own field emitting purely retarded radiation. The advanced part of the interaction would stimulate a response in the electrons of the absorber, and their effect at the source (summed over the whole absorber) would arrive at just the right time and in the right strength to give the required radiation reaction force, without assuming any direct interaction of the electron with its own radiation field. Furthermore, no apparent violation of the principle of causality arises from the use of advanced radiation. Wheeler and Feynman further explored this beautiful theory in articles published in the Reviews of Modern Physics (RMP) in 1945 and 1949.[5] In the first of these articles, no less than four different proofs are presented of the important result concerning the radiative reaction.

Quantizing the Wheeler–Feynman theory (Feynman's PhD thesis): *The Principle of Least Action in Quantum Mechanics*

Having an action-at-a-distance classical theory of electromagnetic interactions without fields, except as an auxiliary device, the question arises as to how to make a corresponding quantum theory. To treat a classical system of interacting particles, there are available analytic methods using generalized coordinates, developed by Hamilton and Lagrange, corresponding canonical transformations, and the principle of least action.[6] The original forms of quantum mechanics, due to Heisenberg, Schrödinger, and Dirac, made use of the Hamiltonian approach and its consequences, especially Poisson brackets. To quantize the electromagnetic field it was represented, by Fourier transformation, as a superposition of plane waves having transverse, longitudinal, and timelike polarizations. A given field was represented as mathematically equivalent to a collection of harmonic oscillators. A system of interacting particles was then described by a Hamiltonian function of three terms representing respectively the particles, the field, and their interaction. Quantization consisted of regarding these terms as Hamiltonian *operators*, the field's Hamiltonian describing a suitable infinite set of quantized harmonic oscillators. The combination of longitudinal and timelike oscillators

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[5] *SP*, p. 35–59 and p. 60–68. The second paper was actually written by Wheeler, based upon the joint work of both authors. It is remarked in these papers that H. Tetrode, W. Ritz, and G. N. Lewis had independently anticipated the absorber idea.

[6] W. Yourgrau and S. Mandelstam give an excellent analytic historical account in *Variational Principles in Dynamics and Quantum Theory* (Saunders, Philadelphia, 3rd edn., 1968).

was shown to provide the (instantaneous) Coulomb interaction of the particles, while the transverse oscillators were equivalent to photons. This approach, as well as the more general approach adopted by Heisenberg and Pauli (1929), was based upon Bohr's correspondence principle.

However, no method based upon the Hamiltonian could be used for the Wheeler–Feynman theory, either classically or quantum mechanically. The principal reason was the use of half-advanced and half-retarded interaction. The Hamiltonian method describes and keeps track of the state of the system of particles and fields at a given time. In the new theory, there are no field variables, and every radiative process depends on contributions from the future as well as from the past! One is forced to view the entire process from start to finish. The only existing classical approach of this kind for particles makes use of the principle of least action, and Feynman's thesis project was to develop and generalize this approach so that it could be used to formulate the Wheeler–Feynman theory (a theory possessing an action, but without a Hamiltonian). If successful, he should then try to find a method to quantize the new theory.[7]

The Introduction to the Thesis

Presenting his motivation and giving the plan of the thesis, Feynman's introductory section laid out the principal features of the (not yet published) delayed electromagnetic action-at-a-distance theory as described above, including the postulate that "fundamental (microscopic) phenomena in nature are symmetrical with respect to the interchange of past and future." Feynman claimed: "This requires that the solution of Maxwell's equation[s] to be used in computing the interactions is to be half the retarded plus half the advanced solution of Lienard and Wiechert." Although it would appear to contradict causality, Feynman stated that the principles of

[7] For a related discussion, including Feynman's PhD thesis, see S. S. Schweber, *QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomonaga* (Princeton University Press, Princeton, 1994), especially pp. 389–397.

the theory "do in fact lead to essential agreement with the results of the more usual form of electrodynamics, and at the same time permit a consistent description of point charges and lead to a unique law of radiative damping .... It is shown that these principles are equivalent to the equations of motion resulting from a principle of least action."

To explain the spontaneous decay of excited atoms and the existence of photons, both seemingly contradicted by this view, Feynman argued that "an atom alone in empty space would, in fact, *not* radiate ... and all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly, and according to quantum mechanical laws."

Two important points conclude the introduction. First, although the Wheeler–Feynman theory clearly furnished its motivation: "It is to be emphasized ... that the work described here is complete in itself without regard to its application to electrodynamics ... [The] present paper is concerned with the problem of finding a quantum mechanical description applicable to systems which in their classical analogue are expressible by a principle of least action, and not necessarily by Hamiltonian equations of motion." The second point is this: "All of the analysis will apply to non-relativistic systems. The generalization to the relativistic case is not at present known."

Classical Dynamics Generalized

The second section of the thesis discusses the theory of functionals and functional derivatives, and it generalizes the principle of least action of classical dynamics. Applying this method to the particular example of particles interacting through the intermediary of classical harmonic oscillators (an analogue of the electromagnetic field), Feynman shows how the coordinates of the oscillators can be eliminated and how their role in the interaction is replaced by a direct delayed interaction of the particles. Before this elimination process, the system consisting of oscillators and particles possesses a Hamiltonian but afterward, when the particles have direct interaction, no

Hamiltonian formulation is possible. Nevertheless, the equations of motion can still be derived from the principle of least action. This demonstration sets the stage for a similar procedure to be carried out in the quantized theory developed in the third and final section of the thesis.

In classical dynamics, the action is given by

$$S = \int L(q(t), \dot{q}(t))dt\,,$$

where L is a function of the generalized coordinates $q(t)$ and the generalized velocities $\dot{q} = dq/dt$, the integral being taken between the initial and final times t_0 and t_1, for which the set of q's have assigned values. The action depends on the paths $q(t)$ taken by the particles, and thus it is a functional of those paths. The *principle of least* action states that for "small" variations of the paths, the end points being fixed, the action S is an extremum, in most cases a minimum. An equivalent statement is that the functional derivative of S is zero. In the usual treatment, this principle leads to the Lagrangian and Hamiltonian equations of motion.

Feynman illustrates how this principle can be extended to the case of a particle (perhaps an atom) interacting with itself through advanced and retarded waves, by means of a mirror. An interaction term of the form $k^2\dot{x}(t)\dot{x}(t+T)$ is added to the Lagrangian of the particle in the action integral, T being the time for light to reach the mirror and return to the particle. (As an approximation, the limits of integration of the action integral are taken as negative and positive infinity.) A simple calculation, setting the variation of the action equal to zero, leads to the equation of motion of the particle. This shows that the force on the particle at time t depends on the particle's motion at times t, $t-T$, and $t+T$. That leads Feynman to observe: "The equations of motion cannot be described directly in Hamiltonian form."

After this simple example, there is a section discussing the restrictions that are needed to guarantee the existence of the usual constants of motion, including the energy. The thesis then treats the more complicated case of particles interacting via intermediate oscillators. It is shown how to eliminate the oscillators and obtain direct delayed action-at-a-distance. Interestingly, by making a suitable choice of the action functional, one can obtain particles either with or without self-interaction.

While still working on formulating the classical Wheeler–Feynman theory, Feynman was already beginning to adopt the over-all space-time approach that characterizes the quantization carried out in the thesis and in so much of his subsequent work, as he explained in his Nobel Lecture:[8]

> *By this time I was becoming used to a physical point of view different from the more customary point of view. In the customary view, things are discussed as a function of time in very great detail. For example, you have the field at this moment, a different equation gives you the field at a later moment and so on; a method, which I shall call the Hamiltonian method, a time differential method. We have, instead [the action] a thing that describes the character of the path throughout all of space and time. The behavior of nature is determined by saying her whole space-time path has a certain character. For the action [with advanced and retarded terms] the equations are no longer at all easy to get back into Hamiltonian form. If you wish to use as variables only the coordinates of particles, then you can talk about the property of the paths — but the path of one particle at a given time is affected by the path of another at a different time . . . . Therefore, you need a lot of bookkeeping variables to keep track of what the particle did in the past. These are called field variables . . . . From the overall space-time point of view of the least action principle, the field disappears as nothing but bookkeeping variables insisted on by the Hamiltonian method.*

Of the many significant contribution to theoretical physics that Feynman made throughout his career, perhaps none will turn out to

[8] "The development of the space-time view of quantum electrodynamics," *SP*, pp. 9–32, especially p. 16.

be of more lasting value than his reformulation of quantum mechanics, complementing those of Heisenberg, Schrödinger, and Dirac.[9] When extended to the relativistic domain and including the quantized electromagnetic field, it forms the basis of Feynman's version of QED, which is now the version of choice of theoretical physics, and which was seminal in the development of the gauge theories employed in the Standard Model of particle physics.[10]

Quantum Mechanics and the Principle of Least Action

The third and final section of the thesis, together with the RMP article of 1949, presents the new form of quantum mechanics.[11] In reply to a request for a copy of the thesis, Feynman said he had not an available copy, but instead sent a reprint of the RMP article, with this explanation of the difference:[12]

> *This article contains most of what was in the thesis. The thesis contained in addition a discussion of the relation between constants of motion such as energy and momentum and invariance properties of an action functional. Further there is a much more thorough discussion of the possible gen-*

[9] The action principle approach was later adopted also by Julian Schwinger. In discussing these formulations, Yourgrau and Mandelstam comment: "One cannot fail to observe that Feynman's principle in particular — and this is no hyperbole — expresses the laws of quantum mechanics in an exemplary neat and elegant manner, notwithstanding the fact that it employs somewhat unconventional mathematics. It can easily be related to Schwinger's principle, which utilizes mathematics of a more familiar nature. The theorem of Schwinger is, as it were, simply a translation of that of Feynman into differential notation." (Taken from Yourgrau and Mandelstam's book [footnote 6], p. 128.)

[10] Although it had initially motivated his approach to QED, Feynman found later that the quantized version of the Wheeler–Feynman theory (that is, QED without fields) could not account for the experimentally observed phenomenon known as vacuum polarization. Thus in a letter to Wheeler (on May 4, 1951) Feynman wrote: "I wish to deny the correctness of the assumption that electrons act only on other electrons . . . . So I think we guessed wrong in 1941. Do you agree?"

[11] R. P. Feynman, "Space-time approach to non-relativistic quantum mechanics," *Rev. Mod. Phys.* **20** (1948) pp. 367–387 included here as an appendix. Also in *SP*, pp. 177–197.

[12] Letter to J. G. Valatin, May 11, 1949.

> *eralization of quantum mechanics to apply to more general functionals than appears in the Review article. Finally the properties of a system interacting through intermediate harmonic oscillators is discussed in more detail.*

The introductory part of this third section of the thesis refers to Dirac's classical treatise for the usual formulation of quantum mechanics.[13]

However, Feynman writes that for those classical systems, which have no Hamiltonian form "no satisfactory method of quantization has been given." Thus he intends to provide one, based on the principle of least action. He will show that this method satisfies two necessary criteria: First, in the limit that $\hbar$ approaches zero, the quantum mechanical equations derived approach the classical ones, including the extended ones considered earlier. Second, for a system whose classical analogue does possess a Hamiltonian, the results are completely equivalent to the usual quantum mechanics.

The next section, "The Lagrangian in Quantum Mechanics" has the same title as an article of Dirac, published in 1933.[14] Dirac presents there an alternative version to a quantum mechanics based on the classical Hamiltonian, which is a function of the coordinates q and the momenta p of the system. He remarks that the Lagrangian, a function of coordinates and velocities, is more fundamental because the action defined by it is a relativistic invariant, and also because it admits a principle of least action. Furthermore, it is "closely connected to the theory of contact transformations," which has an important quantum mechanical analogue, namely, the transformation matrix $(q_t|q_T)$. This matrix connects a representation with the variables q diagonal at time T with a representation having the q's diagonal at time t. In the article, Dirac writes that $(q_t|q_T)$

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[13] P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford University Press, Oxford, 2nd edn., 1935). Later editions contain very similar material regarding the fundamental aspects to which Feynman refers.

[14] P. A. M. Dirac, in *Physikalische Zeitschrift der Sowjetunion*, Band 3, Heft 1 (1933), included here as an appendix. In discussing this material, Feynman includes a lengthy quotation from Dirac's *Principles*, 2nd edn., pp. 124–126.

"corresponds to" the quantity $A(tT)$, defined as

$$A(tT) = \exp\left[i \int_T^t L dt/\hbar\right] .$$

A bit later on, he writes that $A(t_T)$ "is the classical analogue of $(q_t|q_T)$."

When Herbert Jehle, who was visiting Princeton in 1941, called Feynman's attention to Dirac's article, he realized at once that it gave a necessary clue, based upon the principle of least action that he could use to quantize classical systems that do not possess a Hamiltonian. Dirac's paper argues that the classical limit condition for $\hbar$ approaching zero is satisfied, and Feynman shows this explicitly in his thesis. The procedure is to divide the time interval $t - T$ into a large number of small elements and consider a succession of transformations from one time to the next:

$$(q_t|q_T) = \iint \cdots \int (q_t|q_m) dq_m (q_m|q_{m-1}) dq_{m-1} \cdots (q_2|q_1) dq_1 (q_1|q_T) .$$

If the transformation function has a form like $A(tT)$, then the integrand is a rapidly oscillating function when $\hbar$ is small, and only those paths $(q_T, q_1, q_2, \ldots, q_t)$ give an appreciable contribution for which the phase of the exponential is stationary. In the limit, only those paths are allowed for which the action is a minimum; i.e., for which $\delta S = 0$, with

$$S = \int_T^t L dt .$$

For a very small time interval ε, the transformation function takes the form

$$A(t, t + \varepsilon) = \exp i L\varepsilon/\hbar ,$$

where $L = L((Q - q)/\varepsilon, Q)$, and we have let $q = q_t$ and $Q = q_{t+\varepsilon}$. Applying the transformation function to the wave function $\psi(q, t)$ to obtain $\psi(Q, t + \varepsilon)$ and expanding the resulting integral equation to first order in ε, Feynman obtains the Schrödinger equation. His derivation is valid for any Lagrangian containing at most quadratic terms in the velocities. In this way he demonstrates two important points: In the first place, the derivation shows that the usual results of quantum mechanics are obtained for systems possessing a classical Lagrangian from which a Hamiltonian can be derived. Second, he shows that Dirac's $A(tT)$ is not merely an analogue of $(q_t|q_T)$, but is *equal* to it, for a small time ε, up to a normalization factor. For a single coordinate, this factor is $N = \sqrt{2\pi i\varepsilon\hbar/m}$.

This method turns out to be an extraordinarily powerful way to obtain Feynman's path-integral formulation of quantum mechanics, upon which much of his subsequent thinking and production was based. Successive application of infinitesimal transformations provides a transformation of the wave function over a finite time interval, say from time T to time t. The Lagrangian in the exponent can be approximated to first order in ε, and

$$\psi(Q,T) \cong \iint \cdots \int \exp\left\{ \frac{i}{\hbar} \sum_{i=0}^{m} \left[L\left(\frac{q_{i+1}-q_i}{t_{i+1}-t_i}, q_{i+1} \right) (t_{i+1}-t_i) \right] \right\}$$
$$\times\ \psi(q_0,t_0) \frac{\sqrt{g_0}dq_0 \cdots \sqrt{g_m}dq_m}{N(t_1-t_0)\cdots N(T-t_m)}\,,$$

is the result obtained by induction, where $Q = q_{m+1}, T = t_{m+1}$, and the N's are the normalization factors (one for each q) referred to above. In the limit where ε goes to zero, the right-hand side is equal to $\psi(Q,T)$. Feynman writes: "The sum in the exponential resembles $\int_{t_0}^{T} L(q,\dot{q})dt$ with the integral written as a Riemann sum. In a similar manner we can compute $\psi(q_0,t_0)$ in terms of the wave function at a later time ..."

A sequence of q's for each t_i will, in the limit, define a path of the system and each of the integrals is to be taken over the entire range available to each q_i. In other words, the multiple integral is taken over all possible paths. We note that each path is continuous but not, in general, differentiable.

Using the idea of path integrals as in the expression above for $\psi(Q,T)$, Feynman considers expressions at a given time t_0, such as $\langle f(q_0)\rangle = \langle\chi|f(q_0)|\Psi\rangle$, which represents a quantum mechanical matrix element if χ and Ψ are different state functions or an expectation value if they represent the same state (i.e., $\chi = \Psi^*$). Path integrals relate the wave function $\psi(q_0, t_0)$ to an earlier time and the wave function $\chi(q_0, t_0)$ to a later time, which are taken as the distant past and future, respectively. By writing $\langle f(q_0) \rangle$ at two times separated by ε and letting ε approach zero, Feynman shows how to calculate the time derivative of $\langle f(q_t) \rangle$.

The next section of the thesis uses the language of functionals $F(q_i)$, depending on the values of the q's at the sequence of times t_i, to derive the quantum Lagrangian equations of motion from the path integrals. It shows the relation of these equations to q-number equations, such as $pq - qp = \hbar/i$ and discusses the relation of the Lagrangian formulation to the Hamiltonian one for cases where the latter exists. For example, the well-known result is derived that $HF - FH = (\hbar/i)\dot{F}$.

As was the case in the discussion of the classical theory, Feynman extends the formalism to the case of a more general action functional, beginning with the simple example of "a particle in a potential $V(x)$ and which also interacts with itself in a mirror, with half advanced and half retarded waves." An immediate difficulty is that the corresponding Lagrangian function involves two times. As a consequence, the action integral over the finite interval between times T_1 and T_2 is meaningless, because "the action might depend on values of $x(t)$ outside of this range." One can avoid this difficulty by formally letting the interaction vanish at times after large positive T_2 and before large negative T_1. Then for times outside the range of integration the particles are effectively free, so that wave functions can be defined at the endpoints. With this assumption the earlier discussions concerning functionals, operators, etc., can be carried through with the more general action functional.

However, the question as to whether a wave function or other wave-function-like object exists with the generalized Lagrangian is not solved in the thesis (and perhaps has never been solved). Although Feynman shows that much of quantum mechanics can be solved in terms of expectation values and transition amplitudes, at the end it is far from clear that it is possible to drop the very useful notion of the wave function (and if it is possible, it is probably not desirable to do so). A number of the pages of the thesis that follow are concerned with the question of the wave function, with conservation of energy, and with the calculation of transition probability amplitudes, including the development of a perturbation theory.

We shall not discuss these issues here, but continue to the last part of the thesis, where the forced harmonic oscillator is calculated. Based upon the path-integral solution of that problem, particles interacting through an intermediate oscillator are introduced and eventually the oscillators (i.e. the "field variables") are completely eliminated. Enrico Fermi had introduced the method of representing the electromagnetic field as a collection of oscillators and had eliminated the oscillators of longitudinal and timelike polarization to give the instantaneous Coulomb potential, as Feynman points out.[15] That had been the original aim of the thesis, to eliminate *all of the oscillators* (and hence the field) in order to quantize the Wheeler–Feynman action-at-a-distance theory. It turns out, however, that the elimination of all the oscillators was also very valuable in field theory having purely retarded interaction, and led in fact to the overall space-time point of view, to path integrals, and eventually to Feynman diagrams and renormalization.

We will sketch very briefly how Feynman handled the forced oscillator, using the symbol S for the generalized action. He wrote

$$S = S_0 + \int dt \left\{ \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} + \gamma(t)x \right\},$$

where S_0 is the action of the other particles of the system of which the oscillator $[x(t)]$ is a part, and $\gamma(t)x$ is the interaction of the oscillator with the particles that form the rest of the system. If $\gamma(t)$ is a simple function of time (for example $\cos\omega_1 t$) then it represents a given force applied to the oscillator. However, more generally we are dealing with an oscillator interacting with another quantum system and $\gamma(t)$ is a functional of the coordinates of that system. Since the action $S - S_0$ depends quadratically and linearly on $x(t)$, the path integrals

[15] Feynman mentions in this connection Fermi's influential article "Quantum theory of Radiation," *Rev. Mod. Phys.* **4** (1932) pp. 87–132. In this paper, the result is assumed to hold; it was proven earlier by Fermi in "Sopra l'elettrodynamica quantistica," *Rendiconti della R. Accademia Nazionali dei Lincei* **9** (1929) pp. 881–887.

over the paths of the oscillator can be performed when calculating the transition amplitude of the system from the initial time 0 to the final time T. With $x(0) = x$ and $x(T) = x'$, Feynman calls the function so obtained $G_\gamma(x, x'; T)$, obtaining finally the formula for the transition amplitude

$$\langle \chi_T | 1 | \psi_0 \rangle_S = \int \chi_T(Q_m, x) e^{\frac{i}{\hbar} S_0[\ldots Q_i \ldots]} G_\gamma(x, x'; T) \psi_0(Q_0, x') \times dx dx' \frac{\sqrt{g} dQ_m \cdots \sqrt{g} dQ_0}{N_m \cdots N_1},$$

where the Q's are the coordinates of the system other than the oscillator.

By using the last expression in the problem of particles interacting through an intermediate oscillator having $x(0) = \alpha$ and $x(T) = \beta$, Feynman shows that the expected value of a functional of the coordinates of the particles alone (such as a transition amplitude) can be obtained with a certain action that does not involve the oscillator coordinates, but only the constants α and β.[16] This eliminates the oscillator from the dynamics of the problem. Various other initial and/or final conditions on the oscillator are shown to lead to a similar result. A brief section labeled "Conclusion" completes the thesis.

Laurie M. Brown
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The editor (LMB) thanks to Professor David Kiang for his invaluable assistance in copy-editing the retyped manuscript and checking the equations.

[16] In the abstract at the end of the thesis this conclusion concerning the interaction of two systems is summarized as follows: "It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems."