

Chapter 1

Bochner Integral

1.1 Simple functions, measurability

Definition 1.1.1. A function $f : I \rightarrow X$ is called *simple* if there is a finite sequence $E_m \subset I$, $m = 1, \dots, p$ of measurable sets such that

$$E_m \cap E_l = \emptyset \quad \text{for } m \neq l$$

and

$$I = \bigcup_{m=1}^p E_m$$

where

$$f(t) = y_m \in X \quad \text{for } t \in E_m, \quad m = 1, \dots, p,$$

i.e. f is constant on the measurable set E_m .

Denote by $\mathcal{J}(\mu, X) = \mathcal{J}$ the set of all simple functions defined on I .

Clearly \mathcal{J} is a linear space and if f is a simple function then also $\|f\| : I \rightarrow \mathbb{R}$ is a simple function.

Definition 1.1.2. A function $f : I \rightarrow X$ is called *measurable* if there exists a sequence $(f_n), f_n \in \mathcal{J}, n \in \mathbb{N}$ with

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

Clearly, if $f \in \mathcal{J}$ then f is measurable.

Proposition 1.1.3. *If $f : I \rightarrow X$ is measurable then the real function $\|f\|_X : I \rightarrow \mathbb{R}$ is measurable.*

Proof. Let $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ be the sequence corresponding to f . Then $\|f_n\|_X$ are simple real functions for all $n \in \mathbb{N}$ and because

$$|\|f_n(t)\|_X - \|f(t)\|_X| \leq \|f_n(t) - f(t)\|_X$$

for $t \in I$ we conclude that $\lim_{n \rightarrow \infty} \|f_n(t)\|_X = \|f(t)\|_X$ a.e. in I and therefore $\|f\|_X$ is measurable. □

Remark. It has to be mentioned that (in the case $X = \mathbb{R}$) a function $f : I \rightarrow \mathbb{R}$ is measurable in the sense of Definition 1.1.2 if and only if for every finite $a \in \mathbb{R}$ the set $\{t \in I; f(t) > a\}$ (or equivalently $\{t \in I; f(t) \geq a\}$, $\{t \in I; f(t) < a\}$, $\{t \in I; f(t) \leq a\}$) is measurable. For details see e.g. [WZ77], Theorem (4.13).

Definition 1.1.4. A function $f : I \rightarrow X$ is called *weakly measurable* if for each $x^* \in X^*$ the real function $x^*(f) : I \rightarrow \mathbb{R}$ is measurable.

The concepts of measurability and weak measurability are closely related. The relation is given by the well known theorem of Pettis presented below. First we give the following lemma.

Lemma 1.1.5. *Assume that X is a separable Banach space. Then there is a sequence $\{x_m^* \in B(X^*); m \in \mathbb{N}\}$ such that for every $x^* \in B(X^*)$ there exists a subsequence $\{x_k^*; k \in \mathbb{N}\}$ of $\{x_m^* \in B(X^*); m \in \mathbb{N}\}$ such that*

$$\lim_{k \rightarrow \infty} x_k^*(x) = x^*(x)$$

for every $x \in X$.

Proof. Assume that $\{x_n \in X; n \in \mathbb{N}\}$ is a dense sequence in the separable space X .

Consider for $n \in \mathbb{N}$ the mapping

$$x^* \in B(X^*) \rightarrow \varphi_n(x^*) = \{x^*(x_1), \dots, x^*(x_n)\} \in \mathbb{R}^n.$$

The space \mathbb{R}^n with the Euclidean norm is separable and therefore for any fixed $n \in \mathbb{N}$ there is a sequence $\{x_{n,k}^* \in B(X^*); k \in \mathbb{N}\}$ such that the set

$$\{\varphi_n(x_{n,k}^*); k \in \mathbb{N}\}$$

is dense in the image $\varphi_n(B(X^*)) \subset \mathbb{R}^n$ of the unit ball $B(X^*)$. This means that for every $x^* \in B(X^*)$ there exists a subsequence (x_{n,n_k}^*) of $\{x_{n,k}^* \in B(X^*); k \in \mathbb{N}\}$ such that

$$|x_{n,n_k}^*(x_i) - x^*(x_i)| < \frac{1}{n}$$

for $i = 1, 2, \dots, n$. Therefore we get

$$\lim_{n \rightarrow \infty} x_{n,n_k}^*(x_i) = x^*(x_i)$$

for every $i \in \mathbb{N}$.

Since the sequence $\{\|x_n^*\|_{X^*}; n \in \mathbb{N}\}$ is bounded and

$$\lim_{n \rightarrow \infty} x_{n,n_k}^*(x_i) = x^*(x_i)$$

for the dense sequence (x_i) in X we obtain

$$\lim_{n \rightarrow \infty} x_{n,n_k}^*(x) = x^*(x)$$

for all $x \in X$ and the lemma is proved. \square

Remark. The final part of the proof is based on the following theorem:

If X is a Banach space then a sequence $x_n^ \in X^*$ weak*-converges to some $x_\infty^* \in X^*$ if and only if the sequence $\{\|x_n^*\|; n \in \mathbb{N}\}$ is bounded and $\lim_{n \rightarrow \infty} x_n^*(x) = x_\infty^*(x)$ on a dense subset in X . (See [Y65], p. 188.)*

Let us also mention that the conclusion of Lemma 1.1.5 is in fact the weak* separability of the ball $B(X^*)$ in X^* . This

means that if the Banach space X is separable then $B(X^*)$ is weak*-separable.

Theorem 1.1.6. (Pettis) *A function $f : I \rightarrow X$ is measurable if and only if f is weakly measurable and almost everywhere separable valued, i.e. there is a set $N \subset I$, $\mu(N) = 0$ such that the set*

$$\{f(t); t \in I \setminus N\} \subset X$$

is separable.

Proof. Let $f : I \rightarrow X$ be measurable. Then there is a sequence $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\|_X = 0 \quad (1.1.1)$$

for almost all $t \in I$.

If $x^* \in X^*$ then (1.1.1) implies

$$\lim_{n \rightarrow \infty} x^*(f_n(t)) = x^*(f(t))$$

a.e. in I .

Since $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ we obtain that $x^*(f_n) : I \rightarrow \mathbb{R}$ is a simple real function for every $x^* \in X^*$. Therefore $x^*(f)$ is measurable for each $x^* \in X^*$ and f is weakly measurable by definition.

We will use the following result (see [DS] Theorem III.6.12 and its corollaries).

Theorem 1.1.7. (Egoroff's theorem) *Let $f_n : I \rightarrow X$, $n \in \mathbb{N}$ be a sequence of measurable functions such that*

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\|_X = 0$$

almost everywhere in I .

Then for every $\eta > 0$ there is a measurable set $H \subseteq I$ such that $\mu(I \setminus H) < \eta$ and

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\|_X = 0$$

uniformly on H .

Thus for each $n \in \mathbb{N}$ there is a measurable set $E_n \subset I$ such that $\mu(E_n) < \frac{1}{n}$ and $\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\|_X = 0$ uniformly on $I \setminus E_n$.

Since $f_n \in \mathcal{J}$, the range $f_n(I) \subset X$ of f_n is finite for every $n \in \mathbb{N}$ and it follows that $\bigcup_{n \in \mathbb{N}} f_n(I)$ is countable. Hence for every $n \in \mathbb{N}$ the set $f(I \setminus E_n)$ is separable and

$$f\left(\bigcup_{n \in \mathbb{N}} (I \setminus E_n)\right) = \bigcup_{n \in \mathbb{N}} f(I \setminus E_n) \quad (1.1.2)$$

is separable as well.

Using the fact that $\mu(E_n) < \frac{1}{n}$, $n \in \mathbb{N}$ we have

$$\bigcap_{n \in \mathbb{N}} E_n = I \setminus \bigcup_{n \in \mathbb{N}} (I \setminus E_n)$$

and

$$\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \mu\left(I \setminus \bigcup_{n \in \mathbb{N}} (I \setminus E_n)\right) = 0.$$

Putting $N = \bigcap_{n \in \mathbb{N}} E_n$ we get by (1.1.2) the separability of the set $\{f(t); t \in I \setminus N\}$.

Let us prove the converse. Without loss of generality assume that the full range $f(I)$ of the function f is separable. Therefore the space X can be also assumed to be separable (X can be taken as the smallest closed linear subspace containing $f(I)$). Assume that $x_n \in X$, $n \in \mathbb{N}$ is dense in X .

First we show that the function $\|f(t)\|_X$ is measurable.

For $a \geq 0$ and $x^* \in X^*$ consider the sets

$$A = \{t \in I; \|f(t)\|_X \leq a\}$$

and

$$A_{X^*} = \{t \in I; |x^*(f(t))| \leq a\}.$$

We have

$$A \subset \bigcap_{x^* \in B(X^*)} A_{X^*}$$

and since by the Hahn-Banach Theorem for every fixed $t \in I$ there exists $x_0^* \in X^*$ with $\|x_0^*\|_{X^*} = 1$ such that $x_0^*(f(t)) = \|f(t)\|_X$ we have also

$$\bigcap_{x^* \in B(X^*)} A_{X^*} \subset A$$

and consequently

$$A = \bigcap_{x^* \in B(X^*)} A_{X^*}.$$

According to Lemma 1.1.5 we have

$$\bigcap_{x^* \in B(X^*)} A_{X^*} = \bigcap_{n=1}^{\infty} A_{x_n^*}$$

where x_n^* is given by this lemma and therefore

$$A = \bigcap_{n=1}^{\infty} A_{x_n^*}.$$

The sets $A_{x_n^*}$, $n \in \mathbb{N}$ are measurable by the weak measurability of f (cf. Remark after Proposition 1.1.3) and henceforth the set $A \subset I$ is measurable. This yields the measurability of the function $\|f(t)\|_X$ on I (see the same Remark).

Let $\{y_n \in f(I); n \in \mathbb{N}\}$ be a dense set in $f(I)$.

Similarly as we have shown the measurability of $\|f(t)\|_X$ above we can show that the functions $g_n : t \in I \rightarrow \|f(t) - y_n\|_X \in \mathbb{R}$, $n \in \mathbb{N}$ are measurable.

Taking a fixed $k \in \mathbb{N}$ put

$$E_n^k = \left\{ t \in I; g_n(t) < \frac{1}{k} \right\} = \left\{ t \in I; \|f(t) - y_n\|_X < \frac{1}{k} \right\}.$$

The measurability of $g_n : I \rightarrow \mathbb{R}$ yields that $E_n^k \subset I$ are measurable sets and because for every $t \in I$ there is an $n \in \mathbb{N}$ such that $\|f(t) - y_n\|_X < \frac{1}{k}$ we get $\bigcup_{n=1}^{\infty} E_n^k = I$.

Define

$$B_n^k = E_n^k \setminus \bigcup_{j < n} E_j^k, \quad n \in \mathbb{N}, k \in \mathbb{N}.$$

$B_n^k \subset I$ are measurable sets, $B_n^k \cap B_m^k = \emptyset$ for $n \neq m$, $\bigcup_{n=1}^{\infty} B_n^k = I$ and

$$\sum_{n=1}^{\infty} \mu(B_n^k) = \mu(I) < \infty.$$

Hence for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\sum_{n=n_k+1}^{\infty} \mu(B_n^k) < \frac{1}{k}.$$

Define

$$h_k(t) = y_n \quad \text{if } t \in B_n^k$$

and $h_k(t) = 0$ otherwise. For $t \in I$ we have $\|f(t) - h_k(t)\|_X < \frac{1}{k}$ and of course

$$\lim_{k \rightarrow \infty} \|f(t) - h_k(t)\|_X = 0$$

uniformly in I .

The range of h_k is countable by the definition.

Let us set $g_k(t) = h_k(t)$ for $t \in \bigcup_{n=1}^{n_k} B_n^k$ and $g_k(t) = 0$ otherwise. Then $g_k \in \mathcal{J}$ and $\lim_{k \rightarrow \infty} \|f(t) - g_k(t)\|_X = 0$ a.e. in I and this gives the measurability of f . □

Looking at the proof of the Pettis theorem 1.1.6 we can use the countable valued functions h_k to present the following.

Corollary 1.1.8. *A function $f : I \rightarrow X$ is measurable if and only if*

$$\lim_{n \rightarrow \infty} h_n(t) = f(t)$$

uniformly for almost all $t \in I$ where (h_n) is a sequence of countable valued measurable functions.

Remark. The Pettis measurability theorem can be found in classical books on vector integration (e.g. [DU77]) sometimes with a shorter proof. We use essentially the approach given in the book [Y65] because Lemma 1.1.5 will play a certain role later in our considerations.

Proposition 1.1.9. *If $f : I \rightarrow X$ is measurable then there is a bounded measurable $g : I \rightarrow X$ and a measurable $h : I \rightarrow X$ with*

$$h(t) = \sum_{n=1}^{\infty} x_n \chi_{E_n}(t), \quad x_n \in X, \quad n \in \mathbb{N}, \quad t \in I,$$

where $E_n \subset I$, $n \in \mathbb{N}$ are pairwise disjoint measurable sets, such that $f = g + h$.

Proof. Using the Pettis theorem 1.1.6 we can suppose that the range $f(I)$ of f is a separable subset in X with $\{x_n, n \in \mathbb{N}\}$ being an at most countable dense subset in $f(I)$.

Define

$$E_n = \left\{ t \in I; f(t) \in (x_n + B(X)) \setminus \bigcup_{k=1}^{n-1} (x_k + B(X)) \right\}.$$

Then $I \subset \bigcup_{n=1}^{\infty} E_n$, $E_n \cap E_m = \emptyset$ for $m, n \in \mathbb{N}$, $m \neq n$. Put

$$h(t) = \sum_{n=1}^{\infty} x_n \chi_{E_n}(t)$$

for $t \in I$. $h : I \rightarrow X$ is measurable and if $t \in E_n \cap I$ then

$$f(t) - h(t) = f(t) - x_n \in B(X),$$

i.e. $\|f(t) - h(t)\|_X \leq 1$. Putting $g(t) = f(t) - h(t)$ we have $\|g(t)\|_X \leq 1$ and $f(t) = g(t) + h(t)$, $t \in I$.

□

Proposition 1.1.10. *If X is a separable Banach space then $f : I \rightarrow X$ is measurable if and only if f is weakly measurable.*

Proof. For a separable space X the range $\{f(t); t \in I\} \subset X$ of f is automatically separable and the statement follows immediately from the Pettis measurability theorem 1.1.6. □

1.2 The integral of simple functions

It is a simple task to define the integral of a simple function. Assume that $f : I \rightarrow X$ is a simple function given by Definition 1.1.1.

Define the *integral of $f : I \rightarrow X$* as

$$\int_I f = \sum_{m=1}^p y_m \mu(E_m) = \sum_{m=1}^p f(E_m) \mu(E_m). \quad (1.2.1)$$

If $A \subset I$ is measurable and $f \in \mathcal{J}$ then define

$$f_A(t) = f(t) \text{ if } t \in A$$

and

$$f_A(t) = 0 \text{ if } t \notin A,$$

i.e. $f_A = f \cdot \chi_A$.

It is easy to see that the function f_A is again simple and we set

$$\int_A f = \int_I f_A.$$

The integral of simple functions $f \in \mathcal{J}$ defined in this way is evidently a linear mapping $\int : \mathcal{J} \rightarrow X$.

If A, B are disjoint measurable sets in I then from the linearity of the integral and from the obvious identity $f_{A \cup B} = f_A + f_B$ we have

$$\int_{A \cup B} f = \int_A f + \int_B f. \quad (1.2.2)$$

Remark. In the special case when $X = \mathbb{R}$ and $f \leq g$ where $f, g \in \mathcal{J}$ we have

$$\int_I f \leq \int_I g. \quad (1.2.3)$$

If $f \geq 0$ and $A \subset B$, then

$$\int_A f \leq \int_B f. \quad (1.2.4)$$

For the integral of a function $f \in \mathcal{J}$ of the form presented in Definition 1.1.1 and a measurable $A \subset I$ we have

$$\left\| \int_A f \right\|_X \leq \int_A \|f\|_X \leq \sup_{t \in I} \|f(t)\|_X \mu(A) \quad (1.2.5)$$

because

$$\begin{aligned} \left\| \int_A f \right\|_X &= \left\| \sum_{m=1}^p y_m \mu(A \cap E_m) \right\|_X \leq \sum_{m=1}^p \|y_m\|_X \mu(A \cap E_m) \\ &= \int_A \|f\|_X \leq \max_m \|y_m\|_X \sum_{m=1}^p \mu(A \cap E_m) = \sup_{t \in I} \|f(t)\|_X \mu(A) \end{aligned}$$

and $\bigcup_{m=1}^p (A \cap E_m) = A$.

For a given $f \in \mathcal{J}$ define

$$\|f\|_1 = \int_I \|f\|_X. \quad (1.2.6)$$

The mapping $\|\cdot\|_1 : \mathcal{J} \rightarrow \mathbb{R}$ has the following properties:

$$\|f\|_1 \geq 0 \text{ for every } f \in \mathcal{J}, \quad (1.2.7)$$

$$\|af\|_1 = |a| \|f\|_1 \text{ for every } f \in \mathcal{J} \text{ and } a \in \mathbb{R}, \quad (1.2.8)$$

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1 \text{ for every } f, g \in \mathcal{J}. \quad (1.2.9)$$

By $\|\cdot\|_1$ from (1.2.6) a seminorm on \mathcal{J} is given; the implication $\|f\|_1 = 0 \Rightarrow f(t) = 0$ for all $t \in I$ does not hold. To see this it suffices to take $A \subset I$ such that $\mu(A) = 0$ and a function f for which $f(t) = 0$ provided $t \notin A$.

The triangle inequality (1.2.9) can be shown using a decomposition of the interval I into measurable sets with respect to which each of the functions f and g is simple, i.e. f and g have constant values at each measurable component of the decomposition; the inequality (1.2.9) then results from the triangle inequality valid in the Banach space X .

The seminorm $\|\cdot\|_1$ defined above for elements of \mathcal{J} is called the *L-seminorm*.

1.3 Bochner integral

Let us now consider sequences of simple functions $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ with the seminorm $\|\cdot\|_1$ given in the previous section.

Definition 1.3.1. A sequence (f_q) , $f_q \in \mathcal{J}$, $q = 1, 2, \dots$ is called *L-zero* if

$$\lim_{q \rightarrow \infty} \|f_q\|_1 = 0.$$

Two sequences (f_q) , (g_q) $f_q, g_q \in \mathcal{J}$, $q = 1, 2, \dots$ are called *equivalent* if their difference $(f_q - g_q)$ is *L-zero*.

A sequence $(f_q) = (f_q)_{q=1}^\infty$, $f_q \in \mathcal{J}$, $q = 1, 2, \dots$ is called *L-Cauchy* if for every $\varepsilon > 0$ there is an $N = N_\varepsilon \in \mathbb{N}$ such that

$$\|f_q - f_r\|_1 < \varepsilon \text{ for } q, r \geq N_\varepsilon.$$

We will consider the completion of the linear space \mathcal{J} of simple functions on I with respect to the *L-seminorm* $\|\cdot\|_1$.

The completion of \mathcal{J} is given as the space of equivalence classes of *L-Cauchy* sequences of functions from \mathcal{J} . For details concerning the concept of the completion of a seminormed space see e.g. [L93].

The set of L -Cauchy sequences of simple functions has the structure of a linear space, i.e., if (f_q) and (g_q) are L -Cauchy sequences of simple functions and $a \in \mathbb{R}$ then both $(f_q + g_q)$ and (af_q) are L -Cauchy sequences of simple functions.

The following statement is fundamental for the construction given below.

Lemma 1.3.2. *Let (f_q) be an L -Cauchy sequence of simple functions defined on I . Then there is a subsequence (g_k) of (f_q) , which converges pointwise almost everywhere to some function $f : I \rightarrow X$ and for every $\varepsilon > 0$ there is a measurable $E \subset I$ with $\mu(E) < \varepsilon$ such that this subsequence converges uniformly on $I \setminus E$.*

Proof. Since the sequence (f_q) is L -Cauchy, for every $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ such that if $q, r \geq N_k$, then

$$\|f_q - f_r\|_1 < \frac{1}{2^{2k}}.$$

Without loss of generality it can be assumed that $N_k < N_{k+1}$. We set

$$g_k = f_{N_k};$$

then

$$\|g_m - g_n\|_1 = \|f_{N_m} - f_{N_n}\|_1 < \frac{1}{2^{2n}}$$

for $m \geq n$.

Next we define for $t \in I$ the series

$$g_1(t) + \sum_{k=1}^{\infty} (g_{k+1}(t) - g_k(t))$$

and show that it converges absolutely for almost all $t \in I$ to an element in X and that this convergence is uniform except for a set with arbitrarily small measure.

For $n \in \mathbb{N}$ denote

$$M_n = \left\{ t \in I; \|g_{n+1}(t) - g_n(t)\|_X \geq \frac{1}{2^n} \right\}.$$

Then

$$\begin{aligned} \frac{1}{2^n} \cdot \mu(M_n) &= \int_{M_n} \frac{1}{2^n} \leq \int_{M_n} \|g_{n+1}(t) - g_n(t)\|_X \\ &\leq \int_I \|g_{n+1}(t) - g_n(t)\|_X = \|g_{n+1} - g_n\|_1 < \frac{1}{2^{2n}} \end{aligned}$$

and this yields

$$\mu(M_n) < \frac{1}{2^n}.$$

Let us define

$$Z_n = M_n \cup M_{n+1} \cup \dots.$$

Then $Z_{n+1} \subset Z_n$, $n \in \mathbb{N}$ and

$$\mu(Z_n) \leq \sum_{j=n}^{\infty} \mu(M_j) < \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}}.$$

For $t \notin Z_n$ and $k \geq n$ we have

$$\|g_{k+1}(t) - g_k(t)\|_X < \frac{1}{2^k}$$

and therefore the series $\sum_{k=n}^{\infty} (g_{k+1}(t) - g_k(t))$ converges absolutely and uniformly for $t \notin Z_n$.

Assume that $\varepsilon > 0$ is given. Putting $N = Z_k$, we have for sufficiently large k

$$\mu(N) = \mu(Z_k) < \frac{1}{2^{k-1}} < \varepsilon$$

and this leads to the assertion that the series $\sum_{k=n}^{\infty} (g_{k+1}(t) - g_k(t))$ converges absolutely and uniformly on $I \setminus N$.

If we take $M = \bigcap Z_n$, then evidently $\mu(M) = 0$ and if $t \notin M$, then $t \notin Z_n$ for some n . Therefore the series

$g_1(t) + \sum_{k=1}^{\infty} (g_{k+1}(t) - g_k(t))$ converges for $t \notin M$ and this means that $\lim_{k \rightarrow \infty} g_k(t) = \lim_{k \rightarrow \infty} f_{N_k}(t)$ exists for almost all $t \in I$ and the sequence $g_k(t) = f_{N_k}(t)$ converges uniformly on $I \setminus N$.

□

Lemma 1.3.3. a) *If (f_q) is an L -Cauchy sequence of simple functions then the limit $\lim_{q \rightarrow \infty} \int_I f_q$ exists.*

b) *If (f_q) and (g_q) are equivalent L -Cauchy sequences of simple functions then*

$$\lim_{q \rightarrow \infty} \int_I f_q = \lim_{q \rightarrow \infty} \int_I g_q. \quad (1.3.1)$$

c) *If (f_q) and (g_q) are L -Cauchy sequences of simple functions which converge almost everywhere to a function $f : I \rightarrow X$ then (f_q) and (g_q) are equivalent and (1.3.1) holds.*

Proof. The existence of the limit in a) is easy to show. Indeed, for simple functions f_q we have (see (1.2.5))

$$\begin{aligned} \left\| \int_I f_q - \int_I f_r \right\|_X &= \left\| \int_I (f_q - f_r) \right\|_X \\ &\leq \int_I \|f_q - f_r\|_X = \|f_q - f_r\|_1. \end{aligned}$$

This means that the sequence of integrals $\int_I f_q \in X$, $q \in \mathbb{N}$ is a Cauchy sequence and therefore it is convergent, i.e. the limit $\lim_{q \rightarrow \infty} \int_I f_q$ exists.

For proving b) let $\varepsilon > 0$ be given. Then by a) and the equivalence of the L -Cauchy sequences (f_q) and (g_q) there is an $N \in \mathbb{N}$ such that for $r > N$ we have

$$\|f_r - g_r\|_1 = \int_I \|f_r - g_r\|_X < \varepsilon,$$

$$\left\| \int_I f_r - \lim_{q \rightarrow \infty} \int_I f_q \right\|_X < \varepsilon, \quad \left\| \int_I g_r - \lim_{q \rightarrow \infty} \int_I g_q \right\|_X < \varepsilon$$

and

$$\begin{aligned}
& \left\| \lim_{q \rightarrow \infty} \int_I f_q - \lim_{q \rightarrow \infty} \int_I g_q \right\|_X \\
& \leq \left\| \lim_{q \rightarrow \infty} \int_I f_q - \int_I f_r \right\|_X + \left\| \int_I f_r - \int_I g_r \right\|_X + \left\| \int_I g_r - \lim_{q \rightarrow \infty} \int_I g_q \right\|_X \\
& < 2\varepsilon + \int_I \|f_r - g_r\|_X < 3\varepsilon,
\end{aligned}$$

and b) is proved.

For the proof of c) let us set $h_q = f_q - g_q$ and assume that $\varepsilon > 0$ is given. It is clear that $\lim_{q \rightarrow \infty} h_q(t) = 0$ for almost all $t \in I$ and that the sequence h_q is L -Cauchy, i.e. there is an $N \in \mathbb{N}$ such that for $r, q \geq N$ we have

$$\|h_q - h_r\|_1 < \varepsilon.$$

This implies by a) that the sequences of integrals $\int_I h_q$ and $\int_I \|h_q\|_X$ are convergent. It remains to show that

$$\lim_{q \rightarrow \infty} \int_a^b \|h_q\|_X = 0.$$

Define

$$M = \{t \in I; h_N(t) \neq 0\} \subset I.$$

For $q \geq N$ we have

$$\begin{aligned}
\int_{I \setminus M} \|h_q\|_X &= \int_{I \setminus M} \|h_q - h_N\|_X \\
&\leq \int_a^b \|h_q - h_N\|_X = \|h_q - h_N\|_1 < \varepsilon
\end{aligned}$$

because $h_N(t) = 0$ for $t \in I \setminus M$. By Lemma 1.3.2 there exists a subset $Z \subset M$ with

$$\mu(Z) < \frac{\varepsilon}{\sup_{t \in I} \|h_N(t)\|_X + 1}$$

and a subsequence h_{q_s} which converges to zero uniformly on the set $M \setminus Z$. Hence there is an $s_0 \in \mathbb{N}$, $s_0 \geq N$ such that for $s \geq s_0$ and for $t \in M \setminus Z$ we have

$$\|h_{q_s}(t)\|_X < \frac{\varepsilon}{\mu(I)}.$$

Therefore

$$\int_{M \setminus Z} \|h_{q_s}(t)\|_X < \frac{\varepsilon \mu(M \setminus Z)}{\mu(I)} \leq \varepsilon$$

provided $s \geq s_0$. For $s \geq s_0$ we also have

$$\begin{aligned} \int_Z \|h_{q_s}(t)\|_X &\leq \int_Z \|h_{q_s}(t) - h_N(t)\|_X + \int_Z \|h_N(t)\|_X \\ &\leq \|h_{q_s} - h_N\|_1 + \sup_{t \in I} \|h_N(t)\|_X \mu(Z) \\ &< \varepsilon + \frac{\varepsilon}{\sup_{t \in I} \|h_N(t)\|_X + 1} \sup_{t \in I} \|h_N(t)\|_X < 2\varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} \|h_{q_s}\|_1 &= \int_a^b \|h_{q_s}(t)\|_X \\ &= \int_{I \setminus M} \|h_{q_s}(t)\|_X + \int_{M \setminus Z} \|h_{q_s}(t)\|_X + \int_Z \|h_{q_s}(t)\|_X \\ &< \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon \end{aligned}$$

and because

$$\left\| \int_I h_{q_s} \right\|_X \leq \int_I \|h_{q_s}\|_X,$$

we obtain $\lim_{s \rightarrow \infty} \int_I \|h_{q_s}(t)\|_X = 0$ and therefore also $\lim_{q \rightarrow \infty} \int_I \|h_q(t)\|_X = 0$, and c) is proved. \square

Definition 1.3.4. Denote by \mathcal{B} the set of all functions $f : I \rightarrow X$ for which there is an L -Cauchy sequence f_q , $q \in \mathbb{N}$ of simple functions which converges to f almost everywhere in I , i.e.

$$\lim_{q \rightarrow \infty} \|f_q(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

We say in this case that the L -Cauchy sequence $f_q \in \mathcal{J}$, $q \in \mathbb{N}$ determines the function $f \in \mathcal{B}$.

By a) from Lemma 1.3.3 it is easy to see that to every L -Cauchy sequence (f_q) of simple functions a value $x_{(f_q)} \in X$ can be assigned by the relation

$$x_{(f_q)} = \lim_{q \rightarrow \infty} \int_I f_q.$$

Using b) from Lemma 1.3.3 we can see that the same value $x_{(f_q)} \in X$ belongs to all L -Cauchy sequences which are equivalent to the sequence (f_q) .

This allows us now to present the following concept.

Definition 1.3.5. For $f \in \mathcal{B}$ define

$$\int_I f = \lim_{q \rightarrow \infty} \int_I f_q \tag{1.3.2}$$

where (f_q) is an arbitrary sequence of simple functions which determines $f \in \mathcal{B}$.

The value $\int_I f$ given by (1.3.2) is called the *Bochner integral of the function f* .

If necessary the more detailed notation $(\mathcal{B}) \int_I f$ will be used for this concept of integral.

The set of functions \mathcal{B} is called the set of *Bochner integrable functions*.

It is easy to see that the set \mathcal{B} is linear.

By (1.2.1) the integral was defined in a very natural way for simple functions while by the relation (1.3.2) this integral is extended to functions $f \in \mathcal{B}$.

The correctness of this definition is clear by Lemma 1.3.3.

In our presentation we follow the lines given in [L93] by S. Lang but the reader can find the Bochner integral in many books, e.g. [M78] or in general books on functional analysis, e.g. [DS].

Lemma 1.3.6. *If $f \in \mathcal{B}$ and (f_q) is an L -Cauchy sequence of simple functions which determines f , then $\|f\|_X$ is integrable and the sequence $(\|f_q\|_X)$ determines the real function $\|f\|_X$ in the sense of the set \mathcal{B} .*

In this case we have

$$\int_I \|f\|_X = \lim_{q \rightarrow \infty} \int_I \|f_q\|_X = \lim_{q \rightarrow \infty} \|f_q\|_1. \quad (1.3.3)$$

Moreover,

$$\left\| \int_I f \right\|_X \leq \int_I \|f\|_X. \quad (1.3.4)$$

Proof. Since

$$|\|f_q(t)\|_X - \|f_r(t)\|_X| \leq \|f_q(t) - f_r(t)\|_X, \quad t \in I,$$

we get

$$\begin{aligned} \|\|f_q\|_X - \|f_r\|_X\|_1 &= \int_I |\|f_q(t)\|_X - \|f_r(t)\|_X| \\ &\leq \int_I \|f_q(t) - f_r(t)\|_X = \|f_q - f_r\|_1 \end{aligned}$$

and this means that the sequence $\|f_q\|_X$ of real-valued simple functions is L -Cauchy. Moreover,

$$\lim_{q \rightarrow \infty} \|f_q(t)\|_X = \|f(t)\|_X$$

for almost all $t \in I$ and consequently $\|f\|_X : I \rightarrow \mathbb{R}$ is integrable by Definition 1.3.5 and $\|f_q\|_X$, $q \in \mathbb{N}$ determines $\|f\|_X$ where (1.3.3) holds.

Since by (1.2.5) for $f_q \in \mathcal{J}$ we have

$$\left\| \int_A f_q \right\|_X \leq \int_A \|f_q\|_X,$$

(1.3.2) and (1.3.3) can be used for obtaining (1.3.4) by passing to the limits with $q \rightarrow \infty$ on both sides of this inequality. \square

By Lemma 1.3.3 we know that $\lim_{q \rightarrow \infty} \|f_q\|_1$ does not depend on the choice of the L -Cauchy sequence (f_q) which determines the same f ; therefore the seminorm $\|\cdot\|_1$ defined for simple functions $f \in \mathcal{J}$ can be extended to functions $f \in \mathcal{B}$ by the relation

$$\|f\|_1 = \int_a^b \|f(t)\|_X = \lim_{q \rightarrow \infty} \|f_q\|_1. \quad (1.3.5)$$

In this way $\|\cdot\|_1 : \mathcal{B} \rightarrow \mathbb{R}$ is defined and the following holds:

$$\|f\|_1 \geq 0 \text{ for every } f \in \mathcal{B}, \quad (1.3.6)$$

$$\|af\|_1 = |a|\|f\|_1 \text{ for every } f \in \mathcal{B} \text{ and } a \in \mathbb{R}, \quad (1.3.7)$$

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1 \text{ for every } f, g \in \mathcal{B}. \quad (1.3.8)$$

These relations are immediate consequences of the analogous relations (1.2.7) - (1.2.9) for $\|\cdot\|_1$ given on \mathcal{J} , showing that $\|\cdot\|_1$ is a seminorm on \mathcal{B} .

Lemma 1.3.7. *If $f \in \mathcal{B}$ and (f_n) is an L -Cauchy sequence of simple functions determining f , then*

$$\lim_{r \rightarrow \infty} \|f_r - f\|_1 = 0.$$

Proof. Since (f_q) is an L -Cauchy sequence of elements $f_q \in \mathcal{J}$ which converges almost everywhere to f , for every $\varepsilon > 0$ there

is $N_\varepsilon \in \mathbb{N}$ such that

$$\|f_r - f_q\|_1 < \varepsilon \quad (1.3.9)$$

provided $r, q > N_\varepsilon$. Let us fix $r > N_\varepsilon$ and put $g_q = f_r - f_q \in \mathcal{J}$ for $q \in \mathbb{N}$.

Then $\lim_{q \rightarrow \infty} g_q(t) = f_r(t) - f(t) \in \mathcal{B}$ for almost all $t \in I$ and because $\|g_l - g_k\|_1 = \|f_l - f_k\|_1$ the sequence (g_q) is L -Cauchy and determines $f_r - f \in \mathcal{B}$.

Hence

$$\|f - f_r\|_1 = \lim_{q \rightarrow \infty} \|g_q\|_1 = \lim_{q \rightarrow \infty} \|f_q - f_r\|_1 < \varepsilon$$

and this implies $\lim_{r \rightarrow \infty} \|f_r - f\|_1 = 0$. □

Corollary 1.3.8. *If $f \in \mathcal{B}$ then for every $\varepsilon > 0$ there is a simple function $g_\varepsilon \in \mathcal{J}$ such that*

$$\|f - g_\varepsilon\|_1 < \varepsilon, \quad (1.3.10)$$

i.e. the set \mathcal{J} of simple functions is dense in \mathcal{B} with respect to the seminorm $\|\cdot\|_1$.

Lemma 1.3.9. *The space \mathcal{B} equipped with the seminorm $\|\cdot\|_1$ is complete.*

Proof. Assume that $g_q \in \mathcal{B}$, $q \in \mathbb{N}$ is a Cauchy sequence with respect to the seminorm $\|\cdot\|_1$. By Corollary 1.3.8 for every $q \in \mathbb{N}$ there exists a simple function $f_q \in \mathcal{J}$ such that

$$\|g_q - f_q\|_1 < \frac{1}{q}.$$

Hence

$$\|f_q - f_r\|_1 \leq \|f_q - g_q\|_1 + \|g_q - g_r\|_1 + \|g_r - f_r\|_1 < \frac{1}{q} + \frac{1}{r} + \|g_q - g_r\|_1$$

and therefore the sequence (f_q) is L -Cauchy. By Lemma 1.3.2 the sequence (f_q) contains a subsequence (f_{q_s}) which converges almost everywhere in I to a certain function $f : I \rightarrow X$ and this

subsequence is L -Cauchy. Hence $f \in \mathcal{B}$. For this subsequence (f_{q_s}) we have

$$\|g_{q_s} - f\|_1 \leq \|g_{q_s} - f_{q_s}\|_1 + \|f_{q_s} - f\|_1$$

and the subsequence (g_{q_s}) of (g_q) converges in the seminorm $\|\cdot\|_1$ to f by Lemma 1.3.7. This implies that also the original sequence (g_q) converges in this seminorm to $f \in \mathcal{B}$ and henceforth \mathcal{B} is complete. \square

Using Lemma 1.3.9 we can see easily that the following holds.

Corollary 1.3.10. *A function $f : I \rightarrow X$ belongs to \mathcal{B} if and only if there is a sequence $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ such that*

$$\lim_{n \rightarrow \infty} f_n(t) = f(t)$$

for almost all $t \in I$ and

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = \lim_{n \rightarrow \infty} \int_I \|f_n - f\|_X = 0.$$

By this corollary we get that $f \in \mathcal{B}$ is necessarily measurable. On the other hand, this corollary in fact gives another definition of Bochner integrability which is equivalent to Definition 1.3.4 (and Definition 1.3.5 can be used for defining the integral).

Definition 1.3.11. *$f : I \rightarrow X$ is Bochner integrable if there is a sequence of simple functions $f_n : I \rightarrow X$, $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ a.e. in I and*

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = \lim_{n \rightarrow \infty} \int_I \|f_n - f\|_X = 0.$$

Let us note that the following holds.

Theorem 1.3.12. *If $f : I \rightarrow X$ is such that $f(t) = 0$ for almost all $t \in I$ then $f \in \mathcal{B}$ and $\int_I f = 0$.*

Proof. The L -Cauchy sequence of simple functions from Definition 1.3.4 can be chosen as functions which are identically zero. \square

Corollary 1.3.13. *If $f : I \rightarrow X$ is Bochner integrable and $g : I \rightarrow X$ is such that $f(t) = g(t)$ for almost all $t \in I$ then g is Bochner integrable and $\int_I f = \int_I g$.*

Proof. Since $g = g - f + f$ and $g - f$ is Bochner integrable by Theorem 1.3.12, we obtain the statement immediately. \square

Corollary 1.3.13 makes it possible to identify functions which are equal almost everywhere as is usual in the Lebesgue theory.

Remark 1.3.14. For the case $X = \mathbb{R}$, i.e. for $f : I \rightarrow \mathbb{R}$, the definition of Bochner integrability and the Bochner integral (Definition 1.3.5 or Definition 1.3.11) give an alternative approach to *Lebesgue integrability* and the *Lebesgue integral*. This means that $f : I \rightarrow \mathbb{R}$ is Bochner integrable in the sense of Definition 1.3.5 if and only if f is Lebesgue integrable and the two integrals of f have the same value.

1.4 Properties of Bochner integrable functions and of the Bochner integral

From the definition of the class \mathcal{B} it is clear that every $f \in \mathcal{B}$ is measurable in the sense of Definition 1.1.2.

By the Pettis measurability theorem 1.1.6, if $f \in \mathcal{B}$ then f is also weakly measurable and almost everywhere separable valued.

For a given measurable set $E \subset I$ and $f \in \mathcal{B}$ we define

$$\int_E f = \int_I \chi_E \cdot f = \lim_{n \rightarrow \infty} \int_I \chi_E \cdot f_n,$$

where $f_n \in \mathcal{J}$, $n \in \mathbb{N}$ determines f .

This definition makes sense because $\chi_E \cdot f_n$, $n \in \mathbb{N}$ is evidently a sequence of simple functions which determines $\chi_E \cdot f$.

Let $f : I \rightarrow X$ be a countable valued measurable function of

the form

$$f(t) = \sum_{m=1}^{\infty} y_m \chi_{E_m}(t), \quad t \in I, \quad (1.4.1)$$

where $E_m \subset I$, $m \in \mathbb{N}$ is measurable, $E_m \cap E_l = \emptyset$ for $m \neq l$, $y_m \in X$, $m \in \mathbb{N}$.

Lemma 1.4.1. *A countable valued measurable function $f : I \rightarrow X$ of the form (1.4.1) is Bochner integrable if*

$$\sum_{m=1}^{\infty} \|y_m\|_X \mu(E_m) < \infty.$$

Proof. Define for $l \in \mathbb{N}$ functions

$$f_l(t) = \sum_{m=1}^l y_m \chi_{E_m}(t), \quad t \in I.$$

Then $f_l \in \mathcal{J}$ for every $l \in \mathbb{N}$ and $\lim_{l \rightarrow \infty} f_l(t) = f(t)$ for $t \in I$.

For $t \in I$ and $k < l$ we have by definition

$$\|f_l(t) - f_k(t)\|_X = \left\| \sum_{m=k+1}^l y_m \chi_{E_m}(t) \right\|_X$$

and since $\left\| \sum_{m=k+1}^l y_m \chi_{E_m}(t) \right\|_X = \sum_{m=k+1}^l \|y_m\|_X \chi_{E_m}(t)$ we have

$$\|f_l - f_k\|_1 = \sum_{m=k+1}^l \|y_m\|_X \mu(E_m).$$

Now we can see that the sequence (f_l) is L -Cauchy if and only if the series

$$\sum_{m=1}^{\infty} \|y_m\|_X \mu(E_m)$$

converges. In this case the series $\sum_{m=1}^{\infty} y_m \chi_{E_n}$ converges in X to f and by definition we have $f \in \mathcal{B}$ and

$$\int_I f = \sum_{m=1}^{\infty} y_m \mu(E_n)$$

and also

$$\int_I \|f\|_X = \sum_{m=1}^{\infty} \|y_m\|_X \mu(E_n).$$

□

Corollary 1.4.2. *A countable valued measurable function $f : I \rightarrow X$ for which $\|f(t)\|_X \leq g(t)$ a.e. in I with $g \in \mathcal{B}$ is Bochner integrable.*

Proof. Using the sequence (f_l) from the proof of Lemma 1.4.1 we can see that $\|f_l\|_1 \leq \int_I g < \infty$ for every $l \in \mathbb{N}$ and therefore the condition given in Lemma 1.4.1 is satisfied.

□

Theorem 1.4.3. *A measurable function $f : I \rightarrow X$ is Bochner integrable if and only if $\|f\|_X : I \rightarrow \mathbb{R}$ is Bochner integrable.*

Proof. If $f \in \mathcal{B}$ then Lemma 1.3.6 implies the integrability of $\|f\|_X$.

Assume that $\|f\|_X$ is Bochner integrable. Since f is measurable (see Corollary 1.3.10), by Corollary 1.1.8 for every $k \in \mathbb{N}$ there is a countable valued measurable function f_k of the form

$$f_k(t) = \sum_{m=1}^{\infty} y_{k,m} \chi_{E_{k,m}}(t), \quad t \in I, \quad (1.4.2)$$

where $E_{k,m} \subset I$, $m \in \mathbb{N}$ is measurable, $E_{k,m} \cap E_{k,l} = \emptyset$ for $m \neq l$, $y_{k,m} \in X$, $m \in \mathbb{N}$ and f_k has the following property:

there exists $N \subset I$, $\mu(N) = 0$ such that for every $k \in \mathbb{N}$ we have

$$\|f(t) - f_k(t)\|_X < \frac{1}{2k\mu(I)} \quad (1.4.3)$$

for $t \in I \setminus N$.

Hence

$$\|f_k(t)\|_X \leq \|f(t)\|_X + \|f(t) - f_k(t)\|_X < \|f(t)\|_X + \frac{1}{2k\mu(I)}$$

a.e. in I and, since $\mu(I) < \infty$, Corollary 1.4.2 implies that f_k is Bochner integrable and

$$\int_I \|f_k\|_X = \sum_{n=1}^{\infty} \|y_{k,m}\|_X \mu(E_{k,m}) < \infty.$$

Choose an $r_k \in \mathbb{N}$ such that

$$\sum_{n=r_k+1}^{\infty} \|y_{k,m}\|_X \mu(E_{k,m}) < \frac{1}{2k}.$$

Since $\|f - f_k\|_X$ is measurable and (1.4.3) holds the function $\|f - f_k\|_X$ is integrable and

$$\int_I \|f - f_k\|_X < \frac{1}{2k\mu(I)} \mu(I) = \frac{1}{2k}.$$

Put $g_k = \sum_{n=1}^{r_k} y_{k,m} \chi_{E_{k,m}}$. Then $g_k \in \mathcal{J}$ and

$$f_k = g_k + \sum_{n=r_k+1}^{\infty} y_{k,m} \chi_{E_{k,m}}.$$

We have also

$$\begin{aligned} \|f - g_k\|_1 &= \int_I \|f - g_k\|_X \\ &\leq \int_I \|f - f_k\|_X + \int_I \|f_k - g_k\|_X < \frac{1}{2k} + \sum_{n=r_k+1}^{\infty} \|y_{k,m}\|_X \mu(E_{k,m}) < \frac{1}{k}. \end{aligned}$$

and therefore $f \in \mathcal{B}$.

□

Corollary 1.4.4. *If $f : I \rightarrow X$ is measurable and bounded by an integrable function $g : I \rightarrow \mathbb{R}$, i.e. $\|f(t)\|_X \leq g(t)$ for almost all $t \in I$, then f is Bochner integrable.*

Proposition 1.4.5. *Let $f : I \rightarrow X$ be measurable of the form*

$$f = g + \sum_{n=1}^{\infty} x_n \chi_{E_n} \quad (1.4.4)$$

where $g : I \rightarrow X$ is measurable and bounded, E_n are pairwise disjoint measurable subsets of I , $x_n \in X$, $n \in \mathbb{N}$ (see Proposition 1.1.9).

Then f is Bochner integrable if and only if x_n and E_n , $n \in \mathbb{N}$ can be chosen such that the series $\sum_{n=1}^{\infty} x_n \cdot \mu(E_n)$ converges absolutely in X , and in this case we have

$$(\mathcal{B}) \int_E f = (\mathcal{B}) \int_E g + \sum_{n=1}^{\infty} x_n \cdot \mu(E \cap E_n) \quad (1.4.5)$$

for every measurable $E \subset I$.

Proof. Assume that $f \in \mathcal{B}$ is of the form (1.4.4). Since g is bounded we have $g \in \mathcal{B}$ (see Corollary 1.4.4) and also $f - g = \sum_{n=1}^{\infty} x_n \cdot \chi_{E_n} \in \mathcal{B}$.

By Theorem 1.4.3 we have $\int_I \left\| \sum_{n=1}^{\infty} x_n \cdot \chi_{E_n} \right\|_X < \infty$ but this means, because of $E_n \cap E_m = \emptyset$, $m \neq n$, that

$$\int_I \left\| \sum_{n=1}^{\infty} x_n \cdot \chi_{E_n} \right\|_X = \sum_{n=1}^{\infty} \|x_n\|_X \cdot \mu(E_n) < \infty$$

and $\sum_{n=1}^{\infty} x_n \cdot \mu(E_n)$ is absolutely convergent in X .

Conversely, if g is bounded and the series $\sum_{n=1}^{\infty} x_n \cdot \mu(E_n)$ converges absolutely, then $g \in \mathcal{B}$ by Corollary 1.4.4 and $\sum_{n=1}^{\infty} x_n \cdot \mu(E_n) \in \mathcal{B}$ by Lemma 1.4.1. Hence $f = g + h \in \mathcal{B}$.

□