I. PRELIMINARIES

1. Introduction

According to the "principle of functoriality", "Galois" representations $\rho : L_F \to {}^LG$ of the hypothetical Langlands group L_F of a global field F into the complex dual group ${}^LG$ of a reductive group $\mathbf{G}$ over F should parametrize "packets" of automorphic representations of the adèle group $\mathbf{G}(\mathbb{A})$. Thus a map $\lambda : {}^LH \to {}^LG$ of complex dual groups should give rise to lifting of automorphic representations π_H of $\mathbf{H}(\mathbb{A})$ to those π of $\mathbf{G}(\mathbb{A})$.

Here we prove the existence of the expected lifting of automorphic representations of the projective symplectic group of similitudes $\mathbf{H} = \mathrm{PGSp}(2)$ to those on $\mathbf{G} = \mathrm{PGL}(4)$. The image is the set of the self-contragredient representations of PGL(4) which are not lifts of representations of the rank two split orthogonal group SO(4).

The global lifting is defined by means of local lifting. We define the local lifting in terms of character relations. This permits us to introduce a definition of packets and quasi-packets of representations of PGSp(2) as the sets of representations that occur in these relations. Our main local result is that packets exist and partition the set of tempered representations. We give a detailed description of the structure of packets.

Our global results include a detailed description of the structure of the global packets and quasi-packets (the latter are almost everywhere non-tempered). We obtain a *multiplicity one theorem for the discrete spectrum of* PGSp(2), a *rigidity theorem for packets and quasi-packets*, determine all *counterexamples to the naive Ramanujan conjecture*, compute the *multiplicity of each member in a packet or quasi-packet in the discrete spectrum*, conclude that *in each local tempered packet there is precisely one generic representation*, and that *in each global packet which lifts to a generic representation of* PGL(4) *there is precisely one representation which is generic everywhere.* The latter representation is generic if it lifts to a properly induced representation of $\mathrm{PGL}(4, \mathbb{A})$.

We also prove the lifting from SO(4) to PGL(4). This amounts to establishing a product of two representations of GL(2) with central characters whose product is 1. Our rigidity theorem for SO(4) amounts to a strong rigidity statement for a pair of representations of $GL(2, \mathbb{A})$.

Our method is based on an interplay of global and local tools, e.g. the trace formula and the fundamental lemma. We deal with all, not only generic or tempered, representations.

2. Statement of Results

2a. Homomorphisms of Dual Groups

Let $\mathbf{G}$ be the projective general linear group PGL(4) = PSL(4) over a number field F. Our initial purpose is to determine the automorphic representations π (Borel-Jacquet [BJ], Langlands [L4]) of $\mathbf{G}(\mathbb{A})$, $\mathbb{A}$ is the ring of adèles of F, which are self-contragredient: $\pi \simeq \check{\pi}$, equivalently (Bernstein-Zelevinski [BZ1]), θ-invariant: $\pi \simeq {}^{\theta}\pi$. Here θ, $\theta(g) = J^{-1}{}^{t}g^{-1}J$, is the involution defined by

$$J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ${}^{t}g$ denotes the transpose of $g \in \mathbf{G}$, and ${}^{\theta}\pi(g) = \pi(\theta(g))$. According to the principle of functoriality (Borel [Bo1], Arthur [A2]) these automorphic representations are essentially described by representations of the Weil group W_F of F into the dual group $\widehat{G} = SL(4, \mathbb{C})$ of $\mathbf{G}$ which are $\hat{\theta}$-invariant, namely representations of W_F into centralizers $Z_{\widehat{G}}(\hat{s}\hat{\theta})$ of $\mathrm{Int}(\hat{s})\hat{\theta}$ in $\widehat{G}$. Here $\hat{\theta}$ is the dual involution $\hat{\theta}(\hat{g}) = J^{-1}{}^{t}\hat{g}^{-1}J$, and $\hat{s}$ is a semisimple element in $\widehat{G}$. These centralizers are the duals of the twisted (by $\hat{s}\hat{\theta}$) endoscopic groups (Kottwitz-Shelstad [KS]). In fact these are the connected components of the identity of the duals of the twisted endoscopic groups $Z_{\widehat{G}}(\hat{s}\hat{\theta}) \times W_F$. But in our case the endoscopic groups are split so the product of $Z_{\widehat{G}}(\hat{s}\hat{\theta})$ with the Weil group W_F is direct. Hence it suffices for us to work here with the connected component of the identity.

A twisted endoscopic group is called *elliptic* if its dual is not contained in a proper parabolic subgroup of $\widehat{G}$. Representations of nonelliptic endoscopic groups can be reduced by parabolic induction to known ones of smaller rank groups. For our $\widehat{G}$, up to conjugacy the elliptic twisted endoscopic groups have as duals the symplectic group $\widehat{H} = Z_{\widehat{G}}(\hat{\theta}) = \mathrm{Sp}(2,\mathbb{C})$ and the special orthogonal group $\widehat{C} = Z_{\widehat{G}}(\hat{s}\hat{\theta}) =$ "$\mathrm{SO}(4,\mathbb{C})$"

$$= \mathrm{SO}\left(\left(\begin{smallmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{smallmatrix}\right), \mathbb{C}\right) = \left\{g \in \mathrm{SL}(4,\mathbb{C}); g\hat{s}J^t g = \hat{s}J = \left(\begin{smallmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{smallmatrix}\right)\right\},$$

which consists of all $A \otimes B = \left(\begin{smallmatrix} aB & bB \\ cB & dB \end{smallmatrix}\right)$, where

$$\left(A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), B\right) \in [\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})]/\mathbb{C}^\times$$

satisfy $\det A \cdot \det B = 1$. Here $z \in \mathbb{C}^\times$ embeds as the central element (z, z^{-1}), $\hat{s} = \mathrm{diag}(-1,1,-1,1)$ and $\omega = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The group $\widehat{H}$ is the dual group of the simple F-group $\mathbf{H} = \mathrm{PSp}(2) = \mathrm{PGSp}(2)$, the projective group of symplectic similitudes, which can also be denoted by the shorter symbol PGp(2). It is the quotient of

$$\mathrm{GSp}(2) = \{(g,\lambda) \in \mathrm{GL}(4) \times \mathbb{G}_m;\ {}^t gJg = \lambda J\}$$

by its center $\{(\lambda, \lambda^2)\} \simeq \mathbb{G}_m$. Since λ is uniquely determined by g (we write $\lambda = \lambda(g)$), we view GSp(2) as a subgroup of GL(4) and PGSp(2) of PGL(4).

The group $\widehat{C}$ is the dual group of the special orthogonal group ("SO(4)")

$$\mathbf{C} = \{(g_1, g_2) \in \mathrm{GL}(2) \times \mathrm{GL}(2); \det g_1 = \det g_2\}/\mathbb{G}_m.$$

Here $z \in \mathbb{G}_m$ embeds as the central element (z, z). Also we write

$$[\mathrm{GL}(2) \times \mathrm{GL}(2)]'/\,\mathrm{GL}(1)$$

for $\mathbf{C}$, where the prime indicates that the two factors in GL(2) have equal determinants.

The principle of functoriality suggests that automorphic discrete spectrum representations of $\mathbf{H}(\mathbb{A})$ and $\mathbf{C}(\mathbb{A})$ parametrize (or lift to) the θ-invariant automorphic discrete spectrum representations of the group of $\mathbb{A}$-valued points, $\mathbf{G}(\mathbb{A})$, of $\mathbf{G}$. Our main purpose is to describe this lifting, or parametrization. In particular we define tensor products of two automorphic forms of $\mathrm{GL}(2,\mathbb{A})$ the product of whose central characters is 1. Moreover we describe the automorphic representations of the projective symplectic group of similitudes of rank two, $\mathrm{PGSp}(2,\mathbb{A})$, in terms of θ-invariant representations of $\mathrm{PGL}(4,\mathbb{A})$.

Motivation for the theory of automorphic forms is attractively explained in some articles by S. Gelbart, see, e.g. [G]. For a more technical introduction see part 3, "Background", of this volume. It is based on a course I gave at the Ohio State University in 2003. It gives most definitions used in this work, from adèles to Weil and L-groups, to twisted endoscopy, and a proof of (Emil) Artin's conjecture for two dimensional Galois representations with image A_4, S_4 in $\mathrm{PGL}(2,\mathbb{C})$.

2b. Unramified Liftings

We proceed to explain how the liftings are defined, first for unramified representations.

An irreducible admissible representation π of an adèle group $\mathbf{G}(\mathbb{A})$ is the restricted tensor product $\otimes\pi_v$ of irreducible admissible ([BZ1]) representations π_v of the groups $\mathbf{G}(F_v)$ of F_v-points of $\mathbf{G}$, where F_v is the completion of F at the place v of F. Almost all the local components π_v are unramified, that is contain a (necessarily unique up to a scalar multiple) nonzero K_v-fixed vector. Here K_v is the standard maximal compact subgroup of $\mathbf{G}(F_v)$, namely the group $\mathbf{G}(R_v)$ of R_v-points, R_v being the ring of integers of the nonarchimedean local field F_v; $\mathbf{G}$ is defined over R_v at almost all nonarchimedean places v. For such v, an irreducible unramified $\mathbf{G}(F_v)$-module π_v is the unique unramified irreducible constituent in an unramified principal series representation $I(\eta_v)$, normalizedly induced (thus induced in the normalized way of [BZ2]) from an unramified character η_v of the maximal torus $\mathbf{T}(F_v)$ of a Borel subgroup $\mathbf{B}(F_v)$ of $\mathbf{G}(F_v)$ (extended trivially to the unipotent radical $\mathbf{N}(F_v)$ of $\mathbf{B}(F_v)$). The space of $I(\eta_v)$ consists of the smooth functions $\phi:\mathbf{G}(F_v)\to\mathbb{C}$ with

$$\phi(ank)=(\delta_v^{1/2}\eta_v)(a)\phi(k),\quad k\in K_v,\quad n\in\mathbf{N}(F_v),\quad a\in\mathbf{T}(F_v),$$

$\delta_v(a)=\det[\mathrm{Ad}(a)|\,\mathrm{Lie}\,\mathbf{N}(F_v)]$, and the $\mathbf{G}(F_v)$-action is $(g\cdot\phi)(h)=\phi(hg)$, $g,h\in\mathbf{G}(F_v)$.

The character η_v is unramified, thus it factors as $\eta_v:\mathbf{T}(F_v)/\mathbf{T}(R_v)\to$

$\mathbb{C}^\times$. As $X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T}) \simeq \mathbf{T}(F_v)/\mathbf{T}(R_v)$, η_v lies in

$$\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times),$$

where $\widehat{T}$ is the maximal torus in the Borel subgroup $\widehat{B}$ of $\widehat{G}$, both fixed in the definition of the (complex) dual group $\widehat{G}$ (Borel [Bo1], Kottwitz [Ko2]). Now

$$\mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times) = X_*(\widehat{T}) \otimes \mathbb{C}^\times = \widehat{T} \subset \widehat{G}.$$

Thus the unramified irreducible $\mathbf{G}(F_v)$-module π_v determines a conjugacy class $t(\pi_v) = t(I(\eta_v))$ in $\widehat{G}$ represented by the image of η_v in $\widehat{T}$. This class $t(\pi_v)$ is called the Langlands parameter of the unramified π_v.

In the case of $\mathbf{G} = \mathrm{GL}(n)$, take $\mathbf{B}$ to be the group of upper triangular matrices, $\mathbf{T}$ the diagonal subgroup, and $\eta_v(a_1, \dots, a_n) = \prod \eta_i(a_i)$ $(1 \le i \le n)$. If $\boldsymbol{\pi}_v$ is a generator of the maximal ideal of R_v then $t(I(\eta_v))$ is the class of $\mathrm{diag}(\eta_1(\boldsymbol{\pi}_v), \dots, \eta_n(\boldsymbol{\pi}_v))$ in $\widehat{G} = \mathrm{GL}(n, \mathbb{C})$. If $\mathbf{G} = \mathrm{PGL}(n)$ then $\eta_1 \dots \eta_n = 1$ and $t(I(\eta_v))$ is a class in $\widehat{G} = \mathrm{SL}(n, \mathbb{C})$.

We make the following notational conventions: If the components of η are $\eta_{1v}, \eta_{2v}, \dots$, we write $I(\eta_{1v}, \eta_{2v}, \dots)$ for $I(\eta_v)$. For a representation π and a character χ we write $\chi\pi$ for $g \mapsto \chi(g)\pi(g)$, and not $\chi \otimes \pi$, reserving the notation $\pi_1 \otimes \pi_2$, or $\pi_1 \times \pi_2$, for products on different groups: $(h, g) \mapsto \pi_1(h) \otimes \pi_2(g)$ (for example, if (h, g) ranges over a Levi subgroup, the representation normalizedly induced from the representation $\pi_1 \otimes \pi_2$ on the Levi will be denoted by $I(\pi_1, \pi_2)$ or $\pi_1 \times \pi_2$, depending on the context). We prefer the notation $\pi_1 \times \pi_2$ for a representation of a group which is a product of two groups, such as our $C = \mathrm{SO}(4, F)$. By a representation we mean an irreducible one, unless otherwise is specified.

2c. The Lifting from SO(4) to PGL(4)

We next describe our results on our secondary lifting λ_1, from $\mathbf{C} = \mathrm{SO}(4)$ to $\mathbf{G} = \mathrm{PGL}(4)$.

We now return to $\mathbf{G} = \mathrm{PGL}(4)$, θ and $\mathbf{C} = [\mathrm{GL}(2) \times \mathrm{GL}(2)]'/\mathrm{GL}(1)$. Note that an irreducible unramified $\mathrm{GL}(2, F_v)$-module π_{1v} is parametrized by a conjugacy class $t(\pi_{1v})$ in $\mathrm{GL}(2, \mathbb{C})$ (the Langlands parameter of the representation; its eigenvalues are called the Hecke eigenvalues of the representation). An unramified irreducible representation $\pi_{1v} \times \pi_{2v}$ of $\mathbf{C}(F_v)$ is parametrized by a class $t(\pi_{1v}) \times t(\pi_{2v})$ in

$$[\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})]''/\mathbb{C}^\times \simeq \mathrm{SO}\left(\left(\begin{smallmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{smallmatrix}\right), \mathbb{C}\right) = \widehat{C} \subset \widehat{G}.$$

(Double prime means $\det g_1 \cdot \det g_2 = 1$). If π_{iv} is the unramified constituent of

$$I(\eta_{iv}), \quad t(\pi_{iv}) = \operatorname{diag}(\eta_{i1}, \eta_{i2}), \quad \eta_{ij} = \eta_{ijv}(\boldsymbol{\pi}_v), \quad \eta_{11}\eta_{12}\eta_{21}\eta_{22} = 1,$$

we define the "lift" $\pi_{1v} \boxtimes \pi_{2v} = \lambda_1(\pi_{1v} \times \pi_{2v})$ of $\pi_{1v} \times \pi_{2v}$ with respect to the dual group homomorphism $\lambda_1 : \widehat{C} = \mathrm{SO}(4, \mathbb{C}) \hookrightarrow \widehat{G} = \mathrm{SL}(4, \mathbb{C})$ (the natural embedding) to be the unramified irreducible constituent π_v of the $\mathrm{PGL}(4, F_v)$-module $I(\eta_v)$ parametrized by the class

$$t(\pi_v) = \operatorname{diag}(\eta_{11}\eta_{21}, \eta_{11}\eta_{22}, \eta_{12}\eta_{21}, \eta_{12}\eta_{22})$$

in $\widehat{G} = \mathrm{SL}(4, \mathbb{C})$. In different notations,

$$\lambda_1(I(a_1, a_2) \times I(b_1, b_2)) = I(a_1b_1, a_1b_2, a_2b_1, a_2b_2) \qquad (a_i, b_i \in \mathbb{C}^\times),$$

provided that $a_1a_2b_1b_2 = 1$. Note that the inverse image under λ_1 of $I(a_1b_1, a_1b_2, b_1a_2, a_2b_2)$ consists only of

$$\chi I(a_1, a_2) \times \chi^{-1} I(b_1, b_2) \quad \text{and} \quad \chi I(b_1, b_2) \times \chi^{-1} I(a_1, a_2)$$

where χ is any character of $F_v^\times$. Thus, λ_1 is two-to-one unless $\pi_{1v} = \check{\pi}_{2v}$ (the contragredient of π_{2v}), where λ_1 is injective on the set of orbits of multiplication by χ in $\mathrm{Hom}(F_v^\times, \mathbb{C}^\times)$.

The rigidity theorem for the discrete spectrum automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ asserts that discrete spectrum automorphic representations $\pi_1 = \otimes \pi_{1v}$ and $\pi_2 = \otimes \pi_{2v}$ which have $\pi_{1v} \simeq \pi_{2v}$ for almost all places v of F are equivalent (Jacquet-Shalika [JS], Moeglin-Waldspurger [MW1]). Moreover they are even equal, by the multiplicity one theorem for $\mathrm{GL}(n)$ (Shalika [Shal]). Representations of $\mathrm{PGL}(n, \mathbb{A})$ (or $\mathrm{PGL}(n, F_v)$) are simply representations of $\mathrm{GL}(n, \mathbb{A})$ (or $\mathrm{GL}(n, F_v)$) with trivial central character (since $H^1(F, \mathbb{G}_m) = \{0\}$), and the rigidity theorem applies then to $\mathrm{PGL}(n)$. Both multiplicity one theorem, and the rigidity theorem for packets (the latter asserts that $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ must lie in the same packet if $\pi_v \simeq \pi'_v$ for almost all v) hold for $\mathrm{SL}(2)$ ([F3]) and fail for $\mathrm{SL}(n)$, $n \geq 3$ (Blasius [Bla]).

The rigidity theorem holds for $\mathbf{C} = \mathrm{SO}(4)$; this is the content of the assertion that the lifting λ_1 is injective, made in the second paragraph of the following theorem. The first paragraph asserts that the lifting exists. By an *elliptic* representation we mean one whose character (Harish-Chandra [H]) is not identically zero on the set of elliptic elements.

2.1 THEOREM (SO(4) TO PGL(4)). *Let* $\pi_1 = \otimes\pi_{1v}$, $\pi_2 = \otimes\pi_{2v}$ *be discrete spectrum automorphic representations of* $\mathrm{GL}(2,\mathbb{A})$ *whose central characters* ω_1, ω_2 *are equal, and whose components at two places* v_1, v_2 *are elliptic. Then there* **exists** *an automorphic representation* $\pi = \lambda_1(\pi_1 \times \check{\pi}_2)$ *of* $\mathrm{PGL}(4,\mathbb{A})$ *with* $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{2v})$ *for almost all* v.

We have $\lambda_1(\chi_1\pi_1 \times \chi_2\pi_2) = \chi_1\chi_2\lambda_1(\pi_1 \times \pi_2)$ *for* $\chi_i : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ *with* $(\chi_1\chi_2)^2 = 1$.

If $\pi_1 = \pi_E(\mu_1)$, $\pi_2 = \pi_E(\mu_2)$ *are cuspidal monomial representations of* $\mathrm{GL}(2,\mathbb{A})$ *associated with characters* μ_1, μ_2 *of* $\mathbb{A}_E^\times/E^\times$ *where* E *is a quadratic extension of* F *such that the restriction of* $\mu_1\mu_2$ *to* $\mathbb{A}^\times$ *is* 1, *then* $\lambda_1(\pi_E(\mu_1) \times \pi_E(\mu_2)) = I_{(2,2)}(\pi_E(\mu_1\overline{\mu}_2), \pi_E(\mu_1\mu_2))$.

If $\{\pi_1, \pi_2\}$ *are cuspidal but not of the form* $\{\pi_E(\mu_1), \pi_E(\mu_2)\}$, *and* $\pi_1 \neq \chi\pi_2$ *for any quadratic character* χ *of* $\mathbb{A}^\times/F^\times$, *then* $\pi_1 \boxtimes \pi_2$ *is cuspidal.*

If π_1 *is the trivial representation* $\mathbf{1}_2$ *and* π_2 *is a cuspidal representation of* $\mathrm{PGL}(2,\mathbb{A})$, *then* $\lambda_1(\mathbf{1}_2 \times \pi_2)$ *is the discrete spectrum noncuspidal* $\mathrm{PGL}(4,\mathbb{A})$*-module* $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. *Here* $\nu(x) = |x|$, *and* J *is the quotient of the representation* $I(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ *normalizedly induced from the parabolic subgroup of type* $(2,2)$ *of* $\mathrm{PGL}(4)$.

The global map λ_1 *is* **injective** *on the set of pairs* $\pi_1 \times \check{\pi}_2$ *with* $\omega_1 = \omega_2$ *up to the equivalence* $\pi_1 \times \check{\pi}_2 \simeq \chi\pi_1 \times \chi^{-1}\check{\pi}_2$, χ *a character of* $\mathbb{A}^\times/F^\times$, *and* $\pi_1 \times \check{\pi}_2 \simeq \check{\pi}_2 \times \pi_1$.

The injectivity means that if $\pi_1, \pi_2, \pi_1^0, \pi_2^0$ are discrete spectrum automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ with central characters $\omega_1, \omega_2, \omega_1^0, \omega_2^0$ satisfying $\omega_1\omega_2 = 1 = \omega_1^0\omega_2^0$, each of which has elliptic components at least at the three places v_1, v_2, v_3, and if for each v outside a fixed finite set of places of F there is a character χ_v of $F_v^\times$ such that the set $\{\pi_{1v}\chi_v, \pi_{2v}\chi_v^{-1}\}$ is equal to the set $\{\pi_{1v}^0, \pi_{2v}^0\}$ (up to equivalence of representations), then there is a character χ of $\mathbb{A}^\times/F^\times$ such that the set $\{\pi_1\chi, \pi_2\chi^{-1}\}$ is equal to the set $\{\pi_1^0, \pi_2^0\}$. In particular, starting with a pair π_1, π_2 of automorphic discrete spectrum representations of $\mathrm{GL}(2,\mathbb{A})$ with $\omega_1\omega_2 = 1$, we cannot get another such pair by interchanging a set of their components π_{1v}, π_{2v} and multiplying π_{1v} by a local character and π_{2v} by its inverse, unless we interchange π_1, π_2 and multiply π_1 by a global character and π_2 by its inverse.

A considerably weaker result, where the notion of equivalence is generated only by $\pi_{1v} \times \check{\pi}_{2v} \simeq \check{\pi}_{2v} \times \pi_{1v}$ but not by $\pi_{1v} \times \check{\pi}_{2v} \simeq \chi_v\pi_{1v} \times \chi_v^{-1}\check{\pi}_{2v}$, follows also on using the Jacquet-Shalika [JS] theory of L-functions, comparing the poles at $s = 1$ of the partial, product L-functions

$$L^V(s, \pi_1^0 \times \check{\pi}_1) L^V(s, \pi_2^0 \times \check{\pi}_1) = L^V(s, \pi_1 \times \check{\pi}_1) L^V(s, \pi_2 \times \check{\pi}_1).$$

Our global results are complemented and strengthened by very precise local results. If $\pi \simeq {}^\theta\pi$ there is an intertwining operator A with $A\pi(g) = \pi(\theta(g))A$ for all g. By Schur's lemma we may assume that $A^2 = 1$. Then A is unique up to a sign. We put $\pi(\theta) = A$ and $\pi(f \times \theta) = \pi(f)A$. We define λ_1-lifting locally by means of character relations:

$$\lambda_1(\pi_1 \times \check{\pi}_2) = \pi \quad \text{if} \quad \operatorname{tr} \pi(f \times \theta) = \operatorname{tr}(\pi_1 \times \check{\pi}_2)(f_C)$$

for all matching functions f, f_C (and a suitable choice of A). This definition is compatible with the one given above for purely induced π_1 and π_2 and unramified representations. We have $\lambda_1(I_2(\mu, \mu') \times \check{\pi}_2) = I_4(\mu\check{\pi}_2, \mu'\check{\pi}_2)$ (the central character of the $\mathrm{GL}(2, F)$-module π_2 is $\mu\mu'$). The local and global results are closely analogous.

2d. Special Cases of the Lifting from SO(4)

Let us describe some special cases of the lifting λ_1. When $\pi_2 = \check{\pi}_1$ is the contragredient of π_1, $\lambda_1(\pi_1 \times \check{\pi}_1)$ is the $\mathrm{PGL}(4, \mathbb{A})$-module normalizedly induced from the maximal parabolic of type (3,1) and the $\mathrm{PGL}(3, \mathbb{A})$-module $\mathrm{Sym}^2(\pi_1)$ on the GL(3)-factor of the Levi subgroup (extended trivially to the GL(1)-factor of the Levi, and to the unipotent radical). Here $\mathrm{Sym}^2(\pi_1)$ is the symmetric square lifting from GL(2) to PGL(3) ([F3]). Indeed, if the local component π_{1v} of π_1 at v is unramified then $t(\pi_{1v}) = \mathrm{diag}(a, b)$ (thus π_{1v} is a constituent of $I_2(a, b)$), $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{1v})$ has $t(\pi_v) = \mathrm{diag}(a/b, 1, 1, b/a)$ (thus π_v is a constituent of $I_4(I_3(a/b, 1, b/a), 1)$, and $I_3(a/b, 1, b/a)$ is the symmetric square lifting of $I_2(a, b)$). We write I_n to emphasize that the representation is of the group $\mathrm{GL}(n)$, and e.g. $I_{(3,1)}(\pi_3, \pi_1)$ to indicate the representation of GL(4) induced from its maximal parabolic subgroup of type (3,1). However, the results of [F3] are stronger, in lifting representations of $\mathrm{SL}(2, \mathbb{A})$ to $\mathrm{PGL}(3, \mathbb{A})$ and consequently providing new results such as multiplicity one for SL(2).

Although we do not obtain here a new proof of the existence of the symmetric square lift of discrete spectrum representations of $\mathrm{PGL}(2, \mathbb{A})$, we do obtain new character identities, relating the θ-twisted character of $I_{(3,1)}(\mathrm{Sym}^2 \pi_2, 1)$ with that of $\pi_2 \times \check{\pi}_2$. Clearly in this case the lift λ_1 is injective: if

$$\lambda_1(\pi_1 \times \check{\pi}_2) = \lambda_1(\pi_0 \times \check{\pi}_0) \quad (= I_{(3,1)}(\mathrm{Sym}^2(\pi_0), 1))$$

then $\pi_1 = \pi_2 = \pi_0\chi$ for some character χ of $\mathbb{A}^\times/F^\times$.

In particular, if π_1 is a one dimensional representation $g \mapsto \chi(\det g)$ of $\mathrm{GL}(2, \mathbb{A})$, then $\lambda_1(\pi_1 \times \check{\pi}_1) = I_{(3,1)}(\mathbf{1}_3, 1)$ is the representation of $\mathrm{PGL}(4, \mathbb{A})$ normalizedly induced from the trivial representation of the maximal parabolic subgroup of type (3,1). An alternative purely local computation of this twisted character is developed in [FZ].

Let $\pi_1 = \pi(\mu)$ be a cuspidal monomial representation of $\mathrm{GL}(2, \mathbb{A})$ associated with a character μ of $\mathbb{A}_E^\times/E^\times$ where E is a quadratic extension of F (denote by σ the nontrivial element of $\mathrm{Gal}(E/F)$). Then

$$\mathrm{Sym}^2 \pi_1 = I_{(2,1)}(\pi(\mu/{}^\sigma\mu), \chi_{E/F}),$$

where $\chi_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times N_{E/F}\mathbb{A}_E^\times$ ($N_{E/F}$ is the norm map from E to F). Moreover,

$$\lambda_1(\pi(\mu) \times \check{\pi}(\mu)) = I_{(2,1,1)}(\pi(\mu/{}^\sigma\mu), \chi_{E/F}, 1)$$

is an induced representation from the parabolic subgroup of type (2,1,1) of PGL(4). Note that the central character of the $\mathrm{GL}(2, \mathbb{A})$-module $\pi(\mu)$ is $\chi_{E/F} \cdot \mu|\mathbb{A}^\times$, for any character μ of $\mathbb{A}_E^\times/E^\times$. If $\pi(\mu)$ is a $\mathrm{PGL}(2, \mathbb{A})$-module we have that the restriction of μ to $\mathbb{A}_F^\times/F^\times$ is $\chi_{E/F}$, nontrivial but trivial on $F^\times N_{E/F}\mathbb{A}_E^\times$.

If $\pi_1 = \pi_E(\mu_1)$, $\pi_2 = \pi_E(\mu_2)$, cuspidal monomial representations of $\mathrm{GL}(2, \mathbb{A})$ associated with characters μ_1, μ_2 of $\mathbb{A}_E^\times/E^\times$ where E is a quadratic extension of F such that the restriction of $\mu_1\mu_2$ to $\mathbb{A}^\times$ is 1, then

$$\lambda_1(\pi_E(\mu_1) \times \pi_E(\mu_2)) = I_{(2,2)}(\pi_E(\mu_1\overline{\mu}_2), \pi_E(\mu_1\mu_2)).$$

Indeed

$$W_{E/F} = \langle z, \sigma; z \in C_E, \sigma z \sigma^{-1} = \overline{z}, \sigma^2 \in C_F - N_{E/F}C_E \rangle$$

where $C_E = \mathbb{A}_E^\times/E^\times$ (globally, and $E^\times$ locally), and the representation corresponds to

$$z \mapsto \begin{pmatrix} \mu_1(z) & 0 \\ 0 & \mu_1(\overline{z}) \end{pmatrix} \times \begin{pmatrix} \mu_2(z) & 0 \\ 0 & \mu_2(\overline{z}) \end{pmatrix}$$

$$\overset{\lambda_1}{\mapsto} \begin{pmatrix} \mu_1\mu_2 & & & 0 \\ & \mu_1\overline{\mu}_2 & & \\ & & \mu_2\overline{\mu}_1 & \\ 0 & & & \overline{\mu}_1\overline{\mu}_2 \end{pmatrix} \overset{(13)}{\mapsto} \begin{pmatrix} \mu_2\overline{\mu}_1 & & & 0 \\ & \mu_1\overline{\mu}_2 & & \\ & & \mu_1\mu_2 & \\ 0 & & & \overline{\mu}_1\overline{\mu}_2 \end{pmatrix},$$

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ \mu_1(\sigma^2) & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ \mu_2(\sigma^2) & 0 \end{pmatrix}$$

$$\overset{\lambda_1}{\mapsto} \begin{pmatrix} 0 & & & 1 \\ & & \mu_2(\sigma^2) & \\ & \mu_1(\sigma^2) & & \\ 1 & & & 0 \end{pmatrix} \overset{(13)}{\mapsto} \begin{pmatrix} 0 & \mu_1(\sigma^2) & & \\ \mu_2(\sigma^2) & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix},$$

where $\mu_1\mu_2(\sigma^2) = 1$ and $\overline{\mu}_i(z) = \mu_i(\overline{z}), \mu_1\overline{\mu}_1\mu_2\overline{\mu}_2 = 1$ and $\mu_i(z)$ are abbreviated to μ_i in the line of z. When $\mu_1 = \mu_2^{-1}$ we have $\pi(\mu_1\overline{\mu}_2) = \pi(\mu_1/\overline{\mu}_1)$ and $\pi(\mu_1\mu_2) = I(\chi_{E/F}, 1)$. Thus

$$\lambda_1(\pi(\mu_1) \times \check{\pi}(\mu_1)) = I_{(2,1,1)}(\pi(\mu_1/\overline{\mu}_1), \chi_{E/F}, 1) = I_{(3,1)}(\mathrm{Sym}^2(\pi(\mu_1)), 1).$$

Note that if $\mu : \mathbb{A}_E^\times \to \mathbb{C}^\times$ has $(\mu/\overline{\mu})^2 = 1 \neq \overline{\mu}/\mu$ then there are quadratic extensions E_2, E_3 and characters $\mu_i : \mathbb{A}_{E_i}^\times/E_i^\times \to \mathbb{C}^\times$ with $\pi_{E_i}(\mu_i) = \pi_E(\mu)$.

Another interesting special case is when π_1 is taken to be the trivial representation $\mathbf{1}_2$ of $\mathrm{PGL}(2,\mathbb{A})$ while π_2 is a cuspidal representation of $\mathrm{PGL}(2,\mathbb{A})$. Then $\lambda_1(\mathbf{1}_2 \times \pi_2)$ is the discrete spectrum noncuspidal representation $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ of $\mathrm{PGL}(4,\mathbb{A})$, the quotient of the normalizedly induced $I(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ from the parabolic of type (2,2) of PGL(4). Here $\nu(x) = |x|$. Indeed, $\mathbf{1}_2$ is the quotient of the induced $I(\nu^{1/2}, \nu^{-1/2})$. Hence

$$t(\lambda_1(\mathbf{1}_{2v} \times \pi_{2v})) \quad \text{is} \quad (t(\nu_v^{1/2}\pi_{2v}), t(\nu_v^{-1/2}\pi_{2v})). \quad \text{Then} \quad \lambda_1(\mathbf{1}_{2v} \times \pi_{2v})$$

is the quotient $J(\nu_v^{1/2}\pi_{2v}, \nu_v^{-1/2}\pi_{2v})$ of the induced $I(\nu_v^{1/2}\pi_{2v}, \nu_v^{-1/2}\pi_{2v})$ for all v where π_{2v} is unramified. Hence it is $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ globally by the rigidity theorem for this noncuspidal discrete spectrum ([MW1]).

On the set of pairs $\pi_1 \times \pi_2$ such that at least one of π_1 or π_2 is one dimensional, the lifting λ_1 is injective. Indeed, a discrete spectrum representation of $\mathrm{GL}(2,\mathbb{A})$ with a one-dimensional component is necessarily one-dimensional. If π_2 is not cuspidal but rather trivial, then the quotient $J(\nu^{1/2}\mathbf{1}_2, \nu^{-1/2}\mathbf{1}_2)$ of $I_4(\nu^{1/2}\mathbf{1}_2, \nu^{-1/2}\mathbf{1}_2)$ is not discrete spectrum, but the induced $I_4(\mathbf{1}_3)$ from the trivial representation of the (3,1)-parabolic; this is $\lambda_1(\mathbf{1}_2 \times \mathbf{1}_2)$.

2e. The Lifting from PGSp(2) to PGL(4)

We now turn to the study of our main lifting λ, and of the automorphic representations of the F-group $\mathbf{H} = (\mathrm{PSp}(2) =)\,\mathrm{PGSp}(2) = \mathrm{GSp}(2)/\mathbb{G}_m$, where the center $\mathbb{G}_m$ of

$$\mathrm{GSp}(2) = \{g \in \mathrm{GL}(4);\ {}^t gJg = \boldsymbol{\lambda} J, \quad \exists \boldsymbol{\lambda} = \boldsymbol{\lambda}(g) \in \mathbb{G}_m\}$$

consists of the scalar matrices. Its dual group is $\widehat{H} = \mathrm{Sp}(2,\mathbb{C}) = Z_{\widehat{G}}(\widehat{\theta}) \subset \widehat{G} = \mathrm{SL}(4,\mathbb{C})$, where $\widehat{\theta}(g) = J^{-1}{}^t g^{-1} J$. It has a single elliptic endoscopic group $\mathbf{C}_0$ different than $\mathbf{H}$ itself. Thus

$$\widehat{C}_0 = Z_{\widehat{H}}(\widehat{s}_0) = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d \end{pmatrix} \in \widehat{H} \right\} \simeq \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}),$$

where $\widehat{s}_0 = \mathrm{diag}(-1,1,1,-1)$, and $\mathbf{C}_0 = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$. Write λ_0 for the embedding $\widehat{C}_0 \hookrightarrow \widehat{H}$, and λ for the embedding $\widehat{H} \hookrightarrow \widehat{G}$.

The embedding $\lambda_0 : \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \widehat{H} = \mathrm{Sp}(2,\mathbb{C})$ defines the "endoscopic" lifting

$$\lambda_0 : \pi_2(\mu_1, \mu_1^{-1}) \times \pi_2(\mu_2, \mu_2^{-1}) \mapsto \pi_{\mathrm{PGSp}(2)}(\mu_1, \mu_2).$$

Here $\pi_2(\mu_i, \mu_i^{-1})$ is the unramified irreducible constituent of the normalizedly induced representation $I(\mu_i, \mu_i^{-1})$ of $\mathrm{PGL}(2, F_v)$ (μ_i are unramified characters of $F_v^\times$, $i = 1, 2$); $\pi_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ is the unramified irreducible constituent of the $\mathrm{PGSp}(2, F_v)$-module $I_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ normalizedly induced from the character $n \cdot \mathrm{diag}(\alpha, \beta, \gamma, \delta) \mapsto \mu_1(\alpha/\gamma)\mu_2(\alpha/\beta)$ of the upper triangular subgroup of $\mathrm{PGSp}(2, F_v)$ (n is in the unipotent radical, $\alpha\delta = \beta\gamma$).

The embedding $\lambda : \widehat{H} = \mathrm{Sp}(2,\mathbb{C}) \hookrightarrow \mathrm{SL}(4,\mathbb{C}) = \widehat{G}$ defines the lifting λ which maps the unramified irreducible representation $\pi_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ of $\mathrm{PGSp}(2, F_v)$ to $\pi_4(\mu_1, \mu_2, \mu_2^{-1}, \mu_1^{-1})$, an unramified irreducible representation of $\mathrm{PGL}(4, F_v)$.

The composition $\lambda \circ \lambda_0 : \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \widehat{G} = \mathrm{SL}(4,\mathbb{C})$ takes $\pi_2(\mu_1, \mu_1^{-1}) \times \pi_2(\mu_2, \mu_2^{-1})$ to

$$\pi_4(\mu_1, \mu_2, \mu_2^{-1}, \mu_1^{-1}) = \pi_4(\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}),$$

namely the unramified irreducible $\mathrm{PGL}(2, F_v) \times \mathrm{PGL}(2, F_v)$-module $\pi_2 \times \pi_2'$ to the unramified irreducible constituent $\pi_4(\pi_2, \pi_2')$ of the $\mathrm{PGL}(4, F_v)$-module $I_4(\pi_2, \pi_2')$ normalizedly induced from the representation $\pi_2 \otimes \pi_2'$ of the parabolic of type (2,2) of $\mathrm{PGL}(4, F_v)$ (extended trivially on the unipotent radical). For example $\lambda \circ \lambda_0$ takes the trivial $\mathrm{PGL}(2, F_v) \times \mathrm{PGL}(2, F_v)$-module $\mathbf{1}_2 \times \mathbf{1}_2$ to the unramified irreducible constituent $\pi_4(\mathbf{1}_2, \mathbf{1}_2)$ of $I_4(\mathbf{1}_2, \mathbf{1}_2)$, and $\mathbf{1}_2 \times \pi_2$ to $\pi_4(\mathbf{1}_2, \pi_2) = \pi_4(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. Note that this last π_4 is traditionally denoted by J.

The definition of lifting is extended from the case of unramified representations to that of any admissible representations. For this purpose we define below norm maps from the set of θ-stable θ-regular conjugacy classes in $G = \mathbf{G}(F)$ to the set of stable conjugacy classes in $H = \mathbf{H}(F)$, and from this to the set of conjugacy classes in $\mathbf{C}_0(F)$, extending the norm maps on the split tori in these groups which are dual to the dual groups homomorphisms λ and λ_0. This is used to define a relation of matching functions f, f_H and f_{C_0} (they have suitably defined matching orbital integrals) and a dual relation of liftings of representations.

To express the lifting results we use the following notations for induced representations of $H = \mathrm{PGSp}(2, F)$. For characters μ_1, μ_2, σ of $F^\times$ with $\mu_1\mu_2\sigma^2 = 1$ we write $\mu_1 \times \mu_2 \rtimes \sigma$ for the H-module normalizedly induced from the character

$$p = mu \mapsto \mu_1(a)\mu_2(b)\sigma(\boldsymbol{\lambda}), \quad m = \mathrm{diag}(a, b, \boldsymbol{\lambda}/b, \boldsymbol{\lambda}/a), \quad u \in U,$$

$a, b, \boldsymbol{\lambda} \in F^\times$, of the upper triangular minimal parabolic of H.

For a $\mathrm{GL}(2, F)$-module π_2 and character μ we write $\pi_2 \rtimes \mu$ for the $\mathrm{PGSp}(2, F)$-module normalizedly induced from the representation

$$p = mu \mapsto \pi_2(g)\mu(\boldsymbol{\lambda}), \quad m = \mathrm{diag}(g, \boldsymbol{\lambda} w {}^t g^{-1} w), \quad u \in U_{(2)}, \quad \boldsymbol{\lambda} \in F^\times$$

(here the product of the central character ω of π_2 with μ^2 is 1) of the Siegel parabolic subgroup (whose unipotent radical $U_{(2)}$ is abelian).

We write $\mu \rtimes \pi_2$, if $\omega\mu = 1$, for the representation of $\mathrm{PGSp}(2, F)$ normalizedly induced from the representation

$$p = mu \mapsto \mu(a)\pi_2(g), \qquad m = \mathrm{diag}(a, g, \boldsymbol{\lambda}(g)/a), \qquad u \in U_{(1)},$$

$\boldsymbol{\lambda}(g) = \det g$, of the Heisenberg parabolic subgroup (whose unipotent radical $U_{(1)}$ is a Heisenberg group).

These inductions are normalized by multiplying the inducing representation by the character $p \mapsto |\det(\mathrm{Ad}(p))| \,\mathrm{Lie}\, U|^{1/2}$, as usual. For example,

$$I_H(\mu_1, \mu_2) = \mu_1\mu_2 \times \mu_1/\mu_2 \rtimes \mu_1^{-1}.$$

Note that $\pi \rtimes \sigma \simeq \check{\pi} \rtimes \omega\sigma$ and $\mu(\pi \rtimes \sigma) = \pi \rtimes \mu\sigma$.

Complete results describing reducibility of these induced representations, stated in Sally-Tadic [ST] following earlier work of Rodier [Ro2], Shahidi [Sh2,3], Waldspurger [W1], are recorded in chapter V, section 1, Propositions 2.1-2.3, below. For notations see chapter II, section 4.

For properly induced representations, defining λ- and λ_0-liftings by character relations ($\lambda(\pi_H) = \pi_4$ if $\mathrm{tr}\,\pi_4(f \times \theta) = \mathrm{tr}\,\pi_H(f_H)$ for all matching f, f_H, and $\lambda_0(\pi_1 \times \pi_2) = \pi_H$ if $\mathrm{tr}\,\pi_H(f_H) = \mathrm{tr}(\pi_1 \times \pi_2)(f_{C_0})$ for all matching f_H, f_{C_0}), our preliminary results (obtained by local character evaluations), are that $\omega^{-1} \rtimes \pi_2$ λ-lifts to $\pi_4 = I_G(\pi_2, \check{\pi}_2)$, that $\mu\pi_2 \rtimes \mu^{-1}$ (here $\omega = 1$) λ-lifts to $\pi_4 = I_G(\mu, \pi_2, \mu^{-1})$, and that $I_2(\mu, \mu^{-1}) \times \pi_2$ λ_0-lifts from C_0 to $\mu\pi_2 \rtimes \mu^{-1}$ on $H = \mathrm{PGSp}(2, F)$.

Let χ be a character of $F^\times/F^{\times 2}$. It defines a one-dimensional representation $\chi_H(h) = \chi(\lambda(h))$ of H, which λ-lifts to the one-dimensional representation $\chi(g) = \chi(\det g)$ of G (if $h = Ng$ then $\boldsymbol{\lambda}(h) = \det g$; on diagonal matrices $N(\mathrm{diag}(a, b, c, d)) = \mathrm{diag}(ab, ac, db, dc)$). The Steinberg representation of H λ-lifts to the Steinberg representation of G, and for any character χ of $F^\times$ with $\chi^2 = 1$ we have $\lambda(\chi_H \,\mathrm{St}_H) = \chi\,\mathrm{St}_G$.

2f. Elliptic Representations

Our finer local lifting results concern elliptic representations (whose characters are nonzero on the elliptic set). They follow on using global techniques. Elliptic representations include the cuspidal ones (terminology of [BZ]. These are called "supercuspidal" by Harish-Chandra, who used the word "cuspidal" for what is currently named "discrete series" or "square integrable" representations).

2.2 Local Theorem (PGSp(2) to PGL(4)). (1) *For any unordered pair* π_1, π_2 *of square integrable irreducible representations of* PGL$(2,F)$ *there exists a unique pair* π_H^+, π_H^- *of tempered (square integrable if* $\pi_1 \neq \pi_2$, *cuspidal if* $\pi_1 \neq \pi_2$ *are cuspidal) representations of* H *with*

$$\mathrm{tr}(\pi_1 \times \pi_2)(f_{C_0}) = \mathrm{tr}\,\pi_H^+(f_H) - \mathrm{tr}\,\pi_H^-(f_H),$$
$$\mathrm{tr}\,I_G(\pi_1, \pi_2; f \times \theta) = \mathrm{tr}\,\pi_H^+(f_H) + \mathrm{tr}\,\pi_H^-(f_H)$$

for all matching functions f, f_H, f_{C_0}.

If $\pi_1 = \pi_2$ *is cuspidal,* π_H^+ *and* π_H^- *are the two inequivalent constituents of* $1 \rtimes \pi_1$.

If $\pi_1 = \pi_2 = \sigma\,\mathrm{sp}_2$ *where* σ *is a character of* $F^\times$ *with* $\sigma^2 = 1$, *then* π_H^+ *and* π_H^- *are the two tempered inequivalent constituents* $\tau(\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$, $\tau(\nu^{1/2}\mathbf{1}_2, \sigma\nu^{-1/2})$ *of* $1 \rtimes \sigma\,\mathrm{sp}_2$.

If $\pi_1 = \sigma\,\mathrm{sp}_2$, $\sigma^2 = 1$, *and* π_2 *is cuspidal, then* π_H^+ *is the square integrable constituent* $\delta(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$ *of the induced* $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$; π_H^- *is cuspidal, denoted here by*

$$\delta^-(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2}).$$

If $\pi_1 = \sigma\,\mathrm{sp}_2$ *and* $\pi_2 = \xi\sigma\,\mathrm{sp}_2$, ξ $(\neq 1 = \xi^2)$ *and* σ $(\sigma^2 = 1)$ *are characters of* $F^\times$, *then* π_H^+ *is the square integrable constituent*

$$\delta(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$$

of the induced $\xi\nu^{1/2}\,\mathrm{sp}_2 \rtimes \sigma\nu^{-1/2}$; π_H^- *is cuspidal, denoted here by*

$$\delta^-(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2}).$$

(2) *For every character* σ *of* $F^\times/F^{\times 2}$ *and square integrable* π_2 *there exists a nontempered representation* $\pi_H^\times$ *of* H *such that*

$$\mathrm{tr}(\sigma\mathbf{1}_2 \times \pi_2)(f_{C_0}) = \mathrm{tr}\,\pi_H^\times(f_H) + \mathrm{tr}\,\pi_H^-(f_H),$$
$$\mathrm{tr}\,I_G(\sigma\mathbf{1}_2, \pi_2; f \times \theta) = \mathrm{tr}\,\pi_H^\times(f_H) - \mathrm{tr}\,\pi_H^-(f_H),$$

for all matching f, f_H, f_{C_0}. *Here*

$$\pi_H^- = \pi_H^-(\sigma\,\mathrm{sp}_2 \times \pi_2), \qquad \pi_H^\times = L(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2}).$$

(3) *For any characters* ξ, σ *of* $F^\times/F^{\times 2}$ *and matching* f, f_H, f_{C_0} *we have that* $\operatorname{tr}(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2)(f_{C_0})$ *is*

$$= \operatorname{tr} L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})(f_H) - \operatorname{tr} X(\xi\nu^{1/2}\operatorname{sp}_2, \xi\sigma\nu^{-1/2})(f_H),$$

and $\operatorname{tr} I_G(\sigma\xi\mathbf{1}_2, \sigma\mathbf{1}_2; f \times \theta)$ *is*

$$= \operatorname{tr} L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})(f_H) + \operatorname{tr} X(\xi\nu^{1/2}\operatorname{sp}_2, \xi\sigma\nu^{-1/2})(f_H).$$

Here $X = \delta^-$ *if* $\xi \neq 1$ *and* $X = L$ *if* $\xi = 1$.

(4) *Any* θ*-invariant irreducible square integrable representation* π *of* G *which is not a* λ_1*-lift is a* λ*-lift of an irreducible square integrable representation* π_H *of* H, *thus* $\operatorname{tr}\pi(f \times \theta) = \operatorname{tr}\pi_H(f_H)$ *for all matching* f, f_H. *In particular, the square integrable (resp. nontempered) constituent* $\delta(\xi\nu, \nu^{-1/2}\pi_2)$ *(resp.* $L(\xi\nu, \nu^{-1/2}\pi_2)$*) of the induced representation* $\xi\nu \rtimes \nu^{-1/2}\pi_2$ *of* H, *where* π_2 *is a cuspidal (irreducible) representation of* $\mathrm{GL}(2, F)$ *with central character* $\xi \neq 1 = \xi^2$ *and* $\xi\pi_2 = \pi_2$, λ*-lifts to the square integrable (resp. nontempered) constituent*

$$S(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2) \qquad (\textit{resp.} \qquad J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2))$$

of the induced representation $I_G(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ *of* $G = \mathrm{PGL}(4, F)$.

These character relations permit us to introduce the notion of a packet of an irreducible representation, and of a quasi-packet, over a local field. Thus we say that the *packet* of a representation π_H of H consists of π_H alone unless it is tempered of the form π_H^+ or π_H^- for some pair π_1, π_2 of (irreducible) square integrable representations of $\mathrm{PGL}(2, F)$, in which case the packet $\{\pi_H\}$ is defined to be $\{\pi_H^+, \pi_H^-\}$, and we write $\lambda_0(\pi_1 \times \pi_2) = \{\pi_H^+, \pi_H^-\}$ and $\lambda(\{\pi_H^+, \pi_H^-\}) = I_G(\pi_1, \pi_2)$. Further, we define a *quasi-packet* only for the nontempered (irreducible) representations $\pi_H^\times$ and $L = L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})$, to consist of $\{\pi_H^\times, \pi_H^-\}$ and $\{L, X\}$, $X = X(\xi\nu^{1/2}\operatorname{sp}_2, \xi\sigma\nu^{-1/2})$. We say that $\sigma\mathbf{1}_2 \times \pi_2$ λ_0-lifts to the quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2) = \{\pi_H^\times, \pi_H^-\}$, which in turn λ-lifts to $I_G(\sigma\mathbf{1}_2, \pi_2)$, and similarly, $\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2$ λ_0-lifts to $\lambda_0(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2) = \{L, X\}$ which λ-lifts to $I_G(\sigma\xi\mathbf{1}_2, \sigma\mathbf{1}_2)$.

Conjecturally our packets and quasi-packets coincide with the L-packets and A-packets conjectured to exist by Langlands and Arthur [A2-3].

Using the notations of section V.11 below, we state the analogue of these results in the real case: $F = \mathbb{R}$. For clarity, denote π_1 and π_2

above by π^1 and π^2. In (1), $\pi^1 = \pi_{k_1}$ and $\pi^2 = \pi_{k_2}$, $k_1 \geq k_2 > 0$ and k_1, k_2 are odd, are discrete series representations of $\mathrm{PGL}(2,\mathbb{R})$, and π_H^+ is the generic $\pi^{\mathrm{Wh}}_{k_1,k_2}$, π_H^- is the holomorphic $\pi^{\mathrm{hol}}_{k_1,k_2}$, which are discrete series representations of $\mathrm{PGSp}(2,\mathbb{R})$ when $k_1 > k_2$. When $k_1 = k_2$, π_H^+ is the generic and π_H^- is the nongeneric (tempered) constituents of the induced $1 \rtimes \pi_{2k_1+1}$. There is no special or Steinberg representation of $\mathrm{GL}(2,\mathbb{R})$; the analogue is the lowest discrete series π_1. The π_k are self invariant under twist with sgn. In (2) with $\pi^2 = \pi_{2k+3}$ $(k \geq 0)$, $\pi_H^\times$ is $L(\sigma\nu^{1/2}\pi_{2k+3}, \sigma\nu^{-1/2})$, π_H^- is $\pi^{\mathrm{hol}}_{2k+3,1}$, π_H^+ is $\pi^{\mathrm{Wh}}_{2k+3,1}$. In (3), if $\xi = \mathrm{sgn}$ then X is the tempered $\pi_H^- \subset 1 \rtimes \pi_1$, if $\xi = 1$ then X is $L(\nu^{1/2}\pi_1, \sigma\nu^{-1/2})$. Both of these X, as well as $L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})$, are not cohomological. In (4), π^2 is π_{2k+2}, $L(\xi\nu, \nu^{-1/2}\pi^2)$ is $L(\mathrm{sgn}\nu, \nu^{-1/2}\pi_{2k+2})$, $\delta(\xi\nu, \nu^{-1/2}\pi^2)$ is $\pi^{\mathrm{hol}}_{2k+3,2k+1} \oplus \pi^{\mathrm{Wh}}_{2k+3,2k+1}$.

2g. Automorphic Representations

With these local definitions we can state our *global results.* These global results *are partial, since we work with test functions whose components are elliptic at least at three places,* and consequently we cannot detect automorphic representations which do not have at least three components whose (θ-) characters are nonzero on the (θ-) elliptic set. Thus we fix three places $\{v_1, v_2, v_3\}$ and discuss only $\pi_1 \times \pi_2$, π_H and $\pi = \pi_G$ whose components there are (θ-) elliptic.

Let us explain the reason for this restriction. The (noninvariant) trace formula, as developed by Arthur, involves weighted orbital integrals and logarithmic derivatives of induced representations. Arthur's splitting formula shows that these can be expressed as products of local distributions, which are all invariant (orbital integrals or traces of induced representations) except at most at rank(H) places. Working with test functions $f_H = \otimes f_{Hv}$ with rank(H)+1 components f_{Hv} with $\mathrm{tr}\,\pi_{Hv}(f_{Hv}) = 0$ for every tempered properly induced representation π_{Hv} of H_v (equivalently: f_{Hv} whose orbital integrals vanish on the regular nonelliptic set of H_v), all non elliptic terms vanish. We call such f_{Hv} elliptic. At an additional place we use a regular Iwahori biinvariant component (see [FK1], [FK2], [F2] or [F3;VI]) to annihilate the singular orbital integrals. For the twisted trace formula we use the twisted rank, which is equal to rank(H), to obtain the same vanishing. This removes all complicated terms in the trace formulae comparison. Here *rank* means the F-dimension of a maximal split torus in the derived group, or in the derived group of the group of fixed points of the involution in the twisted case.

For very little effort we can reduce the number of restrictions to two, rather than three. Using elliptic components f_{Hv_1}, f_{Hv_2}, implies that the local factors at each $v \neq v_1$, v_2, in the terms in the trace formula, are invariant. We then use at a third, nonarchimedean, place v_3 a regular-Iwahori function (as in [FK1], [FK2], [F2], [F3;VI]). Similar choice is made for the twisted formula. The geometric sides of the trace formulae consist now of elliptic terms only. As the distributions at v_3 which occur in the trace formula are invariant, such f_{Hv_3} can also be taken to be a spherical function with the same orbital integrals as the Iwahori-regular component. The resulting equality of discrete and continuous measures (the continuous measure comes from the spectral sides), which are invariant distributions in f_{Hv_3}, implies their vanishing by the (standard) argument of "generalized linear independence of characters" (using the Stone-Weierstrass theorem) employed in this context in [FK1], [FK2], [F2], [F3]. To simplify our exposition we do not record this argument here, but our global results can safely be used with two restriction, at v_1, v_2, rather than three.

One can do better, and require that only one component, f_{Hv}, be elliptic, at a single real place v. This argument, explained in Laumon [Lau], requires very extensive referencing to much of Arthur's deep analysis of the distributions appearing in the trace formula. Inclusion of these arguments here would have made this work more complicated than the relatively elementary exposition I wish to present. However, our results are provable for global representations with a single elliptic component at a real place. This suffices for all purposes of studying the decomposition of the ℓ-adic cohomology with compact supports of the Shimura variety associated with our group, and any coefficients, as a Galois-Hecke module ([F7]).

These constraints will be removed once the trace formulae identity is established for general test functions. This is being developed by Arthur. A simpler method, based on regular functions, has been introduced when the rank is one (see [F2;I], [F3;VI], [F4;III]) to establish unconditional comparison of trace formulae. But it has not yet been extended to the higher rank cases.

With this reservation, emphasized by a $*$-superscript in the following

Global Theorem, the discrete spectrum representations of $\mathrm{PGSp}(2,\mathbb{A})$, i.e. $\mathbf{H}(\mathbb{A})$, can now be described by means of the liftings. They consist of two types, stable and unstable. Global packets and quasi-packets define a partition of the spectrum. To define a (global) [quasi-] packet $\{\pi_H\}$, fix a local [quasi-] packet $\{\pi_{Hv}\}$ at each place v of F, such that $\{\pi_{Hv}\}$ contains an unramified member π^0_{Hv} (and then $\{\pi_{Hv}\}$ consists only of π^0_{Hv} in case it is a packet) for almost all v. The [quasi-] *packet* $\{\pi_H\}$ is then defined to consist of all products $\otimes_v\pi'_{Hv}$ with π'_{Hv} in $\{\pi_{Hv}\}$ for all v, and $\pi'_{Hv}=\pi^0_{Hv}$ for almost all v. The [quasi-] packet $\{\pi_H\}$ of an automorphic representation π_H is defined by the local [quasi-] packets $\{\pi_{Hv}\}$ of the components π_{Hv} of π_H at almost all places.

The discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ will be described by means of the λ_0- and λ-liftings. We say that the discrete spectrum $\pi_1\times\pi_2$ λ_0-*lifts* to a packet $\{\pi_H\}$ (or to a member thereof) if $\{\pi_{Hv}\}=\lambda_0(\pi_{1v}\times\pi_{2v})$ for almost all v, and that a packet $\{\pi_H\}$ (or a member of it) λ-*lifts* to an irreducible self-contragredient automorphic representation π if $\lambda(\{\pi_{Hv}\})=\pi_v$ for almost all v. The *unstable* spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ is the set of discrete spectrum representations which are λ_0-lifts; its complement is the *stable* spectrum. A [quasi-] packet whose automorphic members lie in the (un)stable spectrum is called a(n un)stable [quasi-] packet.

2.3 GLOBAL THEOREM* (PGSP(2) TO PGL(4)). *The packets and quasi-packets partition the discrete spectrum of the group* $\mathrm{PGSp}(2,\mathbb{A})$, *thus they satisfy the rigidity theorem: if* π_H *and* π'_H *are discrete spectrum representations locally equivalent at almost all places then their packets or quasi-packets are equal.*

The λ*-lifting is a bijection between the set of packets (resp. quasi-packets) of discrete spectrum representations in the stable spectrum (of* $\mathrm{PGSp}(2,\mathbb{A})$*) and the set of self contragredient discrete spectrum representations of* $\mathrm{PGL}(4,\mathbb{A})$ *which are one dimensional, or cuspidal and not a* λ_1*-lift from* $\mathbf{C}(\mathbb{A})$ *(or residual* $J(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$ *where* π_2 *is a cuspidal representation of* $\mathrm{GL}(2,\mathbb{A})$ *with central character* $\xi\neq 1=\xi^2$ *and* $\xi\pi_2=\pi_2$*).*

The λ_0*-lifting is a bijection between the set of pairs of discrete spectrum representations*

$$\{\pi_1\times\pi_2,\pi_2\times\pi_1;\pi_1\neq\pi_2\}\quad \textit{of}\quad \mathrm{PGL}(2,\mathbb{A})\times\mathrm{PGL}(2,\mathbb{A}),$$

and the set of packets and quasi-packets in the unstable spectrum of the *group* PGSp(2, $\mathbb{A}$). *The* λ*-lifting is a bijection from this last set to the set of automorphic representations* $I_G(\pi_1, \pi_2)$ *of* PGL(4, $\mathbb{A}$), *normalizedly induced from discrete spectrum* $\pi_1 \times \pi_2$ ($\pi_1 \neq \pi_2$) *on the parabolic subgroup with Levi factor of type* (2, 2). *If* $\pi_1 \times \pi_2$ *is cuspidal, its* λ_0*-lift is a packet, otherwise: quasi-packet.*

Each member of a stable packet occurs in the discrete spectrum of the group PGSp(2, $\mathbb{A}$) *with multiplicity one. The multiplicity* $m(\pi_H)$ *of a member* $\pi_H = \otimes \pi_{Hv}$ *of an unstable [quasi-]packet* $\lambda_0(\pi_1 \times \pi_2)$ ($\pi_1 \neq \pi_2$) *is not ("stable", or) constant over the [quasi-]packet. If* $\pi_1 \times \pi_2$ *is cuspidal, it is*

$$m(\pi_H) = \frac{1}{2}(1 + (-1)^{n(\pi_H)}) \quad (\in \{0, 1\}).$$

Here $n(\pi_H)$ *is the number of components* π_{Hv}^- *of* π_H *(it is bounded by the number of places* v *where both* π_{1v} *and* π_{2v} *are square integrable). Each* π_H *with* $m(\pi_H) = 1$ *is cuspidal.*

The multiplicity $m(\pi_H)$ *(in the discrete spectrum of* PGSp(2, $\mathbb{A}$)) *of* $\pi_H = \otimes \pi_{Hv}$ *from a quasi-packet* $\lambda_0(\sigma \mathbf{1}_2 \times \pi_2)$, *where* π_2 *is a cuspidal representation of* PGL(2, $\mathbb{A}$) *and* σ *is a character of* $\mathbb{A}^\times / F^\times \mathbb{A}^{\times 2}$, *is*

$$\frac{1}{2}(1 + \varepsilon(\sigma\pi_2, \frac{1}{2})(-1)^{n(\pi_H)}) \qquad (= 0 \quad \textit{or} \quad 1),$$

where $n(\pi_H)$ *is the number of components* π_{Hv}^- *of* π_H, *and* $\varepsilon(\pi_2, s)$ *is the usual* ε*-factor which appears in the functional equation of the* L*-function* $L(\pi_2, s)$. *In particular* $\pi_H^\times = \otimes \pi_{Hv}^\times$ ($n(\pi_H) = 0$) *is in the discrete spectrum if and only if* $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$.

Finally we have $m(\pi_H) = \frac{1}{2}(1 + (-1)^{n(\pi_H)})$ *for* $\pi_H = \otimes \pi_{Hv}$ *in* $\lambda_0(\sigma\xi \mathbf{1}_2 \times \sigma \mathbf{1}_2)$ *with* $n(\pi_H)$ *components* $\pi_{Hv} = X_v$. *Here* $\pi_H = \otimes L_v$ ($n(\pi_H) = 0$) *is residual.*

2h. Unstable Spectrum

Note that the quasi-packet $\lambda_0(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2)$ is defined by the local quasi-packets

$$\{L_v = L(\nu_v\xi_v, \xi_v \rtimes \sigma_v\nu_v^{-1/2}), \qquad X_v = X(\xi_v\nu_v^{1/2}\,\mathrm{sp}_{2v}, \xi_v\sigma_v\nu_v^{-1/2})\}$$

for every v, where ξ ($\neq 1$), σ are characters of $\mathbb{A}^\times/F^\times$ with $\xi^2 = 1 = \sigma^2$ and ξ_v, σ_v are their components. When ξ_v, σ_v are unramified, this quasi-packet contains the unramified representation $\pi^0_{Hv} = L_v$. Members of this quasi-packet have been studied by means of the theta correspondence by Howe and Piatetski-Shapiro, see, e.g., [PS1], Theorem 2.5. They attracted interest since they violate the naive generalization of the Ramanujan conjecture, which expects the components of a cuspidal representation to be tempered. (The form of the Ramanujan conjecture which is expected to be true asserts that the components of a cuspidal representation of $\mathrm{PGSp}(2,\mathbb{A})$ which λ-lifts to a representation of $\mathrm{PGL}(4,\mathbb{A})$ induced from a cuspidal representation of a Levi subgroup, are tempered.) Members of this quasi-packet are equivalent at almost all places to the quotient of the properly induced representation $\nu\xi \times \xi \rtimes \sigma\nu^{-1/2}$.

Let π_2 be a cuspidal representation of $\mathrm{PGL}(2,\mathbb{A})$, σ a character of $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}$. The packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ contains the constituent $\pi_H^\times = \otimes_v\pi_{Hv}^\times$ of the representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2} \simeq \sigma\nu^{-1/2}\pi_2 \rtimes \sigma\nu^{1/2}$ properly induced from an automorphic representation, hence it is automorphic by [L4]. It is known that $\pi_H^\times$ is residual precisely when $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; hence $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$ in this case.

Let $n(\pi_2)$ denote the number of square integrable components of π_2. The quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ thus consists of $2^{n(\pi_2)}$ (irreducible) representations. If $n(\pi_2) \geq 1$, half of them in the discrete spectrum, all cuspidal if $L(\sigma\pi_2, \frac{1}{2}) = 0$, all but one: $\pi_H^\times = \otimes_v\pi_{Hv}^\times$, are cuspidal if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$. If $n(\pi_2) \geq 1$ and $L(\sigma\pi_2, \frac{1}{2}) = 0$, the automorphic nonresidual $\pi_H^\times$ is cuspidal when $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$.

If π_2 has no square integrable components ($n(\pi_2) = 0$), the packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ consists only of $\pi_H^\times$. This $\pi_H^\times$ is residual if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; cuspidal (by [PS1], Theorem 2.6 and [PS2], Theorem A.2) if $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$; or (automorphic but) not in the discrete spectrum otherwise: $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = -1$. In this last case the λ_0-

lift of $\sigma\mathbf{1}_2 \times \pi_2$ is not in the discrete spectrum, and there is no discrete specrtum representation λ-lifting to $I_G(\sigma\mathbf{1}_2, \pi_2)$.

At a place v where π_{2v} is induced $I(\mu_v, \mu_v^{-1})$, the packet

$$\pi_{Hv} = \lambda_0(\sigma_v\mathbf{1}_2 \times \pi_{2v})$$

is the irreducible induced $\mu_v\sigma_v\mathbf{1}_2 \rtimes \mu_v^{-1}$, which λ-lifts to the induced $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$, and *not* the irreducible induced

$$\sigma_v\mu_v\nu_v^{1/2} \times \sigma_v\mu_v^{-1}\nu_v^{1/2} \rtimes \sigma_v\nu_v^{-1/2} = \sigma_v\mu_v\nu_v^{1/2} \rtimes I(\mu_v^{-1}, \sigma_v\nu_v^{-1/2}),$$

which λ-lifts to the reducible induced $I_G(\mu_v, \sigma_v I(\nu_v^{1/2}, \nu_v^{-1/2}), \mu_v^{-1})$, which has the constituent $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$.

Members of the quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ were studied numerically by H. Saito and N. Kurokawa, and using the theta correspondence by Piatetski-Shapiro and others, see [PS1], Theorem 2.6. They attracted interest since they violate the naive generalization of the Ramanujan conjecture. They are equivalent at almost all places to the quotient of the properly induced representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$.

A discrete specrtum representation π_H with a local component

$$L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$$

(whose packet consists of itself), where π_{2v} is a cuspidal representation with central character $\xi_v \neq 1 = \xi_v^2$ and $\xi_v\pi_{2v} = \pi_{2v}$, is in the packet of $L(\nu\xi, \nu^{-1/2}\pi_2)$. Here π_2 is cuspidal with central character $\xi \neq 1 = \xi^2$, hence $\xi\pi_2 = \pi_2$, whose components at v are π_{2v} and ξ_v. It λ-lifts to $J_G(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. At v with $\xi_v = 1$ the component π_{2v} is induced. If $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\xi_v^2 = 1$ and $\mu_v^2 = 1$ (in particular whenever $\xi_v \neq 1$ and π_{2v} is not cuspidal), then $L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$ is $L_v = L(\nu_v\xi_v, \xi_v \rtimes \mu_v\nu_v^{-1/2})$, which λ-lifts to $I_G(\mu_v\mathbf{1}_2, \mu_v\xi_v\mathbf{1}_2)$, and its packet contains also $X_v = X(\nu_v^{1/2}\xi_v\,\mathrm{sp}_{2v}, \xi_v\mu_v\nu_v^{-1/2})$. Thus the packet of π_H is determined by $\{L_v, X_v\}$ at all v where $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\mu_v^2 = 1 = \xi_v^2$, and by the singleton $\{L_v = L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})\}$ at all other v, where π_{2v} is cuspidal, or $\xi_v = 1$ and $\pi_{2v} = I(\mu_v, \mu_v^{-1})$, $\mu_v^2 \neq 1$. Each member of this infinite packet occurs in the discrete specrtum with multiplicity one, and is cuspidal, with the exception of $L(\nu\xi, \nu^{-1/2}\pi_2) = \otimes_v L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$, which is residual ([Kim], Theorem 7.2). Members of the packet $L(\nu\xi, \nu^{-1/2}\pi_2)$ are considered in the Appendix of [PS1] and its corrigendum.

If π_1 and π_2 are cuspidal but there is no place v where both are square integrable, $\lambda_0(\pi_1 \times \pi_2)$ consists of a single irreducible cuspidal representation. This instance of the lifting λ_0 – where π_i are cuspidal – can also be studied ([Rb]) using the theta correspondence for suitable dual reductive pairs (SO(4), PGSp(2)) for the isotropic and anisotropic forms of the orthogonal group, to describe further properties of the packets, such as their periods.

2i. Generic Representations

Our proof of the existence of the lifting λ uses only the trace formula, orbital integrals and character relations. However, for cuspidal representations π_1, π_2 of PGL$(2, F)$, F local, we get only the character relation

$$\operatorname{tr} I_G(\pi_1, \pi_2; f \times \theta) = (2m+1)[\operatorname{tr} \pi_H^+(f_H) + \operatorname{tr} \pi_H^-(f_H)].$$

Here f on $G = \mathrm{PGL}(4, F)$ and f_H on $H = \mathrm{PGSp}(2, F)$ are any matching functions, and $m = m(\pi_1, \pi_2)$ is a nonnegative integer. To prove multiplicity one theorem for PGSp$(2, \mathbb{A})$ we need the fact that $m = 0$.

Our proof is global. It uses the following results from the theory of the theta correspondence, Whittaker models and Eisenstein series. (1) Ginzburg-Rallis-Soudry [GRS], Theorem A: Each representation $I(\pi_1, \pi_2)$ of PGL$(4, \mathbb{A})$ normalizedly induced from a cuspidal representation $\pi_1 \times \pi_2$ of its $(2,2)$-parabolic, where $\pi_1 \neq \pi_2$ are cuspidal representations of PGL$(2, \mathbb{A})$, is a λ-lift of a unique *generic* cuspidal representation π_H of SO$(5, \mathbb{A})$ = PGSp$(2, \mathbb{A})$. (2) Kudla-Rallis-Soudry [KRS], Theorem 8.1: If π_0 is a locally generic cuspidal representation of Sp$(2, \mathbb{A})$ and the partial degree 5 L-function $L(S, \pi_0, \mathrm{id}_5, s)$ is $\neq 0$ at $s = 1$ then π_0 is (globally) generic. (3) Shahidi [Sh1], Theorem 5.1: If π_0 is a generic cuspidal representation of Sp$(2, \mathbb{A})$, then $L(S, \pi_0, \mathrm{id}_5, s)$ is $\neq 0$ at $s = 1$. See chapter V, section 7, and the final remark in section 6, for further comments. We do not use the assertion (attributed to "a yet to be published result of Jacquet and Shalika") in the Remark following the statement of Theorem 8.1 in [KRS], p. 535 (that a cuspidal representation of GSp(2) is generic iff it lifts to GL(4)), which contradicts – at least as stated – our result that all representations but one in a packet of PGSp(2) are nongeneric, yet they all lift to PGL(4).

Our global proof resembles (but is strictly different from) the second proof of [F4;II], Proposition 3.5, p. 48, which is also based on the theory of generic representations. This Proposition claims the multiplicity one theorem for the discrete spectrum of $U(3, E/F)$. However, the proof of [F4;II], p. 48, is not complete. Indeed, the claim in Proposition 2.4(i) in reference [GP] to [F4;II], that "$L^2_{0,1}$ has multiplicity 1", is interpreted in [F4;II] as asserting that generic representations of U(3) occur in the discrete spectrum with multiplicity one. But it should be interpreted as asserting that irreducible π in $L^2_{0,1}$ have multiplicity one *only in the subspace* $L^2_{0,1}$ of the discrete spectrum. This claim does not exclude the possibility of having a cuspidal π' perpendicular and equivalent to $\pi \subset L^2_{0,1}$.

Multiplicity one for the generic spectrum would follow via this global argument from the statement that a locally generic cuspidal representation is globally generic (multiplicity one implies this statement too). In our case of PGSp(2) we deduce from [KRS], [GRS], [Sh1] that a locally generic cuspidal representation which is equivalent at almost all places to a generic cuspidal representation is globally generic. A proof for U(3) still needs to be written down.

The usage of the theory of generic representations in the proof described above is not natural. A purely local proof of multiplicity one theorem for the discrete spectrum of U(3) based only on character relations is proposed in [F4;II], Proof of Proposition 3.5, p. 47. It is based on Rodier's result [Ro1] that the number of Whittaker models is encoded in the character of the representation near the origin. Details of this proof are given in [F4;IV] in odd residual characteristic in the case of basechange for U(3). It implies that in a tempered packet of representations of $U(3, E/F)$ there is precisely one generic representation, and that each generic packet of discrete spectrum representations of $U(3, \mathbb{A}_E/\mathbb{A}_F)$ – where a generic packet means one which lifts to a generic representation of $GL(3, \mathbb{A}_E)$ – would contain precisely one generic member. Moreover, a locally generic cuspidal representation of $U(3, \mathbb{A}_E/\mathbb{A}_F)$ is generic.

This type of a local argument was introduced in [FK1] in the proof of the metaplectic correspondence and the multiplicity one theorem for the discrete spectrum of the metaplectic group of $GL(n, \mathbb{A})$. We have not carried out this local proof in the case of PGSp(2) as yet.

In the case of PGSp(2) our global proof implies that a local tempered packet contains precisely one generic representation, and that a global packet which lifts to a generic representation of $\mathrm{PGL}(4,\mathbb{A})$ contains precisely one everywhere generic representation. The latter is generic if the packet is unstable (in the image of the lifting λ_0). We do not show that a locally generic cuspidal representation of $\mathrm{PGSp}(2,\mathbb{A})$ which is stable (λ-lifts to a cuspidal representation of $\mathrm{PGL}(4,\mathbb{A})$) is generic.

There is some overlap between our results on the existence of the λ-lifting and the work of [GRS] which asserts that the weak (i.e., in terms of almost all places) lifting establishes a bijection from the set of equivalence classes of (irreducible automorphic) cuspidal *generic* representations of the split group $\mathrm{SO}(2n+1,\mathbb{A})$, to the set of representations of $\mathrm{PGL}(2n,\mathbb{A})$ of the form $\pi = I(\pi_1,\dots,\pi_r)$, normalized induction from the standard parabolic subgroup of type $(2n_1,\dots,2n_r)$, $n = n_1 + \cdots + n_r$, where π_i are cuspidal representations of $\mathrm{GL}(2n_i,\mathbb{A})$ such that $L(S,\pi_i,\Lambda^2,s)$ has a pole at $s=1$ and $\pi_i \neq \pi_j$ for all $i \neq j$, and the partial L-function is defined as a product outside a finite set S where all π_i are unramified. Of course we are concerned only with the case $n=2$, where $\mathrm{PGSp}(2) \simeq \mathrm{SO}(5)$.

Our characterization of the lifting λ is (as in [GRS]) that $I(\pi_1,\pi_2)$, cuspidal representations $\pi_1 \neq \pi_2$ of $\mathrm{PGL}(2,\mathbb{A})$, are in the image; and that self contragredient cuspidal representations π of $\mathrm{PGL}(4,\mathbb{A})$ are in the image of the lifting λ from $\mathrm{PGSp}(2,\mathbb{A})$ $(=\mathrm{SO}(5,\mathbb{A}))$ precisely if they are not in the image of the lifting λ_1 from $\mathrm{SO}(4,\mathbb{A})$. The cuspidal $\pi = \lambda(\pi_H)$, generic π_H, are characterized in [GRS] as the $\pi \simeq \check{\pi}$ such that $L(S,\pi,\Lambda^2,s)^{-1}$ is 0 at $s=1$. Thus the characterization of the cuspidal image of λ here is complementary to but different than that of [GRS].

However, the methods of [GRS] apply only to generic representations, while our methods apply to all representations of PGSp(2). In particular, we can define packets, describe their structure, establish multiplicity one theorem and rigidity theorem for packets of PGSp(2), specify which member in a packet or a quasi-packet is in the discrete spectrum, and we can also λ-lift the nongeneric nontempered (at almost all places) packets to residual self-contragredient representations of $\mathrm{PGL}(4,\mathbb{A})$. Our liftings are proven in terms of all places, not only almost all places. In addition we establish the lifting λ_1 from SO(4) to PGL(4), determine its fibers (that is, prove multiplicity one theorem for SO(4) and rigidity in the sense explained above), and show that any self-contragredient discrete spectrum representation of PGL(4, $\mathbb{A}$) which is not a λ-lift from PGSp(2, $\mathbb{A}$) is a λ_1-lift from SO(4, $\mathbb{A}$).

2j. Orientation

This work is an analogue for (SO(4), PGSp(2), PGL(4)) of [F3], which dealt with (PGL(2), SL(2), PGL(3)), thus with the symmetric square lifting, and of [F4], which dealt with quadratic basechange for the unitary group U(3, E/F), thus with (U(2, E/F), U(3, E/F), GL(3, E)). These works use the twisted – by transpose-inverse (and the Galois action in the unitary groups case) – trace formulae on PGL(4), PGL(3), GL(3, E). They are based on the fundamental lemma: [F5] in our case, [F3;V] and [F4;I] in the other cases. The technique employed in these last works benefited from work of Weissauer [W] and Kazhdan [K1]. The present work, which deals with the applications of the fundamental lemma and the trace formula to character relations, liftings and the definition of packets, is analogous to [F3;IV] and [F4;II].

The trace formula identity is proven in [F3;VI] and [F4;III] for all test functions. Here we deal only with test functions which have at least three elliptic components. The trace formulae identity for a general test function has not yet been proven in our case. Perhaps the method of [AC] could be used for that, as it has been applied in a general rank case. It would be interesting to pursue the elementary techniques of [F3;VI] and [F4;III], and [F2;I], which establish the trace formulae identity for basechange for GL(2) by elementary means, based on the usage of regular, Iwahori test functions. In particular the present work does not develop the trace formula. It only uses a form of it.

Our approach uses the trace formula, developed by Arthur (see [A1]), as envisaged by Langlands e.g. in his work on basechange for GL(2).

Of course Siegel modular forms have been extensively studied by many authors (e.g., Siegel, Maass, Shimura, Andrianov, Freitag, Klingen...) over a long period of time, and several textbooks are available.

As noted above, an important representation theoretic approach alternative to the trace formula, based on the theta correspondence, Weil representation, Howe's dual reductive pairs, L-functions and converse theorems, has been fruitfully developed in our context of the symplectic group by Piatetski-Shapiro, Howe, Kudla, Rallis, Ginzburg, Roberts, Schmidt, Soudry, and others, see, e.g., [PS], [KRS], [GRS], [Rb], [Sch].

A purely local approach to character computations is developed in [FZ].

Our results are used by P.-S. Chan [Ch] to determine the representations of GSp(2) which are invariant under twisting by a quadratic character.

The classification of the automorphic representations of PGSp(2) has applications to the decomposition of the étale cohomology with compact supports and twisted coefficients of the Shimura varieties associated with GSp(2), see [F7]. Our techniques extend to deal with admissible and automorphic representations of GSp(2), but this we do not do here.

The present part is divided into five chapters: I. Introduction, II. Basic Facts, III. Trace Formulae, IV. The Lifting λ_1, V. The Lifting λ. Each is divided into sections. Definitions or propositions are numbered together in each section.

3. Conjectural Compatibility

Our local results are analogous to those of Arthur [A2], who verified them in the real case, and are consistent with his conjectures. We shall assume in this section, not to be used anywhere else in this work, familiarity with [A2], [A3], and briefly highlight some of the definitions and conjectures of [A2] in our context, in our notations (H, C_0 in place of Arthur's G, H). For brevity we write W_F for the Weil group of the local field, but as in [A2], 2.1, this group has to be the motivic Galois group of the conjecturally Tannakian category of tempered representations of all GL(n)'s in the global case, a complex pro-reductive group, or an extension of W_F by a connected compact group ($W_F \times \mathrm{SU}(2,\mathbb{R})$ in the p-adic case).

Thus $\Phi(H/F)$ denotes the set of $\widehat{H}$-conjugacy classes of admissible (in particular, $\mathrm{pr}_2 \circ\phi = \mathrm{id}_{W_F}$) maps

$$\phi : W_F \to {}^LH = \widehat{H} \times W_F \qquad (\widehat{H} = {}^LH^0).$$

It contains the subset $\Phi_{\mathrm{temp}}(H/F)$ defined using the ϕ with bounded $\mathrm{Im}(\mathrm{pr}_1 \circ\phi)$. Note that for a split adjoint group H over F, $\widehat{H}$ is simply connected, and for any semisimple s in $\widehat{H}$, the centralizer $\widehat{C}_0 = Z_{\widehat{H}}(s)$ of s in $\widehat{H}$ specifies the endoscopic group H uniquely (up to isomorphism). Write $S_\phi = S_\phi^H = Z_{\widehat{H}}(\phi(W_F))$ (centralizer in the connected group $\widehat{H}$ of the image of ϕ), $\widehat{Z} = Z_{\widehat{H}}({}^LH) \subset \widehat{H}$, and note that $\mathcal{S}_\phi = S_\phi/S_\phi^0\widehat{Z}$ is a finite abelian group, conjecturally in duality with the packet Π_ϕ to be associated with $\phi \in \Pi_{\text{temp}}(H/F)$ (this is the case when $F = \mathbb{R}$, see [A2]). Arthur [A2] defines a further set $\Psi(H/F)$ of $\widehat{H}$-conjugacy classes of maps $\psi : W_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LH$ such that $\psi|W_F \in \Phi_{\text{temp}}(H/F)$, and a map

$$\psi \mapsto \phi_\psi, \qquad \phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}),$$

which embeds $\Psi(H/F)$ in $\Phi(H/F)$. Each ψ can be viewed as a pair

$$(\phi, \rho) \in (\Phi_{\text{temp}}(H/F) \times \mathrm{Hom}(\mathrm{SL}(2,\mathbb{C}), S_\phi))/\,\mathrm{Int}(S_\phi)$$

(quotient by S_ϕ-conjugacy). Then $\Phi_{\text{temp}}(H/F)$ embeds in $\Psi(H/F)$ as the $(\phi, 1)$. Put

$$S_\psi = S_\psi^H = Z_{\widehat{H}}(\psi(W_F \times \mathrm{SL}(2,\mathbb{C}))).$$

It is equal to

$$Z_{S_{\phi_\psi}}(\rho(\mathrm{SL}(2,\mathbb{C}))),$$

a subgroup of S_{ϕ_ψ}, and there is a surjection $\mathcal{S}_\psi = S_\psi/S_\psi^0\widehat{Z} \twoheadrightarrow \mathcal{S}_{\phi_\psi}$. The group $\mathcal{S}_\psi$ is in duality with the quasi-packet Π_ψ conjecturally associated with ψ. Globally, the quasi-packet Π_ψ contains no discrete spectrum representations of H unless S_ψ is finite.

Let us review the examples of [A2], where $\widehat{H} = \mathrm{Sp}(2,\mathbb{C}) \supset \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d \end{pmatrix} \right\}$. The parameter ψ can be described by the maps

$$(\phi = \phi_1 \times \phi_2, \rho = \rho_1 \times \rho_2) : \; W_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}).$$

If $\phi_i : W_F \to \mathrm{SL}(2,\mathbb{C})$ are irreducible and inequivalent, $\rho = 1$,

$$Z_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{Im}\,\phi_i) = \{\pm I\}, \quad S_{\phi_\psi} = \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \mathcal{S}_{\phi_\psi} = \mathbb{Z}/2,$$

$S_\psi = \mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathcal{S}_\psi = \mathbb{Z}/2$. This is a "classical" tempered case, as $\mathrm{Im}\,\phi_i$ are bounded.

If $\phi_1 = \phi_2$ is irreducible, $\rho = 1$, $S_{\phi_\psi} = \mathrm{O}(2,\mathbb{C}) = S_\psi$ (this group consists of the diag(g, g^*), $g^* = w^t g^{-1} w$, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which commute with $\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$, thus $g^t g = I$), $S_\psi^0 = \mathrm{SO}(2,\mathbb{C})$ and $\mathcal{S}_{\phi_\psi} = \mathcal{S}_\psi = \mathbb{Z}/2$ $(= \langle \mathrm{diag}(w,w) \rangle)$.

These cases correspond to $\lambda_0(\pi_1 \times \pi_2)$, where π_1, π_2 are in the discrete spectrum; a local packet consists of $2 = [\mathbb{Z}/2]$ elements. A global packet in the second case consists of no discrete spectrum representations since $S_\psi = O(2,\mathbb{C})$ is not finite. In the first case, where $\pi_2 \neq \pi_1$, the packet consists of 2^n irreducibles, where n is the number of places where both π_1 and π_2 are square integrable; half of the members in the packet are in the discrete spectrum (one, if $n = 0$).

If ϕ_1 is irreducible and $\mathrm{Im}(\phi_2) \subset \{\pm I\}$, and $\rho = 1 \times \mathrm{id}$, we have $S_{\phi_\psi} = \mathbb{Z}/2 \times \mathbb{C}^\times$ $(= \{\mathrm{diag}(\iota, z, z^{-1}, \iota);\, z \in \mathbb{C}^\times,\, \iota \in \{\pm 1\}\})$, $\mathcal{S}_{\phi_\psi} = \{1\}$, $S_\psi = \mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathcal{S}_\psi = \mathbb{Z}/2$. This is the case of $\lambda_0(\pi_1 \times \phi_2 \mathbf{1}_2)$, where ϕ_2 is a character.

If $\mathrm{Im}\,\phi_i \subset \{\pm I\}$ but $\phi_1 \neq \phi_2$, and $\rho_i = \mathrm{id}$, $S_{\phi_\psi} = \mathbb{C}^\times \times \mathbb{C}^\times$ $(= \{\mathrm{diag}(z, t, t^{-1}, z^{-1});\, z, t \in \mathbb{C}^\times\})$, $\mathcal{S}_{\phi_\psi} = \{1\}$, $S_\psi = \mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathcal{S}_\psi = \mathbb{Z}/2$. This is the case of $\lambda_0(\phi_1 \mathbf{1}_2 \times \phi_2 \mathbf{1}_2)$, where $\phi_1 \neq \phi_2$ are characters of $F^\times/F^{\times 2}$ or $\mathbb{A}^\times/F^\times \mathbb{A}^{\times 2}$.

If $\phi_1 = \phi_2$ with image in $\{\pm I\}$, and $\rho_i = \mathrm{id}$, $S_{\phi_\psi} = \mathrm{GL}(2,\mathbb{C})$ $(= \{\mathrm{diag}(g, g^*);\, g \in \mathrm{GL}(2,\mathbb{C})\})$, $\mathcal{S}_{\phi_\psi} = \{1\}$, $S_\psi = O(2,\mathbb{C})$, $\mathcal{S}_\psi = \mathbb{Z}/2$. This is the case of $\lambda_0(\phi_1 \mathbf{1}_2 \times \phi_1 \mathbf{1}_2)$, whose packet contains no discrete spectrum representations, and indeed $S_\psi = O(2,\mathbb{C})$ is not finite.

In addition we determine that the multiplicity d_ψ of [A2], p. 28, is one.

4. Conjectural Rigidity

This section explains the rigidity theorem for SO(4) via the principle of functoriality. It is based on conversations with J.-P. Serre at Singapore.

4.1 PROPOSITION. *Let η_1, η_2, η'_1, η'_2: $W_F \to \mathrm{GL}(2,\mathbb{C})$ be (irreducible continuous) representations of the Weil group W_F of F which are unramified at almost all places v (so they depend there only on the Frobenius element) with $\eta_1 \otimes \eta_2 | W_{F_v} \simeq \eta'_1 \otimes \eta'_2 | W_{F_v}$ for almost all v and with $\det \eta_1 \cdot \det \eta_2 = \det \eta'_1 \cdot \det \eta'_2$. Then there exists a homomorphism $\chi : W_F \to \mathbb{C}^\times$ such that $\eta'_1 = \chi \eta_1$ and $\eta'_2 = \chi^{-1} \eta_2$, or $\eta'_2 = \chi \eta_2$ and $\eta'_2 = \chi^{-1} \eta_1$.*

Since the subgroup of W_F generated by the Frobenii is dense, we may consider instead a group Γ (instead of W_F), and two representations ρ_i

(instead of $\eta_1 \otimes \eta_2$) which are *locally conjugate*, which means that $\rho_1(\gamma)$ is conjugate to $\rho_2(\gamma)$ for each γ in Γ, or alternatively that the restrictions of ρ_1, ρ_2 to any cyclic subgroup are conjugate. We wish to know whether they are conjugate as representations.

We say that a group G over $\mathbb{C}$ has the *rigidity-property* if for any group Γ, any two locally conjugate representations $\rho_1, \rho_2 : \Gamma \to G(\mathbb{C})$ are conjugate. Variants are naturally defined (for special Γ and ρ). For example, if Γ is finite and $G = \mathrm{GL}(n)$, character theory asserts that locally conjugate $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ are conjugate. The group $G = \mathrm{GL}(n)$ has the rigidity-property for any semisimple continuous representations ρ_1, ρ_2 of the Weil group. On the other hand, the group $\mathrm{PGL}(n, \mathbb{C})$ does not have the rigidity-property since it is the dual group of $\mathrm{SL}(n)$, for which rigidity does not hold.

In our case we wish to know whether locally conjugate ρ_1, ρ_2 into $\mathrm{SO}(4, \mathbb{C})$ are conjugate. They are not, but almost are: they are conjugate in $\mathrm{O}(4, \mathbb{C})$, which is the semidirect product of $\mathrm{SO}(4, \mathbb{C})$ with an element which maps $\eta_1 \otimes \eta_2$ to $\eta_2 \otimes \eta_1$. We proceed to explain this via the group theoretical notion of fusion control.

4.2 DEFINITION. Given groups $G \supset H' \supset H$ we say that H' *controls the fusion* of H in G if for any sets A, B in H and g in G with $gAg^{-1} = B$ there is h in H' with $hah^{-1} = gag^{-1}$ for every a in A, namely $h^{-1}g$ lies in the centralizer $C_G(A)$ of A in G.

4.3 EXAMPLE. Let S be an abelian p-Sylow subgroup in a finite group G, and $N = N_G(S)$ the normalizer of S in G. Then $S \subset N \subset G$ and N controls the fusion of S in G.

PROOF. Since S is abelian and A is a subset of S we have that S is contained in the centralizer $C_G(A)$ of A in G. Hence S is a p-Sylow subgroup of $C_G(A)$. Now the abelian S commutes with any subset B of S, hence $g^{-1}Sg$ commutes with $g^{-1}Bg = A$, and so $g^{-1}Sg$ is a p-Sylow subgroup of $C_G(A)$ for any g in G. Since p-Sylow subgroups are conjugate, there is u in $C_G(A)$ with $g^{-1}Sg = uSu^{-1}$; take $h = gu \in N_G(S)$. Then $hah^{-1} = guau^{-1}g^{-1} = gag^{-1}$ for any a in A. □

4.4 EXAMPLE. Let G be an algebraic reductive group, T a maximal torus and $N = N_G(T)$ the normalizer of T in G. Then $T \subset N \subset G$ and N controls the fusion of T in G.

PROOF. If A is any subset of the abelian T, we have that T lies in the centralizer $C_G(A)$ of A in G. Hence T is a maximal torus in $C_G(A)$. Now T commutes with any of its subsets B, hence $g^{-1}Tg$ commutes with $g^{-1}Bg = A$, and so $g^{-1}Tg$ is a maximal torus in $C_G(A)$. Since maximal tori of a reductive group are conjugate, there exists u in $C_G(A)$ such that $g^{-1}Tg = uTu^{-1}$. Hence $h = gu$ lies in $N_G(T)$ and satisfies $hah^{-1} = guau^{-1} = gag^{-1}$ for any a in A. □

4.5 PROPOSITION. *Let* $S = {}^tS$ *be a symmetric matrix in* $\mathrm{GL}(n,\mathbb{C})$. *Put* $g^* = S{}^tg^{-1}S^{-1}$. *Then the orthogonal group* $\mathrm{O}(S,\mathbb{C}) = \{g \in \mathrm{GL}(n,\mathbb{C}); g = g^*\}$ *controls its own fusion in* $\mathrm{GL}(n,\mathbb{C})$.

PROOF. Suppose that A, B are subsets of $\mathrm{O}(S,\mathbb{C})$ and $g \in \mathrm{GL}(n,\mathbb{C})$ satisfies $gAg^{-1} = B$. For each a in A we have $a^* = a$, hence $g^*ag^{*-1} = (gag^{-1})^* = gag^{-1}$ (as $b = b^*$ for $b = gag^{-1}$). Then $c = g^{-1}g^*$ commutes with each a in A, and $c^{*-1} = S{}^tcS^{-1} = S{}^tg^{*t}g^{-1}S^{-1} = g^{-1}S{}^tg^{-1}S^{-1} = g^{-1}g^* = c$. Let d be a square root of c, thus $c = d^2$. Using the binomial expansion $u^{1/2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(u-1)^n$ for a unipotent matrix u and $(re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$ $(0 \leqslant \theta < 2\pi,\ r > 0)$, the Jordan decomposition $c = su = us$ and diagonalization, we express d as a function $f(c)$ in c, where f satisfies $f(xyx^{-1}) = xf(y)x^{-1}$ and $f({}^tx) = {}^tf(x)$. Then

$$d^{*-1} = S\,{}^tdS^{-1} = Sf({}^tc)S^{-1} = f(S{}^tcS^{-1}) = f(c^{*-1}) = f(c) = d$$

and $h = (gd)^* = g^*d^* = gcd^{-1} = gd$ satisfies $(gd)a(gd)^{-1} = gag^{-1}$, for all a in A. □

4.6 COROLLARY. *If* ρ_1, $\rho_2 : \Gamma \to \mathrm{O}(S,\mathbb{C})$ *are representations of a group* Γ *into the orthogonal group* $\mathrm{O}(S,\mathbb{C})$, *and there is* g *in* $\mathrm{GL}(n,\mathbb{C})$ *with* $\rho_2 = g\rho_1g^{-1}$, *then there is* h *in* $\mathrm{O}(S,\mathbb{C})$ *with* $\rho_2 = h\rho_1h^{-1}$. □

REMARK. The last Proposition and its Corollary hold (with the same proof) for the symplectic group $\mathrm{Sp}(S,\mathbb{C})$, defined using $S = -{}^tS$.

4.7 PROPOSITION. *Let* η_1, η_2, η_1', η_2': $\Gamma \to \mathrm{GL}(2,\mathbb{C})$ *be representations of a group* Γ *with* $\eta_1 \otimes \eta_2 \simeq \eta_1' \otimes \eta_2'$ *in* $\mathrm{GL}(4,\mathbb{C})$ *and* $\det\eta_1 \cdot \det\eta_2 = \det\eta_1' \cdot \det\eta_2'$. *Then there exists a homomorphism* $\chi : \Gamma \to \mathbb{C}^\times$ *such that* $\eta_1' = \chi\eta_1$ *and* $\eta_2' = \chi^{-1}\eta_2$ *or* $\eta_1' = \chi\eta_2$ *and* $\eta_2' = \chi^{-1}\eta_1$.

PROOF. The tensor products $\rho = \eta_1 \otimes \eta_2$ and $\rho' = \eta_1' \otimes \eta_2'$ have images in $\mathrm{SO}(S,\mathbb{C}) \subset \mathrm{O}(S,\mathbb{C})$ where $S = \hat{s}J = \mathrm{antidiag}(-1,1,1,-1)$,

$$\hat{s} = \mathrm{diag}(-1,1,-1,1), \quad J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence ρ and ρ' are equivalent in $\mathrm{O}(S,\mathbb{C}) = \mathrm{SO}(S,\mathbb{C}) \times \langle \iota \rangle$, where $\iota = \mathrm{diag}(1,w,1)$ acts on $a \otimes b$ in $\mathrm{SO}(S,\mathbb{C})$ (a, b in $\mathrm{GL}(2,\mathbb{C})$, $\det ab = 1$) by $\iota : a \otimes b \mapsto b \otimes a$. So ρ is equivalent under $\mathrm{SO}(S,\mathbb{C})$ to ρ' or to ${}^{\iota}\rho' = \eta_2' \otimes \eta_1'$, and (η_1, η_2) is equivalent to $(\chi\eta_1', \chi^{-1}\eta_2')$ or to $(\chi\eta_2', \chi^{-1}\eta_1')$. The map $\chi : \Gamma \to \mathbb{C}^\times$ is a homomorphism since so are the η_i, η_i', $i = 1, 2$. □

We also note the following analogue for the group of similitudes.

4.8 PROPOSITION. *If the representations ρ, $\rho' : \Gamma \to \mathrm{GO}(S,\mathbb{C})$ (of a group Γ into the group of orthogonal similitudes) are conjugate in* $\mathrm{GL}(n,\mathbb{C})$ ($\ni S = {}^tS$) *and have the same factor $\boldsymbol{\lambda}$ of similitudes, then they are conjugate in* $\mathrm{O}(S,\mathbb{C})$.

PROOF. Replacing Γ by the 2-fold cover $\tilde{\Gamma} = \Gamma_{\boldsymbol{\lambda}} \times_{\mathbb{C}^\times, \Box} \mathbb{C}^\times$ (fiber product of $\boldsymbol{\lambda} : \Gamma \to \mathbb{C}^\times$ with $\mathbb{C}^\times \to \mathbb{C}^\times, \Box : z \mapsto z^2$), there is a character $\mu : \tilde{\Gamma} \to \mathbb{C}^\times$ with $\boldsymbol{\lambda} = \mu^2$:

$$\begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{\mu} & \mathbb{C}^\times \\ \downarrow & & \downarrow \Box \\ \Gamma & \xrightarrow{\boldsymbol{\lambda}} & \mathbb{C}^\times \end{array}$$

Then $\mu^{-1}\rho$, $\mu^{-1}\rho' : \tilde{\Gamma} \to \mathrm{O}(S,\mathbb{C})$ are conjugate in $\mathrm{GL}(n,\mathbb{C})$ hence also in $\mathrm{O}(S,\mathbb{C})$, and so ρ, $\rho' : \tilde{\Gamma} \to \mathrm{O}(S,\mathbb{C})$ are conjugate in $\mathrm{O}(S,\mathbb{C})$ and they factorize via $\mathrm{pr}:\tilde{\Gamma} \to \Gamma$. □

We can now return to our initial Proposition 4.1. If the irreducible continuous representations η_1, η_2, η_1', $\eta_2' : W_F \to \mathrm{GL}(2,\mathbb{C})$ are unramified and satisfy $\eta_1 \otimes \eta_2(\mathrm{Fr}_v) \simeq \eta_1' \otimes \eta_2'(\mathrm{Fr}_v)$ for almost all places v, then $\rho = \eta_1 \otimes \eta_2$ and

$$\rho' = \eta_1' \otimes \eta_2' : W_F \to \mathrm{SO}(S,\mathbb{C}) \subset \mathrm{O}(S,\mathbb{C}) \subset \mathrm{GL}(4,\mathbb{C})$$

are conjugate in $\mathrm{GL}(4,\mathbb{C})$ (since the Frobenii are dense in W_F and ρ, ρ' are semisimple). Hence they are conjugate in $\mathrm{O}(S,\mathbb{C})$ and there is a homomorphism $\chi : W_F \to \mathbb{C}^\times$ with $\eta_1' = \chi\eta_1, \eta_2' = \chi^{-1}\eta_2$, or $\eta_1' = \chi\eta_2$, $\eta_2' = \chi^{-1}\eta_1$.

Had we known the Principle of Functoriality, namely that discrete spectrum representations π_i of $\mathrm{GL}(2,\mathbb{A})$ are parametrized by two dimensional representations $\eta_i : \Gamma \to \mathrm{GL}(2,\mathbb{C})$ of a suitable Weil group $\Gamma(=W_F)$, we could conclude the rigidity theorem part of our global theorem about the lifting λ_1 from $\mathbf{C} = \mathrm{SO}(4)$ to $\mathrm{PGL}(4)$. However, this Principle is known only for monomial representations $\eta_i = \mathrm{Ind}(\mu_i; W_{E_i/E_i}, W_{E_i/F})$, induced from characters μ_i of $W_{E_i/E_i} = \mathbb{A}_{E_i}^\times/E_i^\times$, where E_i is a quadratic extension of F. Thus we get an alternative proof – based only on class field theory and the basic group theoretic consideration above – of the special case for monomial representations $\pi_i = \pi(\mu_i)$ stated after that theorem.

Note that the rigidity property, that any locally conjugate ρ, $\rho' : \Gamma \to G(\mathbb{C})$ are conjugate, holds for $G = \mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(n)$ and G_2, and for any connected, simply connected, complex Lie group precisely if it has no direct factors of type $B_n (n \geq 4)$, $D_n (n \geq 4)$, E_n or F_4. For this and related results see Larsen ([Lar]).