

# Numerical Computation of the Marginal Distributions of a Semi-Markov Process

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## 1. Introduction

Industrial devices must be designed to prevent possible severe consequences from the failure of a device component. A way to ameliorate this design is to use probabilistic models of the device, from which technical and economical expectations can be drawn. Let us first give a mathematical background of such models. We consider a semi-Markov process  $(\eta_t)_{t \geq 0}$  taking its values in a finite space  $E$ . Let  $T_0 = 0$  and  $T_n$  ( $n \geq 1$ ) be the successive jump times of this process. We assume that the semi-Markov kernel of the process has a density  $q$  with respect to the Lebesgue measure. This means that, for all  $i_0, i_1, \dots, i_{n-1}, i, j \in E$ , all  $0 < s_1 < \dots < s_n$ , and all bounded

measurable function  $f$  defined on  $\mathbb{R}_+$ , we have

$$\begin{aligned} & \mathbb{E}(1_{\{\eta_{T_{n+1}}=j\}} f(T_{n+1} - T_n)/\eta_0 = i_0, \eta_{T_1} = i_1, T_1 = s_1, \dots, \\ & \quad \eta_{T_{n-1}} = i_{n-1}, T_{n-1} = s_{n-1}, \eta_{T_n} = i, T_n = s_n) \\ &= \mathbb{E}(1_{\{\eta_{T_{n+1}}=j\}} f(T_{n+1} - T_n)/\eta_{T_n} = i) \\ &= \int_{\mathbb{R}_+} f(t) q(i, j, t) dt. \end{aligned}$$

Let us define the transition rates  $a(i, j, t)$  between states  $i$  and  $j$  at time  $t$  by:

$$a(i, j, t) = \frac{q(i, j, t)}{\mathbb{P}(T_1 > t/\eta_0 = i)} = \frac{q(i, j, t)}{\int_t^{+\infty} \sum_{k \in E} q(i, k, u) du}. \quad (1)$$

Let us note that, since the values  $T_n$ ,  $n \in \mathbb{N}_*$ , are jump times, the relation  $0 = q(i, i, t) = a(i, i, t)$  must hold for all  $t \in \mathbb{R}_+$ .

**Remark 1.** An important case is the study of some component with a general failure rate, denoted by  $\lambda(t)$ , and a general repair rate, denoted by  $\mu(t)$ . The state of this component is then described by an alternating renewal process, i.e., a semi-Markov process taking its values in the set  $\mathbb{E} = \{0, 1\}$ , the values 1 and 0, respectively, representing the up-state and the down-state. The transition rates are then given by:

$$a(1, 0, t) = \lambda(t), \quad a(0, 1, t) = \mu(t).$$

The following properties help to understand the meaning of the transition rates  $a$ . It can be shown<sup>1</sup> that:

$$\begin{aligned} & \mathbb{P}(\eta_{T_1} = j, T_1 \leq t/\eta_0 = i) \\ &= \int_0^t a(i, j, s) \exp\left(-\int_0^s \sum_{k \in E} a(i, k, u) du\right) ds. \end{aligned}$$

Therefore we get

$$\mathbb{P}(T_1 > t/\eta_0 = i) = \exp\left(-\int_0^t \sum_{k \in E} a(i, k, u) du\right), \quad (2)$$

meaning that the hazard rate of  $T_1$  knowing  $\{\eta_0 = i\}$  is given by:

$$b(i, t) = \sum_{j \in E} a(i, j, t), \quad \forall t \in \mathbb{R}_+.$$

Thus, giving  $q$  or  $a$  are equivalent, since  $a$  is defined from  $q$  by Eq. (1) and  $q$  is computed from the values of  $a$ , using the relation,

$$q(i, j, t) = a(i, j, t) \exp \left( - \int_0^t \sum_{k \in E} a(i, k, u) du \right).$$

Returning to the example given in Remark 1, the availability of the component is then defined by  $A(t) = \mathbb{P}(\eta_t = 1)$ , that is, one marginal distribution of the process  $(\eta_t)_{t \geq 0}$ .

This chapter presents a new method to approximate the marginal distributions of a general semi-Markov process in the case of any initial distribution (i.e., any distribution of the process at time 0). In Sec. 2, we obtain the equations satisfied by these marginal distributions. Indeed, introducing the variable  $X_t$ , defined as the elapsed time without a jump, the equations fulfilled by the marginal distributions of the Markov process  $(\eta_t, X_t)$  are shown to be the solutions of some transport equations. Since the boundary conditions of these equations are expressed under an integral formulation, no analytical solution can be obtained in the general case. However, in the particular framework of Remark 1, three numerical methods derived from the renewal theory<sup>2</sup> allow a direct computation of the availability (i.e.,  $\mathbb{P}(\eta_t = 1)$ ), using discretization schemes for the resolution of Volterra integral equations. Unfortunately, the third method, which appears to present the best efficiency in most cases, fails in some realistic situations (for example, in the case where the failure rate of a component is much smaller than the repair rate). Moreover, the adaptation of the methods given in Ref. 2 to the general case of semi-Markov processes remains to be carefully studied, since no straightforward implementation seems to hold.

Therefore, in Sec. 3, a new numerical algorithm is shown to deliver a convergent approximation of the marginal distributions in

the general case of semi-Markov processes. Since the equations satisfied by these distributions are transport equations, a finite volume method is used. An advantage of this method is that it gives the possibility to handle the case of any initial distribution, whatever regularity is considered.

Finally, in Sec. 4, numerical examples in comparison with the methods given in Ref. 2, show an admissible but imprecise solution when other methods fail.

## 2. Equations for the Marginal Distributions

### 2.1. The general case

We denote by  $C_b(E \times \mathbb{R}_+)$  the Banach space of all real bounded functions, which are continuous with respect to the real argument and by  $C_b^1(E \times \mathbb{R}_+)$  the class of functions belonging to  $C_b(E \times \mathbb{R}_+)$ , which are continuously differentiable with respect to the real argument and whose derivatives belong to  $C_b(E \times \mathbb{R}_+)$ .

Let  $X_t$  be the elapsed time without a jump at time  $t$ :

$$X_t = t - T_n \quad \text{if } T_n \leq t < T_{n+1}.$$

Then the process  $(\eta_t, X_t)_{t \geq 0}$  is a Markov process, taking its values in  $E \times \mathbb{R}_+$ . In all the following, we assume that  $a$  is continuous. The following proposition can be proven in a more general case.<sup>1,3</sup>

**Proposition 1.** *For all  $h \in C_b^1(E \times \mathbb{R}_+)$ , let us define:*

$$Lh(i, x) = \sum_{j \in E} a(i, j, x)(h(j, 0) - h(i, x)) + \frac{\partial h}{\partial x}(i, x).$$

*Then, the following equation holds:*

$$\mathbb{E}(h(\eta_t, X_t)) = \mathbb{E}(h(\eta_0, X_0)) + \int_0^t \mathbb{E}(Lh(\eta_s, X_s)) ds. \quad (3)$$

Let  $\rho_t$  be the probability distribution of  $(\eta_t, X_t)$ . It can be written  $\rho_t(dx) = (\rho_t(i, dx))_{i \in E}$ . Heuristically, we have

$$\rho_t(i, dx) = \mathbb{P}(\eta_t = i, X_t \in [x, x + dx]),$$

and more precisely, for all  $h \in C_b(E \times \mathbb{R}_+)$

$$\int h d\rho_t = \sum_{i \in E} \int_0^{+\infty} h(i, x) \rho_t(i, dx).$$

Thus, Eq. (3) can also be written as:

$$\begin{aligned} & \sum_{i \in E} \int_0^{+\infty} h(i, x) \rho_t(i, dx) \\ &= \sum_{i \in E} \int_0^{+\infty} h(i, x) \rho_0(i, dx) \\ &+ \sum_{i \in E} h(i, 0) \int_0^t \sum_{j \in E} \int_0^{+\infty} a(j, i, x) \rho_s(j, dx) ds \\ &- \sum_{i \in E} \int_0^t \int_0^{+\infty} b(i, x) h(i, x) \rho_s(i, dx) ds \\ &+ \sum_{i \in E} \int_0^t \int_0^{+\infty} \frac{\partial h}{\partial x}(i, x) \rho_s(i, dx) ds. \end{aligned} \quad (4)$$

Our purpose is to find a numerical approximation of  $\rho_t$ , viewed as the solution of Eq. (4).

## 2.2. Case of an initial distribution given by a density

Let us suppose that the initial distribution has a density with respect to the Lebesgue measure, i.e., for all  $i \in E$ ,  $\rho_0(i, dx)$  can be written:

$$\rho_0(i, dx) = p_0(i, x) dx.$$

It can then be shown that, for all  $i \in E$ , there exists a function  $p_t(i, x)$  such that:

$$\rho_t(i, dx) = p_t(i, x) dx.$$

From Eq. (4), we get that this function verifies:

$$\begin{aligned}
 \int_0^{+\infty} g(x) p_t(i, x) dx &= \int_0^{+\infty} g(x) p_0(i, x) dx \\
 &+ g(0) \int_0^t \sum_j a(j, i, x) p_s(j, x) dx ds \\
 &- \int_0^t \int_0^{+\infty} b(i, x) g(x) p_s(i, x) dx ds \\
 &+ \int_0^t \int_0^{+\infty} g'(x) p_s(i, x) dx ds
 \end{aligned}$$

for all  $i \in E$  and  $g \in C_b^1(\mathbb{R}_+)$ . Assuming that for all  $j \in E$ , the function  $(s, x) \rightarrow p_s(j, x)$  is continuously differentiable, we can integrate by parts the last term of the above equation. We then get, by identification, that the function  $p_t(i, x)$  is, for all  $i \in E$  and  $x \in \mathbb{R}_+$ , the solution of the following system of linear hyperbolic equations:

$$\frac{\partial}{\partial t} p_t(i, x) + \frac{\partial}{\partial x} p_t(i, x) = -b(i, x) p_t(i, x), \quad (5)$$

with the coupled boundary condition

$$\begin{aligned}
 p_t(i, 0) &= \sum_{j \in E} \int_0^{+\infty} a(j, i, x) p_t(j, x) dx \\
 &= \sum_{j \in E} \int_0^{+\infty} a(j, i, x) \rho_t(j, dx). \quad (6)
 \end{aligned}$$

### 2.3. Case of a Dirac initial distribution

Let us now suppose that there exists  $(i_0, x_0) \in E \times \mathbb{R}_+$  such that:

$$\mathbb{P}(\eta_0 = i_0, X_0 = x_0) = 1.$$

In such a case, the measure  $\rho_t(i_0, dx)$  is no more absolutely continuous with respect to the Lebesgue measure. Indeed, the probability

that  $X_t = x_0 + t$ , which means that no jump occurs before time  $t$ , is given by:

$$\alpha(t) = \mathbb{P}(T_1 > t/\eta_0 = i_0, X_0 = x_0) = \exp\left(-\int_0^t b(i_0, x_0 + u)du\right).$$

Then, the marginal distributions are given by:

$$\rho_t(i, dx) = 1_{\{i=i_0\}}\alpha(t)\delta_{x_0+t}(dx) + p_t(i, x)dx. \quad (7)$$

Suppose that the functions  $p_t(j, x)$  are continuously differentiable with respect to  $x$ . Following the same steps as in Sec. 2.2, we obtain that Eq. (5) is satisfied, with the initial condition:

$$p_0(i, x) = 0$$

and the boundary condition:

$$p_t(i, 0) = a(i_0, i, x_0 + t)\alpha(t) + \sum_{j \in E} \int_0^{+\infty} a(j, i, x)p_t(j, x)dx \quad (8)$$

$$= \sum_{j \in E} \int_0^{+\infty} a(j, i, x)\rho_t(j, dx). \quad (9)$$

#### 2.4. Resolution of particular cases using convolution tools

The solution  $p_t(i, x)$  of Eq. (5) satisfies:

$$p_t(i, x) = p_0(i, x - t) \exp\left(-\int_0^t b(i, x - t + u)du\right) \quad \text{if } t < x, \quad (10)$$

$$p_t(i, x) = p_{t-x}(i, 0) \exp\left(-\int_0^x b(i, u)du\right) \quad \text{if } x \leq t. \quad (11)$$

Thanks to Eqs. (6) and (8), for both above particular cases, there exist some functions  $h_i(t)$  such that

$$p_t(i, 0) = h_i(t) + \sum_{j \in E} \int_0^{+\infty} a(j, i, x)p_t(j, x)dx.$$

Using Eqs. (10) and (11), we deduce

$$\begin{aligned}
 p_t(i, 0) &= h_i(t) + \sum_{j \in E} \int_t^{+\infty} a(j, i, x) p_0(j, x - t) \\
 &\quad \times \exp\left(-\int_0^t b(j, x - t + u) du\right) dx \\
 &\quad + \sum_{j \in E} \int_0^t a(j, i, x) p_{t-x}(j, 0) \exp\left(-\int_0^x b(j, u) du\right) dx \\
 &= k_i(t) + \sum_{j \in E} \int_0^t p_{t-x}(j, 0) q(j, i, x) dx, \tag{12}
 \end{aligned}$$

where the functions  $k_i$  are known. The numerical approximation of the solution of Eq. (12) requires some quite complex additional work in the general case.

Let us again consider the framework of the reliability theory such as presented in Remark 1, i.e., the case where  $E = \{0, 1\}$ . Let us suppose that the component is available at time  $t = 0$ , which means the initial distribution is a Dirac mass:

$$\mathbb{P}(\eta_0 = 1, X_0 = 0) = 1.$$

Let us denote by  $f$  (respectively  $g$ ) the probability density function of the duration of working periods (respectively failure periods). Using the above notations, we can write

$$x_0 = 0, \quad p_0 = 0, \quad k_0 = h_0 = \lambda\alpha = f, \quad k_1 = h_1 = 0.$$

Let us denote  $u_i(t) = p_t(i, 0)$ . Equation (12) delivers

$$u_1 = u_0 * g, \quad u_0 = f + u_1 * f,$$

and consequently we get

$$u_1 = f * g + u_1 * f * g, \quad u_0 = f + u_0 * f * g. \tag{13}$$

These equations show that  $u_1$  (respectively  $u_0$ ) is the renewal density associated with the renewal process corresponding to the end of the

repair periods (respectively associated with the component breaking up) (see for example Ref. 4, paragraph 4.4, formula (5) or Ref. 5, formula (6.5)). Let us define  $\bar{F}(t) = \int_t^{+\infty} f(u)du$ . From Eqs. (7) and (11), we deduce that the availability  $A(t) = \mathbb{P}(\eta_t = 1)$  of the component is given by:

$$\begin{aligned} A(t) &= \int_0^{+\infty} \rho_t(1, dx) \\ &= \bar{F}(t) + \int_0^{+\infty} p_t(1, x)dx \\ &= \bar{F}(t) + \int_0^t u_1(t-x) \exp\left(-\int_0^x b(1, u) du\right)dx. \end{aligned}$$

Thanks to Eq. (2), the above equation can be written as:

$$A = \bar{F} + u_1 * \bar{F}. \quad (14)$$

Using Eq. (13), we also have

$$A = \bar{F} + f * g * \bar{F} + u_1 * \bar{F} * f * g = \bar{F} + A * f * g, \quad (15)$$

which is the usual renewal equation for the availability of a component (see Ref. 6, paragraph 4.2.1 or Ref. 5, example 6.45). The implementation of Eqs. (14) and (15) (which respectively corresponds to methods I and II of Ref. 2) seems to fail on some relevant numerical examples, and will not be considered in this paper.

The following section is devoted to a new approach, also based on Eq. (4), which does not require additional regularity assumptions and whose implementation does not depend on the initial conditions.

### 3. An Approximation Using a Finite Volume Method

In this section, we consider a numerical method aimed to globally approximate the measures  $\rho_t$ , solution of the transport equation (4), for all  $t \in [0, T[$ , and for all type of initial condition. The interval

$[0, T[$  is divided into  $N_h$  intervals of length  $h = T/N_h$ , and we have

$$\mathbb{R}_+ \times [0, T[ = \bigcup_{m \geq 0} \bigcup_{n=0}^{N_h-1} [mh, (m+1)h[ \times [nh, (n+1)h[.$$

For all  $i \in E$ , we approximate the measure  $\rho_t(i, dx)$  by the measure  $\bar{u}_t^h(i, x)dx$ , where the function  $\bar{u}$  is equal to a constant, denoted by  $u_n^h(i, m)$ , on each square  $[mh, (m+1)h[ \times [nh, (n+1)h[$ :

$$\bar{u}_t^h(i, x) = u_n^h(i, m) \quad \text{if } (x, t) \in [mh, (m+1)h[ \times [nh, (n+1)h[.$$

The algorithm is initialized by a discretization of the initial distribution  $\rho_0$ :

$$u_0^h(i, m) = \frac{1}{h} \int_{[mh, (m+1)h[} \rho_0(i, dx).$$

Although our scheme approximates the measures  $\rho_t$  for all type of initial data, it can be seen as a numerical approximation of Eq. (5). We thus approximate the quantities  $\partial\rho/\partial t$  and  $\partial\rho/\partial x$  by:

$$\frac{u_{n+1}^h(i, m) - u_n^h(i, m)}{h} \quad \text{and} \quad \frac{u_n^h(i, m) - u_n^h(i, m-1)}{h},$$

which produces the following numerical scheme:

$$u_{n+1}^h(i, m) = \frac{u_n^h(i, m-1)}{1 + hb(i, mh)} \quad \text{for } m \geq 1. \quad (16)$$

Similarly, the boundary conditions are inspired by Eq. (6) or Eq. (9):

$$u_{n+1}^h(i, 0) = \sum_{j \in E} \sum_{m \geq 1} ha(j, i, mh) u_{n+1}^h(j, m). \quad (17)$$

Using the discrete values given by the scheme, we approximate the values  $\mathbb{P}(\eta_t = i) = \int_0^{+\infty} \rho_t(i, dx)$ , for all  $t \in [nh, (n+1)h[$ , by

$$P_n^h(i) = \int_0^{+\infty} \bar{u}_t^h(i, x) dx = h \sum_{m \geq 0} u_n^h(i, m).$$

The weak convergence of the measures  $\bar{u}_t^h(i, x)dx$  to the measures  $\rho_t(i, dx)$  is proven in Ref. 7, using a uniqueness result stated in Ref. 3.

## 4. Numerical Examples

### 4.1. Case of an alternating renewal process

We again consider the case described in Remark 1. We then set  $E = \{0, 1\}$ , and the numerical scheme can be written for  $n \geq 0$  and  $i \in \{0, 1\}$ :

$$u_{n+1}^h(i, m) = \frac{u_n^h(i, m - 1)}{1 + hb(i, mh)} \quad \text{for } m \geq 1,$$

$$u_{n+1}^h(i, 0) = \sum_{m \geq 1} (u_n^h(1 - i, m - 1) - u_{n+1}^h(1 - i, m)).$$

We are interested in the availability  $A(t) = \mathbb{P}(\eta_t = 1)$  of the component. It is known (see for example [1] Formula (4.2) or [2] Proposition 6.51) that the asymptotic availability is equal to:

$$A(\infty) = \frac{m_1}{m_1 + m_2}, \tag{18}$$

where  $m_1$  (respectively  $m_2$ ) is the mean duration of a working period (respectively failure period). We thus compare in Tables 1–3 the values  $A(T)$  provided by the finite volume method, by the third method of Ref. 2, and the value given by Eq. (18). We also compare the

Table 1. Asymptotic availability of Example 1.

	Finite volume method		Method III of Ref. 2	
	A(10 000)	comp. time	A(10 000)	comp. time
$N_h = 2000, h = 5$	0.62444	2.6	0.62465	1.2
$N_h = 10\,000, h = 1$	0.62461	84	0.62465	25

comp. time: computation time (CPU).

Table 2. Asymptotic availability for Example 2.

	Finite volume method		Method III of Ref. 2	
	$A(10\,000)$	comp. time	$A(10\,000)$	comp. time
$N_h = 2000, h = 5$	0.99259	2.8	>1 (n.s.)	1
$N_h = 10\,000, h = 1$	0.99375	85	0.99440 (n.s.)	24
$N_h = 60\,000, h = 0.17$	0.99399	$4 \cdot 10^3$	0.99404	$2.5 \cdot 10^3$

comp. time: computation time (CPU), (n.s.): not stabilized.

Table 3. Asymptotic availability for Example 3.

	Finite volume method		Method III of Ref. 2	
	$A(3000)$	comp. time	$A(3000)$	comp. time
$N_h = 2000, h = 1.5$	0.99775	2.8	0.12763 (n.s.)	1
$N_h = 10\,000, h = 0.3$	0.99880	85	0.85355 (n.s.)	25
$N_h = 60\,000, h = 0.05$	0.99896	$4.1 \cdot 10^3$	0.98851 (n.s.)	$2.3 \cdot 10^3$
$N_h = 120\,000, h = 0.025$	0.99897	$1.7 \cdot 10^4$	0.99527 (n.s.)	$10^4$

comp. time: computation time (CPU), (n.s.): not stabilized.

computing times. When the computed value of the availability does not seem to reach any asymptotic value at large  $t$ , the value  $A(T)$  is followed by “n.s.” for “not stabilized”. In the following figures, the horizontal lines represent this asymptotic availability. In Fig. 12, the value of the availability computed by the finite volume method is plotted in Fig. 12(a) whereas the one computed by Method III of Ref. 2 is plotted in Fig. 12(b).

### Example 1.

The probability distribution of the working periods is a Weibull distribution with a shape parameter  $\beta = 3$  and a mean value equal to

1000. The probability distribution of the failure periods is a Weibull distribution with a shape parameter  $\beta = 3.5$  and a mean value equal to 600. The results are plotted in Figs. 1 and 2. In this case, the application of Eq. (18) gives:  $A(\infty) = 0.62498$ .

In this example, Method III of Ref. 2 is faster and slightly better than the finite volume method.

Figure 3 shows the approximation of the measures  $\rho_{1900}$  by  $\bar{u}_{1900}^h$  for  $h = 1$  (time  $t = 1900$ ) in Fig. 3(a) and  $\rho_{2300}$  by  $\bar{u}_{2300}^h$  (time  $t = 2300$ ) in Fig. 3(b). There are two curves on each figure since both curves  $\bar{u}_t^h(1, \cdot)$  (solid line) and  $\bar{u}_t^h(0, \cdot)$  (dash dot line) are plotted at times  $t = 1900$  and  $2300$ . The approximation of Dirac mass at point  $t$  for  $\rho_t(1, \cdot)$  (see Eq. (7)) exceeds the vertical scale of the figure at the time  $t = 1900$  (the discrete value in the corresponding control volume is equal to  $7.7 \cdot 10^{-3}$ ). At time  $t = 2300$ , this value decreases until  $1.8 \cdot 10^{-4}$ , and it is covered by the thickness of the horizontal axis at large times (it is equal to  $1.6 \cdot 10^{-5}$  for  $t = 2500$ ).

### Example 2.

The probability distribution of the working periods is the same as that of Example 1 but the scale parameter of the failure duration distribution is modified: the probability distribution of the failure periods is defined as a Weibull distribution with a shape parameter  $\beta$  still equal to 3.5 but with a mean value equal to 6. The results are plotted in Figs. 4–6. In this case, the application of Eq. (18) gives:  $A(\infty) = 0.99404$ .

In this example, Method III of Ref. 2 can give inadmissible results (the implementation of the method should then be discussed) whereas the finite volume method, although it demands more computing time, remains robust and gives admissible results.

### Example 3.

The probability distribution of the working periods is a Weibull distribution with a shape parameter  $\beta = 2$  and a mean value equal to 886. The probability distribution of the failure periods is a Weibull distribution with a shape parameter  $\beta = 1.5$  and a mean value equal

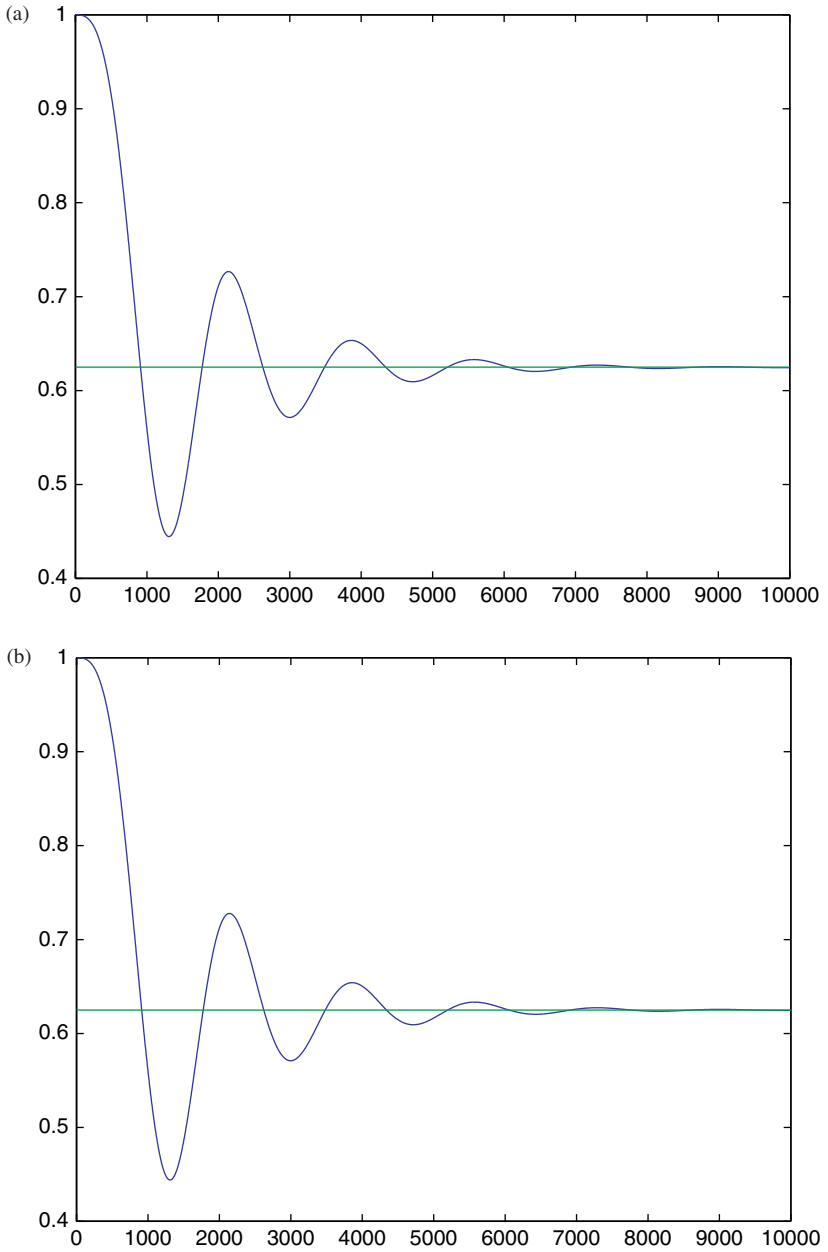


Fig. 1. Availability for Example 1,  $N_h = 2000$ ,  $h = 5$ .

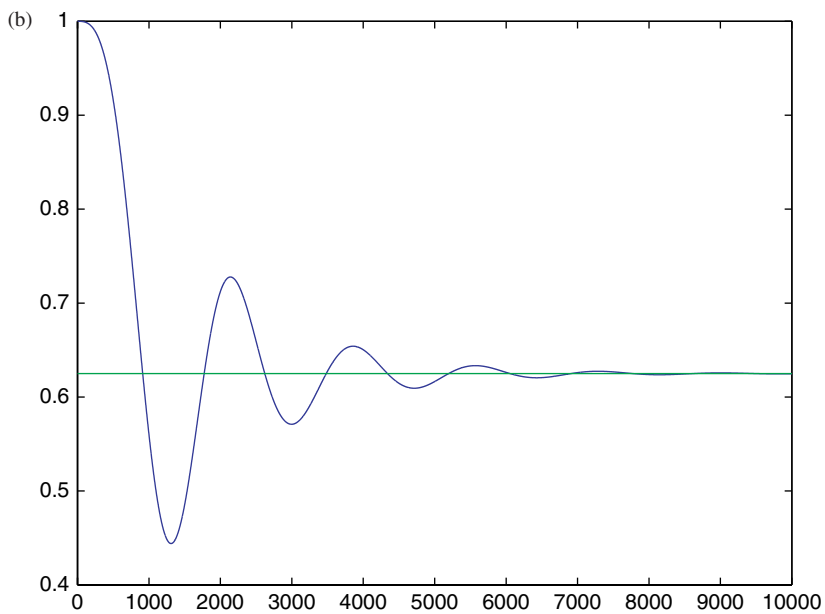
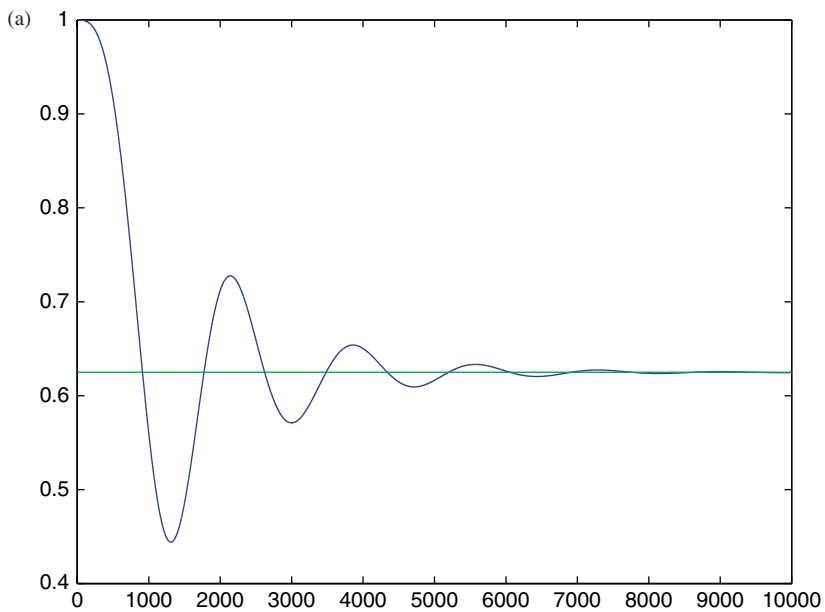


Fig. 2. Availability for Example 1,  $N_h = 10\,000$ ,  $h = 1$ .

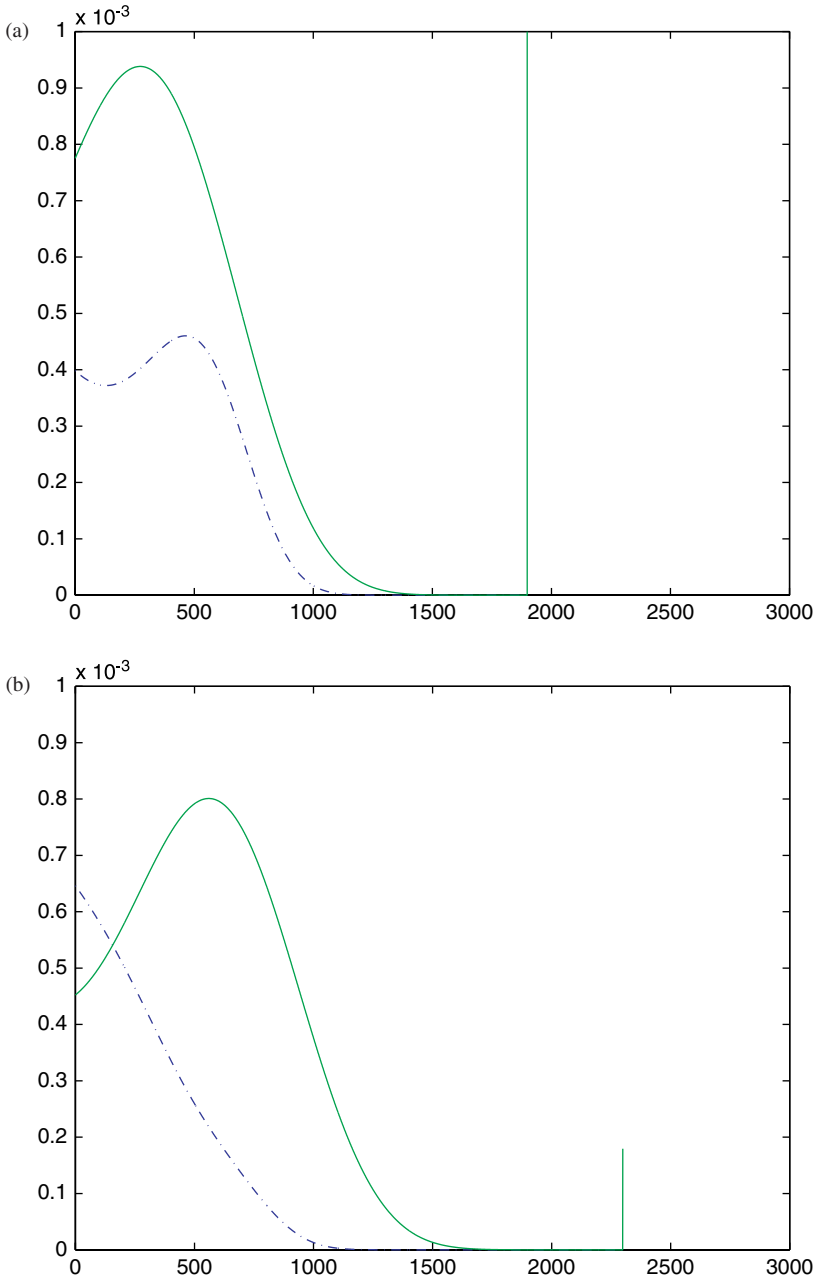


Fig. 3.  $\rho_{1900}$  and  $\rho_{2300}$  for Example 1.

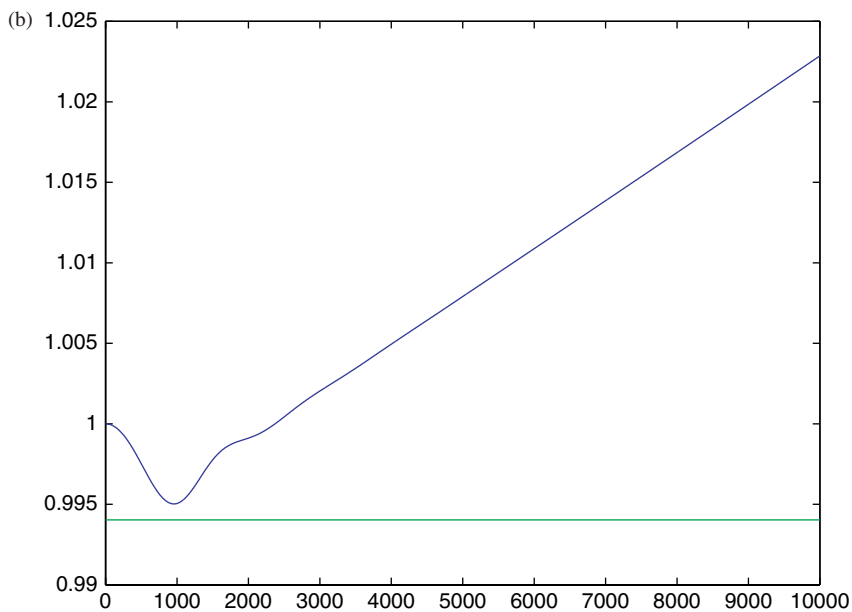
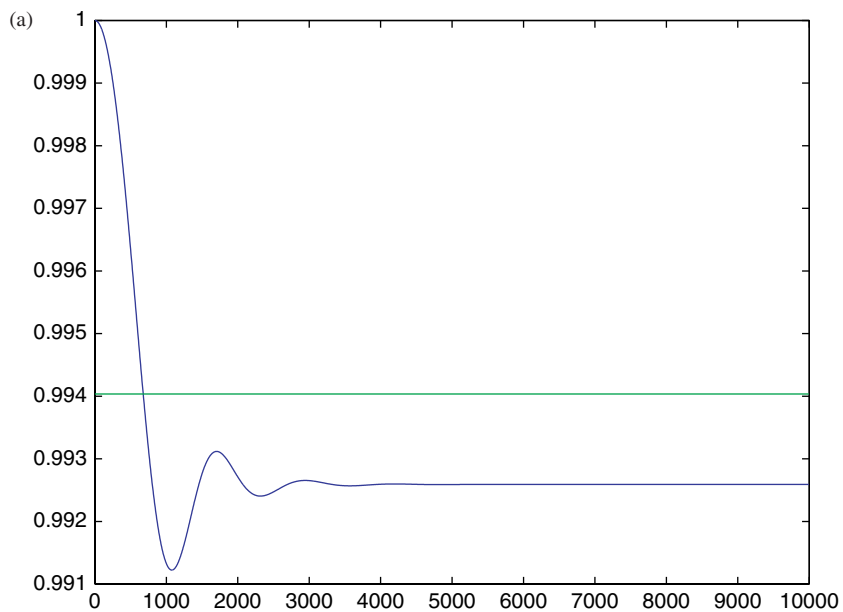


Fig. 4. Availability for Example 2,  $N_h = 2000$ ,  $h = 5$ .

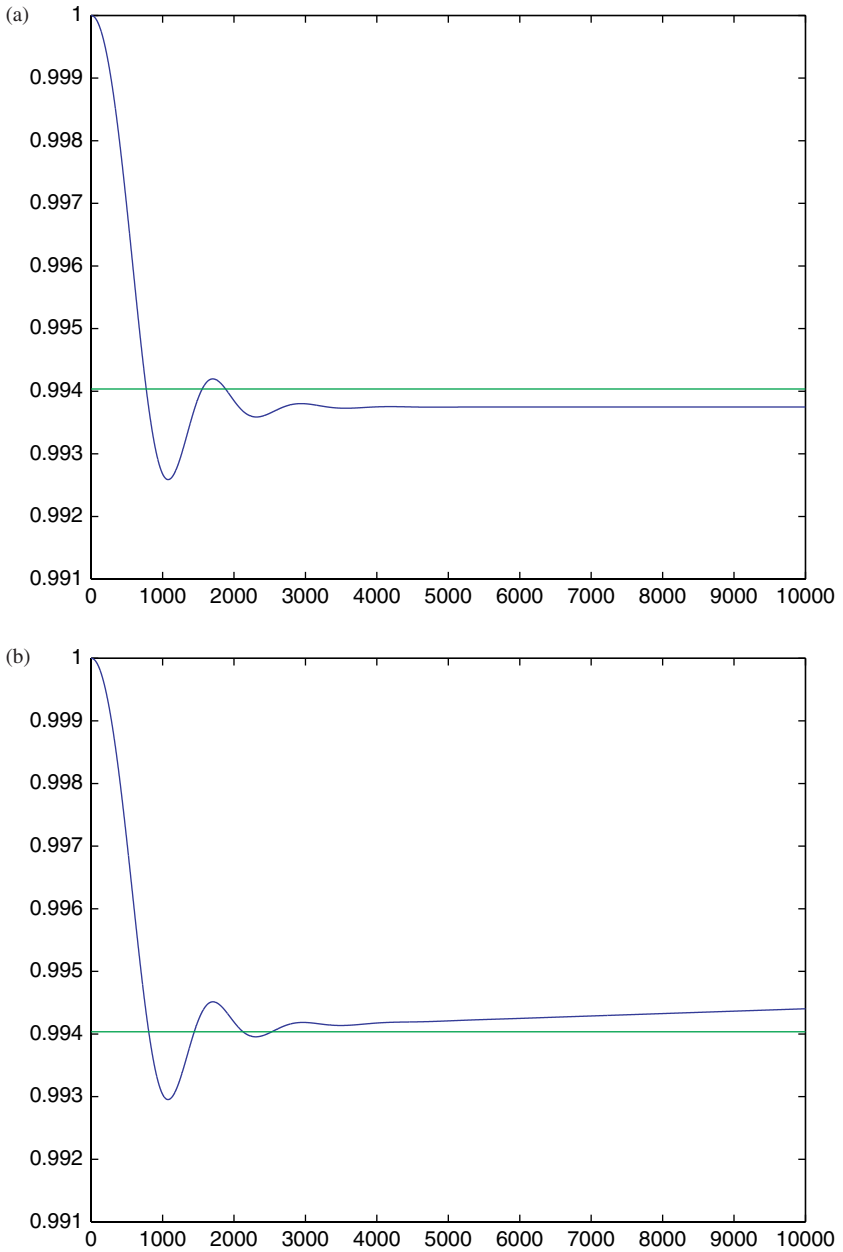


Fig. 5. Availability for Example 2,  $N_h = 10\,000$ ,  $h = 1$ .

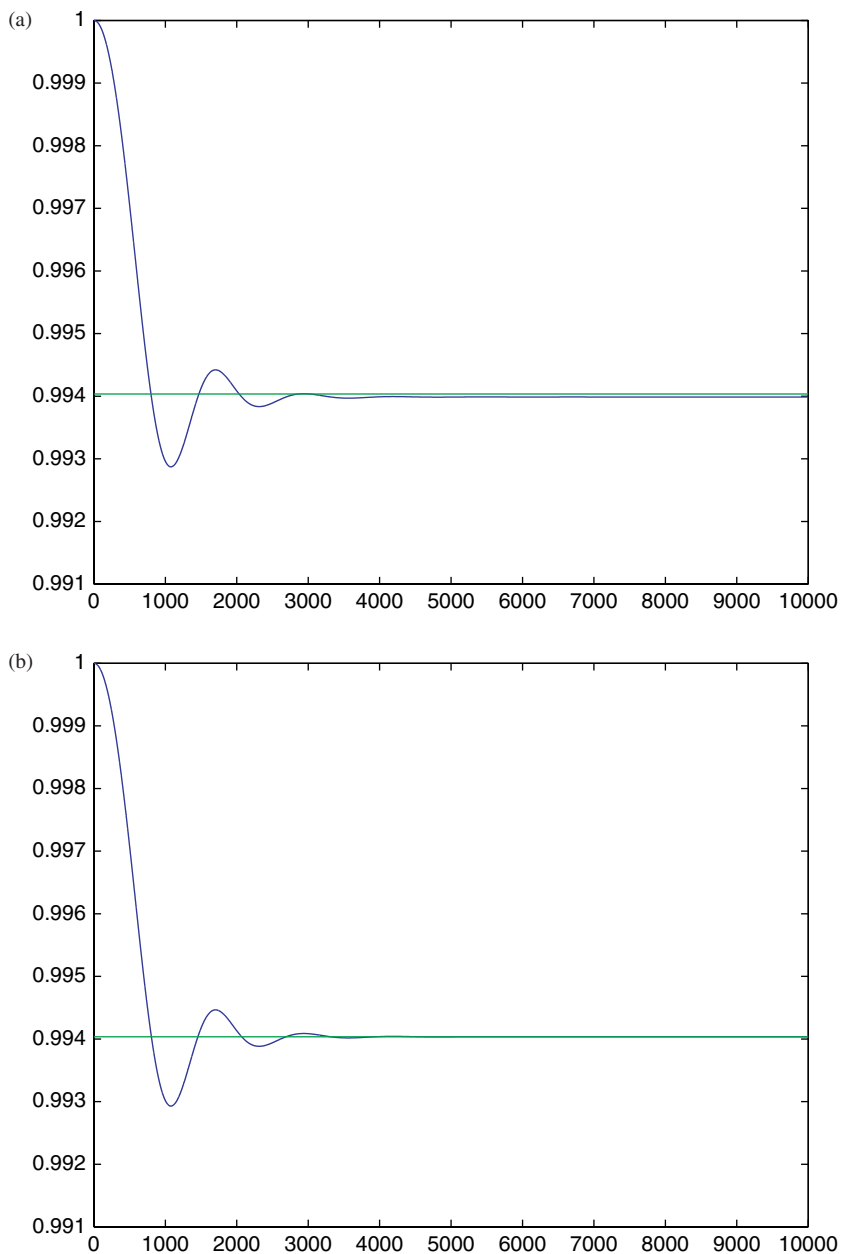


Fig. 6. Availability for Example 2,  $N_h = 60\,000$ ,  $h = 0.17$ .

to 0.903. The results are plotted in Figs. 7–10. In this case, the application of Eq. (18) gives:  $A(\infty) = 0.99898$ .

On this example, our implementation of Method III of Ref. 2 did not allow to obtain relevant values for the availability.

## Other Experiments

We have also studied the numerical results while using log-normal and gamma distributions, comparing them with the so-called “phase method” (see for example Ref. 5: this method is proven to produce good results but it cannot be systemized), and finally we have considered the case of exponential distributions. We recall that in this last case, the availability is given by:

$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t},$$

where  $\lambda$  (respectively  $\mu$ ) is the parameter of the exponential distribution of the working (respectively failure) periods. All these examples seem to indicate that our method is quite robust, it keeps the correct shape of the graph and it delivers the correct convergence speed to the asymptotic availability. In the cases where it does not fail, Method III of Ref. 2 gives a slightly more accurate availability for large  $t$  and it is always faster. But, in the case of contrasted mean working and failure durations (this case occurs in actual reliability studies), our implementation (in MATLAB, using the convolution routine) of Method III of Ref. 2 can give completely wrong results.

### 4.2. Examples with more than two states

In the following two examples, we assume that a system is composed of two components in passive redundancy: usually the first one is working and the second one is at rest. When the first component fails, the second component is started if it is not failed. The second component cannot fail when it is at rest. The system is working if

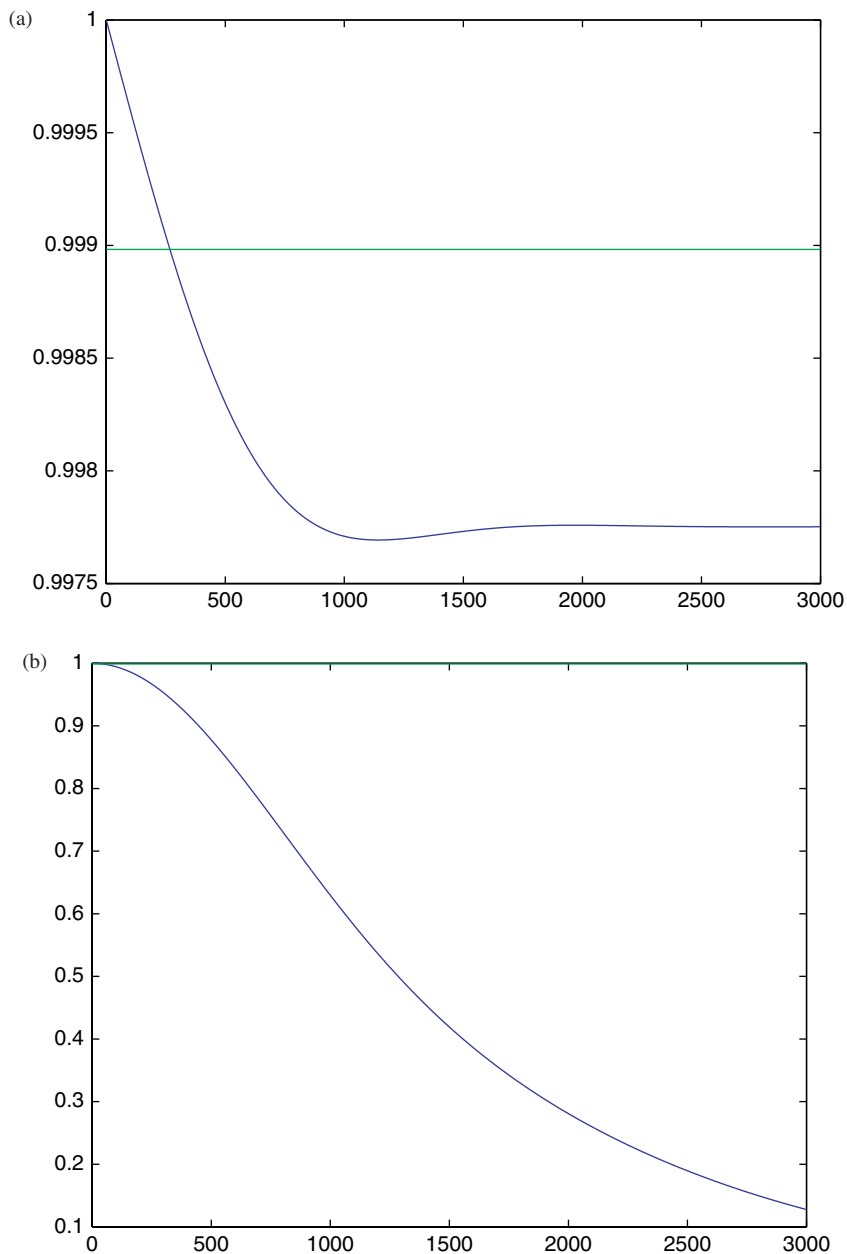


Fig. 7. Availability for Example 3,  $N_h = 2000$ ,  $h = 1.5$ .

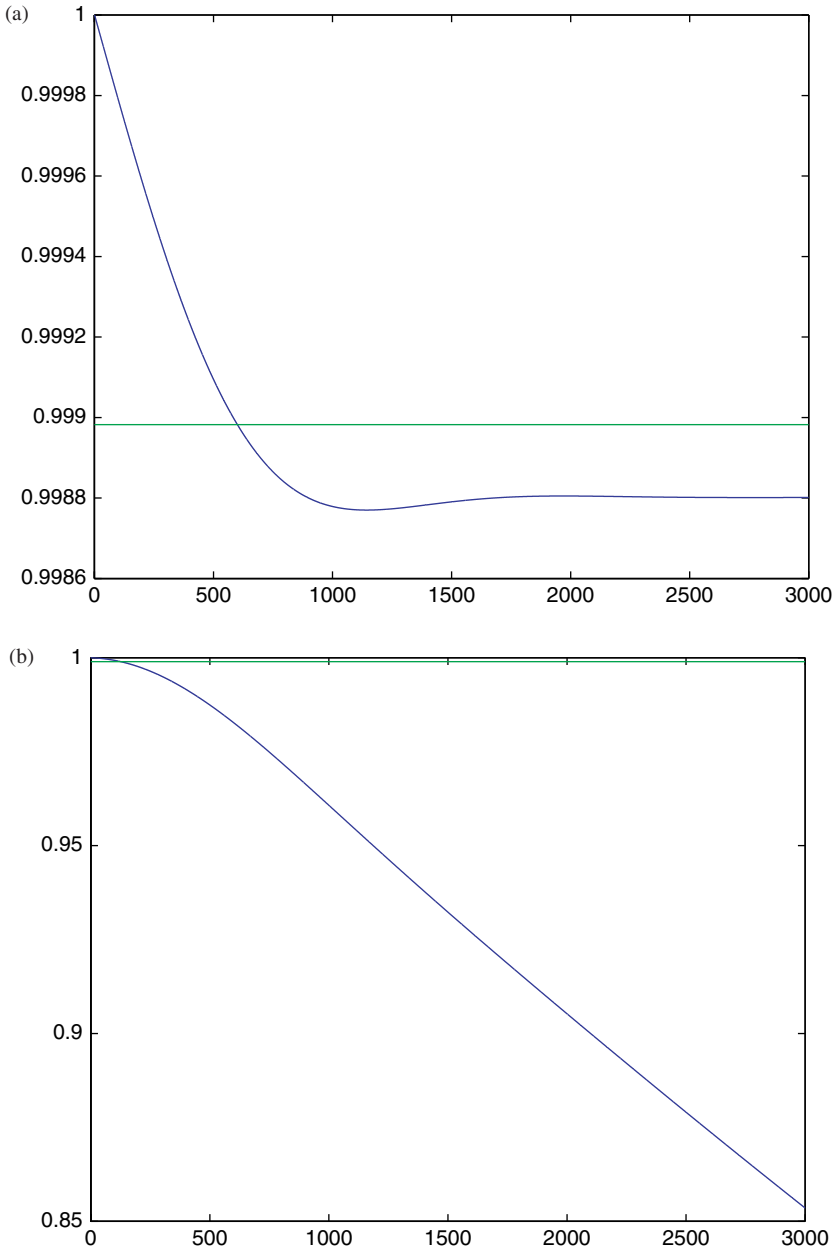


Fig. 8. Availability for Example 3,  $N_h = 10000$ ,  $h = 0.3$ .

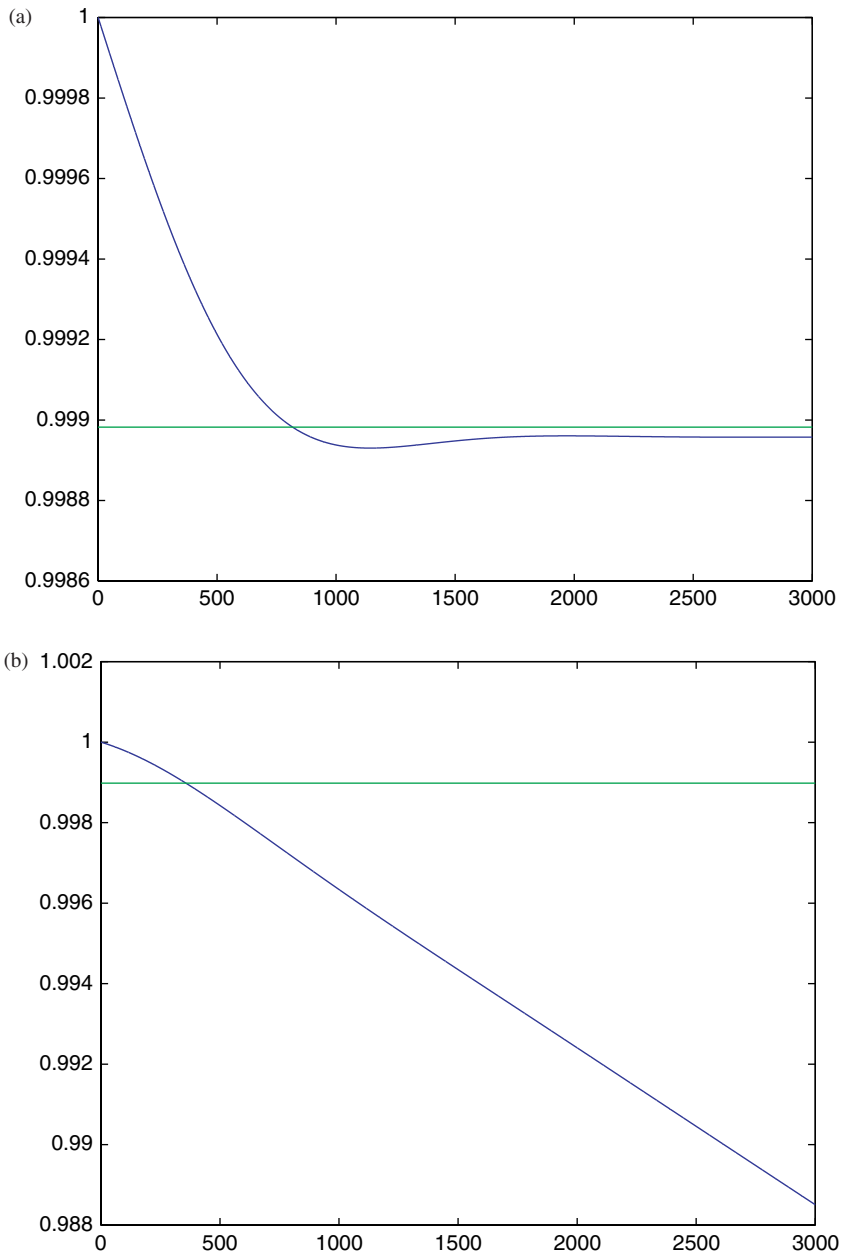


Fig. 9. Availability for Example 3,  $N_h = 60\,000$ ,  $h = 0.05$ .

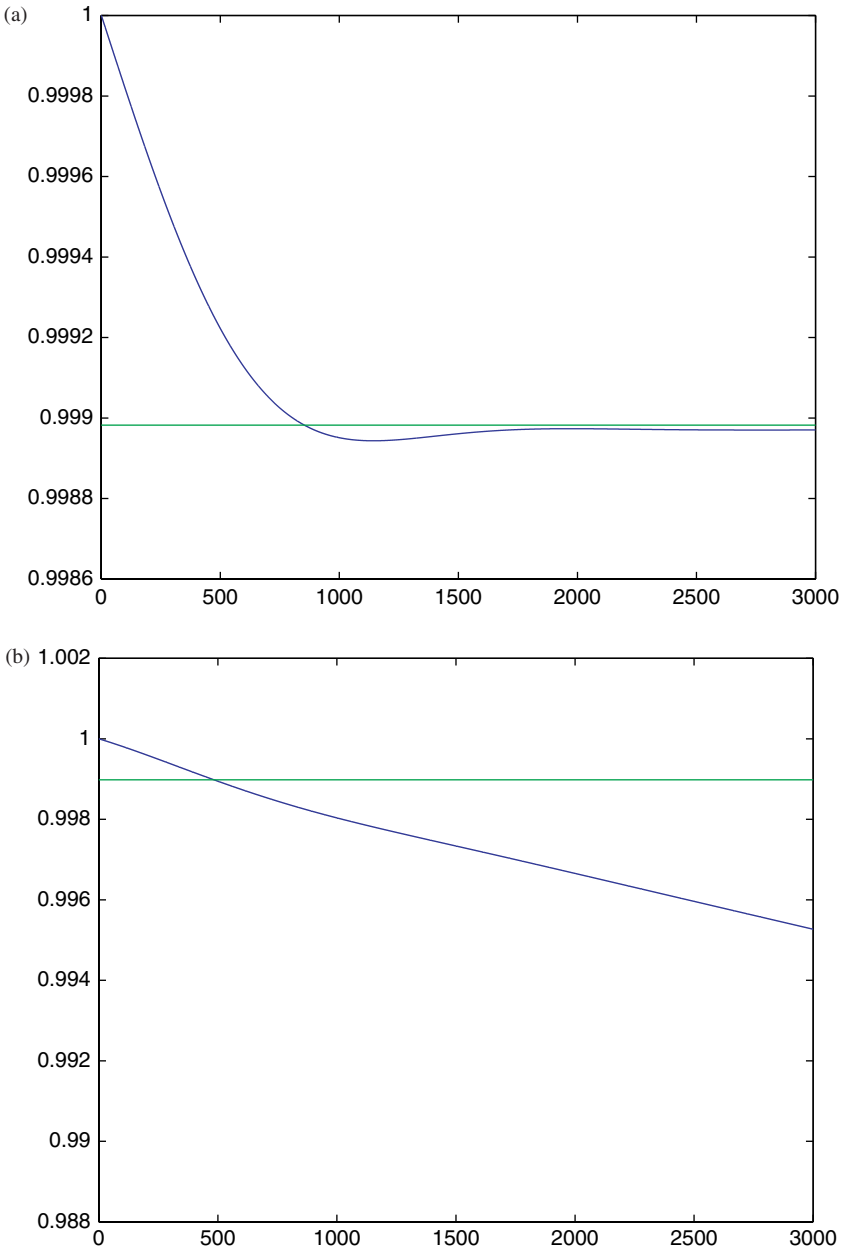


Fig. 10. Availability for Example 3,  $N_h = 120\,000$ ,  $h = 0.025$ .

and only if one component is working. At the end of its repair, a component is as good as new.

**Example 4.**

Component 1 has two types of failure. When a failure of the first type occurs, its repair immediately starts. When a failure of the second type occurs, it is not detected, and no repair is planned, thus the second component is not used and the system fails. Let us assume that the first component is being repaired and that the second one is working: at the end of the repair of the first component, this one is immediately used, the second one is stopped and it is instantaneously upgraded so it becomes as good as new. We are interested in the system reliability, i.e., the probability that the system has no failure during the period  $[0, t]$  since failure states are supposed to be absorbing states. The system has four states:

- State 1: the first component is working and the second one is at rest,
- State 2: the first component is being repaired and the second one is working,
- State 3: the system is out of order because both components are being repaired,
- State 4: the system is out of order because a second type failure of the first component has occurred; it has therefore not been detected and the second component is at rest.

The positive transition rates are as follows:

$a(1, 2, x) = \lambda_1(x)$ : hazard rate of a Weibull distribution with a shape parameter  $\beta = 1.5$  and a mean value equal to 2000,

$a(2, 1, x) = \mu(x)$ : hazard rate of a log-normal distribution with a mean value equal to 102 and a variation coefficient equal to 0.53,

$a(1, 4, x) = \lambda'_1(x)$ : hazard rate of a Weibull distribution with a shape parameter  $\beta = 2$  and a mean value equal to 10 000,

$a(2, 3, x) = \lambda_2(x)$ : hazard rate of a Weibull distribution with a shape parameter  $\beta = 1.5$  and a mean value equal to 1881.

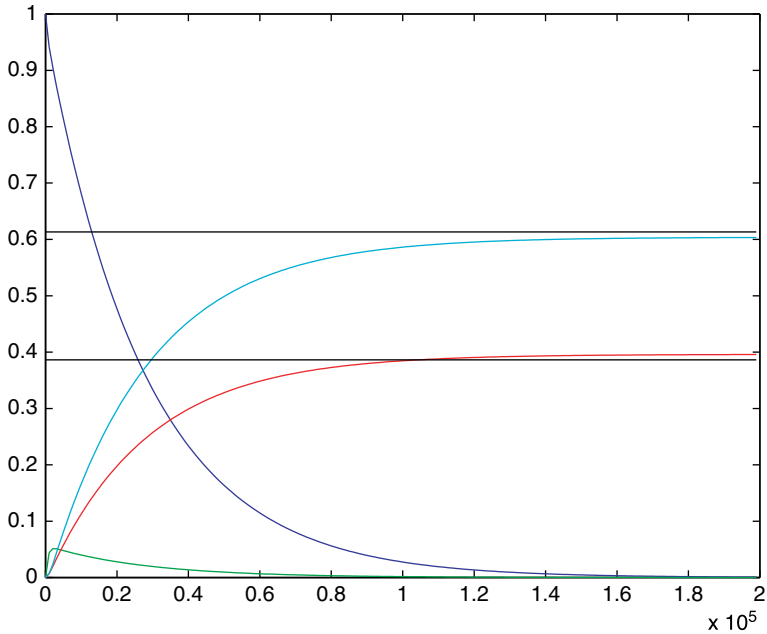


Fig. 11. Example 4.

We assume that at time 0, both components are as good as new, the first one is started and the second one is at rest. In Fig. 11, the probability that the system is in each of the four states is plotted. Horizontal lines again give the asymptotic probability to be in States 3 and 4, respectively (note that the asymptotic probability to be in States 1 and 2 is zero since States 3 and 4 are absorbing states and consequently States 1 and 2 are transient states). As in the previous examples, these asymptotic probabilities indicate the quality of our algorithm since exact formulas are known (see for example Ref. 5, Theorem 10.20 and Remark 10.21).

### Example 5.

Each component has only one type of failure. The system must be stopped before any component is being repaired, consequently, components are being repaired only when both components have failed and they are restarted only when both have been repaired.

The system has three states:

- State 1: the first component is working and the second one is at rest,
- State 2: the first component failed and the second one is working,
- State 3: the system is out of order, both components are being repaired.

The positive transition rates are as follows:

$a(1, 2, x) = \lambda_1(x)$ : hazard rate of a Weibull distribution with a shape parameter  $\beta = 2$  and a mean value equal to 2216,

$a(2, 3, x) = \lambda_2(x)$ : hazard rate of a Weibull distribution with a shape parameter  $\beta = 1.5$  and a mean value equal to 1881,

$a(3, 1, x) = \mu(x)$ : hazard rate of a gamma distribution with a shape parameter  $\alpha = 75$  and a mean value equal to 150.

In Fig. 12, the probability that the system is in each of the three states is plotted. Horizontal lines again give the analytical asymptotic probability to be in each state.

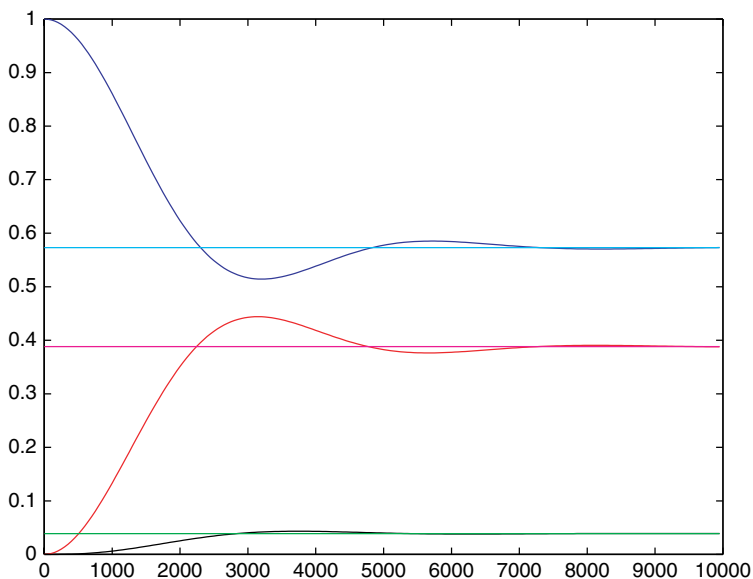


Fig. 12. Example 5.

## 5. Conclusion

We have proposed a new numerical scheme to compute the availability of a component with general failure and repair rates, and more generally to compute the marginal distributions of a semi-Markov process. It is based on a finite volume scheme which requires a discretization in time and in space; a mathematical proof of the convergence of this algorithm when the discretization step tends to 0 is available.

Numerical examples show that this scheme always gives admissible results, although they are less accurate when the rates are contrasted. However, even in this difficult case, the shape of the graph and the convergence speed remain correct. This new scheme thus appears to be usable in reliability studies.

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