Chapter 1

Flows

1.1. What are flows?

Fluid flows are commonly observed phenomena in this world. Giving typical examples, the wind is a flow of the air and the river stream is a flow of water. On the other hand, the motion of clouds or smoke particles floating in the air can be regarded as visualizing the flow that carries them. When we say **flow** of a matter, it implies usually time development of the displacement and deformation of matter. Namely, a number of particles compose the body of matter, and are moving and continuously changing their relative positions, and are evolving with time *always*. Flows are observed in diverse phenomena in addition to the wind and river above: air flows in a living room, flows of *blood* or *respiratory air* in a body, flows of microscopic suspension particles in a chemical test-tube, flows of *bathtub water*, *atmospheric flows* and *sea currents*, *solar wind*, *gas flows in interstellar space*, and so on.

From the technological aspect, vehicles such as ships, aeroplanes, jetliners and rockets utilize flows in order to obtain thrust to move from one place to other while carrying loads. Glider planes or soaring birds use winds passively to get lift.

On the other hand, from the biological aspect, swimming fishes are considered to be using water motions (*eddies*) to get thrust for their motion. Animals such as insects or birds commonly use air flows in order to get lift for being airborne as well as getting thrust for their forward motion. In addition, it is understood that plant seeds, or pollen, often use wind for their purposes. Every living organism has a certain internal system of physiological circulation. In general, it might not be an exaggeration to say that all the living organisms make use of flows in various ways in order to live in this world.

In general, a *material* which constantly deforms itself such as the air or water is called a **fluid**. An elastic solid is deformable as well, however its deformation stops in balance with a force acting on it. Once the body is released from the force, it recovers its original state. A plastic solid is deformed continuously during the application of a force. Once the body is free from the force, it stops deformation (nominally at least). By contrast, the fluid *keeps deforming* even when it is free from force.

A body of *fluid* is composed of innumerably many microscopic molecules. However in a macroscopic world, it is regarded as a body in which mass is continuously distributed. Motion of a fluid, i.e. *flow* of a fluid, is considered to be a mass flow involving its continuous deformation. *Fluid mechanics* studies such flows of fluids, i.e. motions of material bodies of continuous mass distribution, *under fundamental laws of mechanics.*

1.2. Fluid particle and fields

When we consider a fluid flow, it is often useful to use a discrete concept although the fluid itself is assumed to be a continuous body. A *fluid particle* is defined as a mass in a small nearly-spherical volume ΔV, whose diameter is sufficiently small from a macroscopic point of view. However, it is large enough if it is compared with the intermolecular distance, such that the total number N_Δ of molecules in the volume ΔV is sufficiently big so that the statistical description makes sense. In other words, it is a basic assumption that there exists such a volume ΔV enabling to define the concept of a fluid particle. In fact, the study of fluid mechanics is normally carried out at scales of about 10^{-3} mm or larger, whereas the intermolecular scale is 10^{-6} mm or less. At the normal temperature and pressure

(0°C and 760 mmHg), a cube of 1 mm in a gas contains about 2.7×10^{16} molecules.

We consider a monoatomic gas whose molecular mass is m, hence the mass in a small volume ΔV is $\Delta M = mN_\Delta$, and we consider the flow in the (x, y, z) cartesian coordinate frame.

Position $\mathbf{x} = (x, y, z)$ of a fluid particle is defined by the center of mass of the constituent molecules. Density of the fluid ρ is defined by dividing ΔM by ΔV, so that[1]

$$\rho(\mathbf{x}) := \frac{\Delta M}{\Delta V}. \tag{1.1}$$

A α-th molecule constituting the mass moves with its own velocity $\mathbf{u}_\alpha$, where $\mathbf{u}_\alpha$ has three components, say $\left(u^x_\alpha, u^y_\alpha, u^z_\alpha\right)$. The fluid velocity $\mathbf{v}$ at $\mathbf{x}$ is defined by the average value of the molecular velocities, in such a way

$$\mathbf{v}(\mathbf{x}) = \langle \mathbf{u}_\alpha \rangle := \frac{\sum_\alpha m_\alpha \mathbf{u}_\alpha}{\sum_\alpha m_\alpha}, \tag{1.2}$$

where $m_\alpha = m$ (by the assumption), $\sum_\alpha m_\alpha = mN_\Delta = \rho\Delta V$, and $\langle \cdot \rangle$ denotes an average with respect to the molecules concerned. The difference $\tilde{\mathbf{u}}_\alpha = \mathbf{u}_\alpha - \mathbf{v}$ is called the *peculiar velocity* or *thermal velocity*.

In the kinetic theory of molecules, the temperature T is defined by the law that the average of *peculiar* kinetic energy per degree-of-freedom is equal to $\frac{1}{2}kT$, where k is the Boltzmann constant.[2] Each molecule has three degrees of freedom for translational motion. It is assumed that

$$\left\langle \frac{1}{2}m(\tilde{u}^x_\alpha)^2 \right\rangle = \left\langle \frac{1}{2}m(\tilde{u}^y_\alpha)^2 \right\rangle = \left\langle \frac{1}{2}m(\tilde{u}^z_\alpha)^2 \right\rangle = \frac{1}{2}kT.$$

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[1] $A := B$ denotes that A is defined by B.

[2] Boltzmann constant k is a conversion factor between *degree* (Kelvin temperature) and *erg* (energy unit), defined by $k = 1.38 \times 10^{-16}$ erg/deg.

Therefore, we have

$$\frac{3}{2}\,kT(\mathbf{x}) := \left\langle \frac{1}{2} m \tilde{\mathbf{u}}_\alpha^2 \right\rangle = \frac{1}{N_\Delta} \sum_\alpha \frac{1}{2} m \tilde{\mathbf{u}}_\alpha^2. \tag{1.3}$$

On the other hand, *pressure* is a variable defined against a surface element. The pressure p exerted on a surface element ΔS is defined by the momentum flux (i.e. a force) through ΔS. Choosing the x-axis normal to the surface ΔS, the x-component of the pressure force F_x on ΔS acting from the left (smaller x) side to the right (larger x) side would be given by the flux of x-component momentum $m_\beta \tilde{u}_\beta^x$ through ΔS:

$$F_x = p(\mathbf{x})\,\Delta S = \sum_\beta (m_\beta \tilde{u}_\beta^x)\,\tilde{u}_\beta^x\,\Delta S = \Delta S \sum_{\tilde{u}^x} m(\tilde{u}^x)^2 n(\tilde{u}^x), \tag{1.4}$$

where β denotes all the molecules passing through ΔS per unit time, which are contained in the volume element $\tilde{u}_\beta^x \Delta S$, and $n(\tilde{u}^x)$ denotes the number of molecules with $\tilde{u}^x$ in a unit volume. In the kinetic theory, the factor $\sum_{\tilde{u}^x} m(\tilde{u}^x)^2 n(\tilde{u}^x)$ on the right-hand side is expressed by the following two integrals for $\tilde{u}^x > 0$ and $\tilde{u}^x < 0$ respectively:

$$\int_{\tilde{u}^x > 0} m(\tilde{u}^x)^2\, Nf(\tilde{\mathbf{u}})\,\mathrm{d}^3\tilde{\mathbf{u}} + \int_{\tilde{u}^x < 0} m(\tilde{u}^x)^2 Nf(\tilde{\mathbf{u}})\,\mathrm{d}^3\tilde{\mathbf{u}}, \tag{1.5}$$

where the function $f(\tilde{\mathbf{u}})$ denotes the distribution function of the peculiar velocity $\tilde{\mathbf{u}}$, and the number of molecules between $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}} + \mathrm{d}\tilde{\mathbf{u}}$ is given by[3]

$$Nf(\tilde{\mathbf{u}})\,\mathrm{d}^3\tilde{\mathbf{u}}, \quad \text{with} \quad \int_{\text{all }\tilde{\mathbf{u}}} f(\tilde{\mathbf{u}})\,\mathrm{d}^3\tilde{\mathbf{u}} = 1,$$

where N is the total number of molecules in a unit volume. This is interpreted as follows. The first term of (1.5) denotes that a positive momentum $m\tilde{u}^x$ ($\tilde{u}^x > 0$) is absorbed into the right side of ΔS, while the second term denotes that a negative momentum $m\tilde{u}^x$ ($\tilde{u}^x < 0$) is

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[3] More precisely, the number of molecules of the peculiar velocity $(\tilde{u}^x, \tilde{u}^y, \tilde{u}^z)$, which takes values between $\tilde{u}^x$ and $\tilde{u}^x + \mathrm{d}\tilde{u}^x$, $\tilde{u}^y$ and $\tilde{u}^y + \mathrm{d}\tilde{u}^y$ and $\tilde{u}^z$ and $\tilde{u}^z + \mathrm{d}\tilde{u}^z$ respectively, is defined by $Nf(\tilde{u}^x, \tilde{u}^y, \tilde{u}^z)\,\mathrm{d}\tilde{u}^x\,\mathrm{d}\tilde{u}^y\,\mathrm{d}\tilde{u}^z$.

taken out from the right side of ΔS. Both means that the space on the right side has received the same amount of positive momentum. Both terms are combined into one:

$$\sum_{\beta} m_{\beta}(\tilde{u}_{\beta}^{x})^2 = \int_{\text{all } \tilde{\mathbf{u}}} m(\tilde{u}^x)^2 Nf(\tilde{\mathbf{u}})\,\mathrm{d}^3\tilde{\mathbf{u}}$$

$$= Nm\langle(\tilde{u}^x)^2\rangle = NkT, \tag{1.6}$$

since $\frac{1}{2}m\langle(\tilde{u}^x)^2\rangle = \frac{1}{2}kT$, where the average $\langle(\tilde{u}^x)^2\rangle = \langle(\tilde{u}^y)^2\rangle = \langle(\tilde{u}^z)^2\rangle$ is equal to $\frac{1}{3}\langle(\tilde{\mathbf{u}})^2\rangle$ by an isotropy assumption, and $m\,\frac{1}{3}\langle(\tilde{\mathbf{u}})^2\rangle$ is given by kT from (1.3). Thus, from (1.4) and (1.6), we obtain

$$p(\mathbf{x}) = NkT. \tag{1.7}$$

This is known as the **equation of state** of an ideal gas.[4]

The density $\rho(\mathbf{x})$, velocity $\mathbf{v}(\mathbf{x})$, temperature $T(\mathbf{x})$ and pressure $p(\mathbf{x})$ thus defined depend on the position $\mathbf{x} = (x, y, z)$ and the time t smoothly, since the molecular kinetic motion usually works to smooth out discontinuity (if any) by the transport phenomena considered in Chapter 2. Namely, these variables are regarded as continuous and in addition *differentiable* functions of (x, y, z, t). Such variables are called **fields**. This point of view is often called the *continuum hypothesis*.

From a mathematical aspect, *flow* of a fluid is regarded as a continuous sequence of *mappings*. Consider all the fluid particles composing a subdomain B_0 at an initial instant $t = 0$. After an infinitesimal time δt, a particle at $\mathbf{x} \in B$ moves from $\mathbf{x}$ to $\mathbf{x} + \delta\mathbf{x}$:

$$\mathbf{x} \mapsto \mathbf{x} + \delta\mathbf{x} = \mathbf{x} + \mathbf{v}\delta t + O(\delta t^2) \tag{1.8}$$

by the flow field $\mathbf{v}(\mathbf{x}, t)$. Then the domain B_0 may be mapped to $B_{\delta t}$ (say). Subsequent mapping occurs for another δt from $B_{\delta t}$ to $B_{2\delta t}$, and so on. In this way, the initial domain B_0 is mapped one after another smoothly and constantly. At a later time t, the domain

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[4] For a gram-molecule of an ideal gas, N is replaced by $N_A = 6.023 \times 10^{23}$ (Avogadro's constant). The product $N_A K = R$ is called the *gas constant*: $R = 8.314 \times 10^7$ erg/deg. For an ideal gas of molecular weight μ_m, the equation (1.7) reduces to $p = (1/\mu_m)\rho RT$, where $\rho = mN$, $\mu_m = mN_A$ and $R = N_A k$.

B_0 would be mapped to B_t. The map might be differentiable with respect to $\mathbf{x}$, and in addition, for such a map, there is an inverse map. This kind of map is termed a *diffeomorphism* (i.e. differentiable homeomorphism).

1.3. Stream-line, particle-path and streak-line

1.3.1. *Stream-line*

Suppose that a velocity field $\mathbf{v}(\mathbf{x}, t) = (u, v, w)$ is given in a subdomain of three-dimensional Euclidean space $\mathbb{R}^3$, and that, at a given time t, the vector field $\mathbf{v} = (u, v, w)$ is continuous and smooth at every point (x, y, z) in the domain. It is known in the theory of ordinary differential equations in mathematics that one can draw curves so that the curves are tangent to the vectors at all points. Provided that the curve is represented as $(x(s), y(s), z(s))$ in terms of a parameter s, the tangent to the curve is written as $(\mathrm{d}x/\mathrm{d}s,\ \mathrm{d}y/\mathrm{d}s,\ \mathrm{d}z/\mathrm{d}s)$, which should be parallel to the given vector field $(u(x, y, z), v(x, y, z), w(x, y, z))$ by the above definition. This is written in the following way:

$$\frac{\mathrm{d}x}{u(x,y,z)} = \frac{\mathrm{d}y}{v(x,y,z)} = \frac{\mathrm{d}z}{w(x,y,z)} = \mathrm{d}s. \tag{1.9}$$

This system of ordinary differential equations can be integrated for a given initial condition at $s = 0$, at least locally in the neighborhood of $s = 0$. Namely, a curve through the point $P = (x(0), y(0), z(0))$ is determined uniquely.[5] The curve thus obtained is called a *stream-line*. For a set of initial conditions, a family of curves is obtained. Thus, a family of stream-lines are defined at each instant t (Fig. 1.1).

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[5] Mathematically, existence of solutions to Eq. (1.9) is assured by the continuity (and boundedness) of the three component functions of $\mathbf{v}(\mathbf{x}, t)$. For the uniqueness of the solution to the initial condition, one of the simplest conditions is the *Lipschitz condition*: $|\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{y}, t)| \leqq K|\mathbf{x} - \mathbf{y}|$ for a positive constant K.

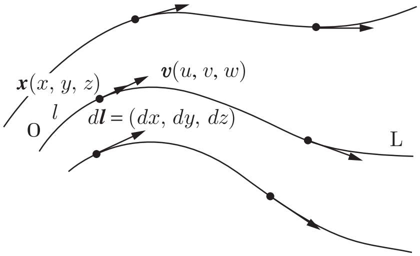

Fig. 1.1. Stream-lines.

1.3.2. *Particle-path* (*path-line*)

Next, let us take a particle-wise point of view. Choosing a fluid particle A, whose position was at $\mathbf{a} = (a, b, c)$ at the time $t = 0$, we follow its subsequent motion governed by the velocity field $\mathbf{v}(\mathbf{x}, t) = (u, v, w)$. Writing its position as $\mathbf{X}_a(t) = (X(t), Y(t), Z(t))$, equations of motion of the particle are

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u(X, Y, Z, t), \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = v(X, Y, Z, t), \\ \frac{\mathrm{d}Z}{\mathrm{d}t} = w(X, Y, Z, t). \tag{1.10}$$

This can be solved at least locally in time, and the solution would be represented as $\mathbf{X}_a(t) = \mathbf{X}(\mathbf{a}, t) = (X(t), Y(t), Z(t))$, where

$$X(t) = X(a, b, c, t), \quad Y(t) = Y(a, b, c, t), \\ Z(t) = Z(a, b, c, t), \tag{1.11}$$

and $\mathbf{X}_a(0) = \mathbf{a}$. For a fixed particle specified with $\mathbf{a} = (a, b, c)$, the function $\mathbf{X}_a(t)$ represents a curve parametrized with t, called the *particle path*, or a path-line. Correspondingly, the particle velocity is given by

$$\mathbf{V}_a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_a(t) = \frac{\partial}{\partial t}\mathbf{X}(\mathbf{a}, t) = \mathbf{v}(\mathbf{X}_a, t). \tag{1.12}$$

This particle-wise description is often called the *Lagrangian description*, whereas the field description such as $\mathbf{v}(\mathbf{x}, t)$ for a point $\mathbf{x}$ and a time t is called the *Eulerian description*.

It is seen that the two equations (1.9) and (1.10) are identical except the fact that the right-hand sides of (1.10) include the time t. Hence if the velocity field is *steady*, i.e. $\mathbf{v}$ does not depend on t, then both equations are equivalent, implying that both stream-lines and particle-paths are identical in steady flows.

1.3.3. *Streak-line*

In most visualizations of flows or experiments, a common practice is to introduce dye or smoke at fixed positions in a fluid flow and observe colored patterns formed in the flow field (Fig. 1.2). Smoke from a chimney is another example of analogous pattern. An instantaneous curve composed of all fluid elements that have passed the same particular fixed point P at previous times is called the *streak-line.*

Denoting the fixed point P by $\mathbf{A}$, the fluid particle that has passed the point $\mathbf{A}$ at a previous time τ will be located at $\mathbf{X} = \mathbf{X}(\boldsymbol{a}_\tau, t)$ at a later time t where $\boldsymbol{a}_\tau$ is defined by $\mathbf{A} = \mathbf{X}(\boldsymbol{a}_\tau, \tau)$. Thus the streak-line at a time t is represented parametrically by the function $\mathbf{X}(\mathbf{A}, t - \tau)$ with the parameter τ.

If the flow field is steady (Fig. 1.3), it is obvious that the streak-line coincides with the particle-path, and therefore with the stream-line. However, if the flow field is time-dependent (Fig. 1.4), then all the three lines are different, and they appear quite differently.

1.3.4. *Lagrange derivative*

Suppose that the temperature field is expressed by $T(\mathbf{x}, t)$ and that the velocity field is given by $\mathbf{v}(\mathbf{x}, t)$, in the way of Eulerian description. Consider a fluid particle denoted by the parameter $\mathbf{a}$ in the flow field and examine how its temperature $T_a(t)$ changes during the motion. Let the particle position be given by $\mathbf{X}_a(t) = (X, Y, Z)$ and its velocity by $\mathbf{v}_a(t) = (u_a, v_a, w_a)$. Then the particle temperature is expressed by

$$T_a(t) = T(\mathbf{X}_a, t) = T(X(t), Y(t), Z(t);\, t).$$

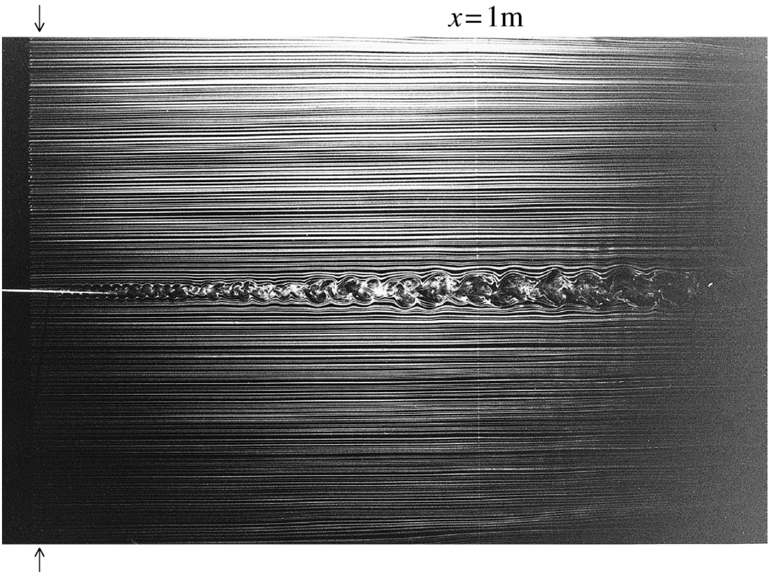

Fig. 1.2. Visualization of the wake behind a thin circular cyinder (of diameter 5 mm) by a *smoke-wire* method. The wake is the central horizontal layer of irregular smoke pattern, and the many parallel horizontal lines in the upper and lower layers show a uniform stream of wind velocity 1 m/s from left to right. The smoke lines originate from equally-spaced discrete points on a vertical straight wire on the left placed at just upstream position of the cylinder (at the point of intersection of the central horizontal white line (from the left) and the vertical line connecting the two arrows out of the frame). Thus, all the smoke lines are *streak-lines.* The illumination is from upward right, and hence the shadow line of the cylinder is visible to downward left on the lower left side. The regular periodic pattern observed in the initial development of the wake is the Kármán vortex street. The vertical white line at the central right shows the distance 1 m from the cylinder. This is placed in order to show how the wake reorganizes to another periodic structure of larger eddies. [As for the wake, see Problem 4.6 (Fig. 4.12), and Fig. 4.7.] The photograph is provided through the courtesy of Prof. S. Taneda (Kyushu University, Japan, 1988). $R_e = 350$ (see Table 4.2).

Hence, the time derivative of the particle temperature is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}T_a(t) &= \frac{\partial T}{\partial t} + \frac{\mathrm{d}X}{\mathrm{d}t}\frac{\partial T}{\partial x} + \frac{\mathrm{d}Y}{\mathrm{d}t}\frac{\partial T}{\partial y} + \frac{\mathrm{d}Z}{\mathrm{d}t}\frac{\partial T}{\partial z} \\ &= \frac{\partial T}{\partial t} + u_a\,\frac{\partial T}{\partial x} + v_a\,\frac{\partial T}{\partial y} + w_a\,\frac{\partial T}{\partial z} \\ &= \left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z}\right) T\,\Big|_{\mathbf{x}=\mathbf{X}_a}.\end{aligned}$$

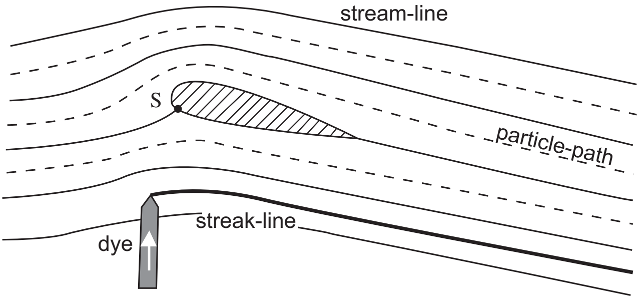

Fig. 1.3. Steady flow: stream-lines (thin solid lines), particle-path (broken lines), and streak-lines (a thick solid line).

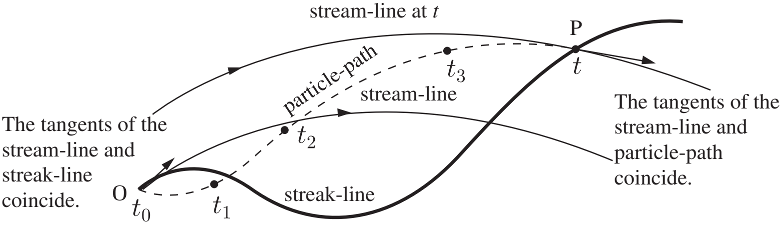

Fig. 1.4. Unsteady flow: stream-lines (thin solid lines), particle-path (a broken line), and streak-line (a thick solid line). The particle P started from the fixed point O at a time t_0 and is now located at P at t after the times t_1, t_2 and t_3.

It is convenient to define the differentiation on the right-hand side by using the operator,

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z} = \partial_t + u\partial_x + v\partial_y + w\partial_z,$$

which is called the *convective derivative*, where $\partial_t := \partial/\partial t$, $\partial_x := \partial/\partial x$, and so on. As is evident from the above derivation, this time derivative denotes the differentiation following the particle motion. This derivative is called variously as the *material derivative*, *convective derivative* or *Lagrange derivative*. Thus, we have $\mathrm{d}T_a/\mathrm{d}t = \mathrm{D}T/\mathrm{D}t\,\big|_{\mathbf{X}_a}$.

Let us introduce the following differential operators,

$$\nabla = (\,\partial_x,\,\partial_y,\,\partial_z\,),$$
$$\mathrm{grad}\, f = (\,\partial_x,\,\partial_y,\,\partial_z\,)\, f,$$

where $f(x, y, z)$ is a differentiable scalar function. The differential operator ∇ with three components is termed the *nabla* operator, and the vector grad$f$ is called the *gradient* of the function $f(\mathbf{x})$. Using ∇ and $\mathbf{v} = (u, v, w)$, one can write

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + u\partial_x + v\partial_y + w\partial_z = \partial_t + \mathbf{v}\cdot\nabla, \tag{1.13}$$

where the dot denotes the inner product [Appendix A.2 and see (A.7)].

Suppose that we have a physical (scalar) field $Q(\mathbf{x}, t)$ such as density ρ, temperature T, etc. Its time derivative following the motion of a fluid particle is given by

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \partial_t Q + (\mathbf{v}\cdot\nabla)Q. \tag{1.14}$$

If the value is invariant during the particle motion, we have

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0. \tag{1.15}$$

1.4. Relative motion

Given a velocity field $\mathbf{v}(\mathbf{x}, t)$, each fluid element moves subject to straining deformation and local rotation. This is shown as follows.

1.4.1. *Decomposition*

In order to represent such a local motion mathematically, we consider a relative motion of fluid in a neighborhood of an arbitrarily chosen point $P = \mathbf{x} = (x_1, x_2, x_3)$, where the velocity is $\mathbf{v} = (v_1, v_2, v_3)$. Writing the velocity of a neighboring point $Q = \mathbf{x} + \mathbf{s}$ as $\mathbf{v} + \delta\mathbf{v}$ at the same instant t, we have

$$\delta\mathbf{v} = \mathbf{v}(\mathbf{x}+\mathbf{s},t) - \mathbf{v}(\mathbf{x},t) = (\mathbf{s}\cdot\nabla)\mathbf{v} + O(s^2),$$

by the Taylor expansion with respect to the separation vector $\mathbf{s} = (s_1, s_2, s_3)$. Writing this in components, we have the following matrix equation,

$$\begin{pmatrix} \delta v_1 \\ \delta v_2 \\ \delta v_3 \end{pmatrix} = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \tag{1.16}$$

where $\partial_k v_i = \partial v_i / \partial x_k$. This can be also written as[6]

$$\delta v_i = \sum_{k=1}^{3} s_k\, \partial_k v_i = s_k\, \partial_k v_i. \tag{1.17}$$

The term $\partial_k v_i$ can be decomposed into a symmetric part e_{ik} and an anti-symmetric part g_{ik} in general (Stokes (1845), [Dar05]), defined by

$$e_{ik} = \frac{1}{2}(\partial_k v_i + \partial_i v_k) = e_{ki}, \tag{1.18}$$

$$g_{ik} = \frac{1}{2}(\partial_k v_i - \partial_i v_k) = -g_{ki}. \tag{1.19}$$

Then one can write as $\partial_k v_i = e_{ik} + g_{ik}$. Using e_{ik} and g_{ik}, the velocity difference δv_i is decomposed as $\delta v_i = \delta v_i^{(s)} + \delta v_i^{(a)}$, where

$$\delta v_i^{(s)} = e_{ik}\, s_k, \tag{1.20}$$

$$\delta v_i^{(a)} = g_{ik}\, s_k. \tag{1.21}$$

These components represent two fundamental modes of relative motion, which we will consider in detail below.

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[6] The summation convention is assumed here, which takes a sum with respect to the repeated indices such as k. Henceforth, summation is meant for such indices without the symbol $\sum_{k=1}^{3}$.

1.4.2. *Symmetric part (pure straining motion)*

The symmetric part is written as

$$\begin{pmatrix} \delta v_1^{(s)} \\ \delta v_2^{(s)} \\ \delta v_3^{(s)} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}. \tag{1.22}$$

Any symmetric (real) matrix can be made diagonal by a coordinate transformation (called the orthogonal transformation, see the footnote 7) to a principal coordinate frame. Using capital letters to denote corresponding variables in the principal frame, the expression (1.22) is transformed to

$$\begin{pmatrix} \delta V_1^{(s)} \\ \delta V_2^{(s)} \\ \delta V_3^{(s)} \end{pmatrix} = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}. \tag{1.23}$$

The diagonal elements E_{11}, E_{22}, E_{33} are the eigenvalues of e_{ik} by the orthogonal transformation $(s_1, s_2, s_3) \to (S_1, S_2, S_3)$.[7] The length is invariant, $|\mathbf{s}| = |\mathbf{S}|$, and in addition, the trace of matrix is invariant:

$$\begin{aligned} E_{11} + E_{22} + E_{33} &= e_{11} + e_{22} + e_{33} \\ &= \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 := \operatorname{div} \mathbf{v}. \end{aligned} \tag{1.24}$$

In the principal frame, we have

$$\delta \mathbf{V}^{(s)} = (E_{11} S_1,\ E_{22} S_2,\ E_{33} S_3), \tag{1.25}$$

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[7] One may write the transformation as $\mathbf{s} = A\mathbf{S}$ and $\delta\mathbf{v}^{(s)} = A\delta\mathbf{V}^{(s)}$, where A is a 3×3 transformation matrix. Then, substituting these into (1.22): $\delta\mathbf{v}^{(s)} = e\mathbf{s}$, one obtains $\delta\mathbf{V}^{(s)} = A^{-1}eA\mathbf{S} = ES$ (where $E = A^{-1}eA$), which corresponds to (1.23). Every orthogonal transformation makes the length invariant by definition, i.e. $|\mathbf{s}|^2 = s_i s_i = A_{ik} S_k A_{il} S_l = A^T_{ki} A_{il} S_k S_l = S_k S_k = |\mathbf{S}|^2$, where A^T is the transpose of A. Namely, the orthogonal transformation is defined by $A^T A = AA^T = I = (\delta_{ik})$ (a unit matrix). Hence, $A^T = A^{-1}$. There exists an orthogonal transformation A which makes $A^{-1}eA$ diagonal. We have $\mathrm{Tr}\{A^{-1}eA\} = \mathrm{Tr}\{A^T eA\} = A_{ji} e_{jk} A_{ki} = \delta_{jk}\, e_{jk} = \mathrm{Tr}\{e\}$.

namely, the velocity component $\delta V_i^{(s)}$ of the symmetric part is proportional to the displacement S_i in the respective axis. This motion $\delta\mathbf{v}^{(s)}$ is termed the *pure straining* motion, and the symmetric tensor e_{ik} is termed the *rate-of-strain tensor*. The trace $\mathrm{div}\,\mathbf{v}$ gives the relative rate of volume change (see Problem 1.2).

1.4.3. *Anti-symmetric part (local rotation)*

The anti-symmetric part is written as

$$(g_{ij}) = \begin{pmatrix} 0 & -g_{21} & g_{13} \\ g_{21} & 0 & -g_{32} \\ -g_{13} & g_{32} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{1.26}$$

where we set $\omega_1 = 2g_{32},\ \omega_2 = 2g_{13},\ \omega_3 = 2g_{21}$. In this way, a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is introduced. Using the original definition, we have

$$\boldsymbol{\omega} = (\partial_2 v_3 - \partial_3 v_2,\ \partial_3 v_1 - \partial_1 v_3,\ \partial_1 v_2 - \partial_2 v_1). \tag{1.27}$$

This is nothing but the curl of the vector $\mathbf{v}$ defined by (A.14) in Appendix A.3, and denoted by

$$\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v} = \nabla \times \mathbf{v}, \quad \omega_i = \varepsilon_{ijk}\partial_j v_k. \tag{1.28}$$

The above relation (1.26) between g and ω suggests the following[8]:

$$g_{ij} = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k. \tag{1.29}$$

Now, from (1.21) and (1.29), the anti-symmetric part is

$$\delta v_i^{(a)} = g_{ij}s_j = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k s_j = \varepsilon_{ikj}\left(\frac{1}{2}\omega_k\right)s_j.$$

In the vector notation, using (A.12), this is written as

$$\delta\mathbf{v}^{(a)} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{s}. \tag{1.30}$$

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[8] For the definition of ε_{ijk}, see Appendix A.1. For example, we have $g_{12} = -\frac{1}{2}(\varepsilon_{121}\omega_1 + \varepsilon_{122}\omega_2 + \varepsilon_{123}\omega_3) = -\frac{1}{2}\omega_3$.

This component of relative velocity describes a rotation of the angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Although $\boldsymbol{\omega}$ depends on $\mathbf{x}$, it is independent of the displacement vector $\mathbf{s}$. Namely, every point $\mathbf{s}$ in the neighborhood of $\mathbf{x}$ rotates with the same angular velocity. Thus, it is found that $\delta\mathbf{v}^{(a)}$ represents *local rigid-body rotation.*

In summary, it is found that *the local relative velocity* $\delta\mathbf{v}$ *consists of a pure straining motion* $\delta\mathbf{v}^{(s)}$ *and a local rigid-body rotation* $\delta\mathbf{v}^{(a)}$.

1.5. Problems

Problem 1.1 Pattern of ink-drift

Suppose that some amount of water is contained in a vessel, and the water is set in motion and its horizontal surface is in smooth motion. Let a liquid-drop of Chinese ink be placed quietly on the flat horizontal surface maintaining a flow with some eddies. The ink covers a certain compact area of the surface.

After a while, some ink pattern will be observed. If a sheet of plain paper (for calligraphy) is placed quietly on the free surface of the water, a pattern will be printed on the paper, which is called the *ink-drift printing* (Fig. 1.5). This pattern is a snap-shot at an instant and consists of a number of curves. What sort of lines are the curves printed on the paper? Are they stream-lines, particle-paths or streak-lines, or other kind of lines?

Problem 1.2 Divergence operator div

Consider a small volume of fluid of a rectangular parallelepiped in a flow field of fluid velocity $\mathbf{v} = (v_x, v_y, v_z)$. The fluid volume V changes under the straining motion. Show that the time-rate of change of volume V per unit volume is given by the following,

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \operatorname{div}\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \qquad (1.31)$$

Fig. 1.5. Ink-drift printing.

Problem 1.3 Acceleration of a fluid particle

Given the velocity field $\mathbf{v}(\mathbf{x}, t)$ with $\mathbf{x} = (x, y, z)$ and $\mathbf{v} = (u, v, w)$. Show that the velocity and acceleration of a fluid particle are given by the following expressions:

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathbf{x} = \mathbf{v}, \quad \text{(Sec. 12.6.2)} \tag{1.32}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathbf{v} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{1.33}$$