Chapter 0

Introduction

Quantum Mechanics deserves the interest of mathematicians not only because it is a very important physical theory, which governs all microphysics, i.e. the physical phenomena at the microscopic scale of 10^{-8} cm., but also because it turned out to be at the root of important developments of modern mathematics.

The first branch of modern mathematics which was strongly influenced by Quantum Mechanics is the theory of *Algebras of Operators in Hilbert Spaces*, the so-called Von Neumann Algebras, whose foundations are due to Von Neumann also in connection with his interests in Quantum Mechanics. The theory of *Von Neumann Algebras*, as well as the related theory of C^* *Algebras* is now a well developed branch of mathematics [Dixmier (1981), (1977); Kadison and Ringrose (1983); Stratila and Zsido' (1979); Takesaki (1979); Jorgensen and Muhly (1987); AMS Proc. (1982)].

A strictly related topic is the study of the representations of the *Weyl algebra*, equivalently of the unitary representations of the *Heisenberg group*, which is at the basis of the canonical formulation of Quantum Mechanics of systems with a finite number of degrees of freedom [Putnam (1967); Bratteli and Robinson (1981); Folland (1989); Garding and Wightman (1954)].

The Schroedinger equation, which governs the time evolution of quantum systems, has motivated the theory of the *Schroedinger Operators* and the *Theory of Scattering*, which are now robust chapters of the theory of partial differential equations [Cycon et al. (1987); Eastham and Kalf (1982); Graffi (1984); Holten and Jensen (1989); Hislop and Sigal (1996); Amrein et al. (1977); Lax and Philips (1989), (1976); La Vita and Marchand (1974); Pearson (1988); Perry (1983); Petkov (1989); Sigal (1983); Yafaev (1992)].

Finally, it is worthwhile to mention that, under general stability conditions, the quantum mechanical time evolution allows an analytic continuation to purely imaginary time and the so-obtained theories (uniquely determined by the real time theories) correspond to stochastic processes.

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Such a deep relation between Quantum Mechanics and *Stochastic Processes* has been at the origin of important developments in the theory of stochastic processes like the Feynman-Kac formula, the theory of functional (or path) integration etc. [Blanchard et al. (1987); Chung and Williams (1982); Glimm and Jaffe (1987); Kac (1980); Roepstorff(1993); Simon (1979)].

Along the lines of the deep philosophical changes, which led from Classical Mechanics to Quantum Mechanics, quite recently new steps were taken in the frontier developments of classical analysis and geometry, giving rise to the corresponding non-commutative (or quantum) extensions. These new developments were given the names of *Quantum Calculus, Non-Commutative Integration, Non-Commutative Geometry, Non-Commutative Harmonic Analysis, Quantum Probability, etc.* Even the discovery of *Quantum Groups*, a rapidly growing theory, is due to the influence of Quantum Mechanics [Araki (1993); Biane (1995); Connes (1992), (1994), (1995); Kassel (1995); Kirillov (1995); Madore (1995); Manin (1988); Meyer (1992); Parthasarathy (1992)].

In conclusion, Quantum Mechanics, as a very important physical theory, was not only a source of concrete and special mathematical problems arising in the solution of particular physical problems, but also provided a body of general mathematical structures which strongly influenced the development of modern mathematics. To better appreciate this role, it may be worthwhile to recall the strict relation between the development of Mechanics and Mathematics. Indeed, the origin of Calculus or classical Analysis can be traced back to Newton and Leibnitz, who discovered a mathematical language for the foundations of mechanics. The underlying idea is that physical quantities are described by functions of space and time (and possibly of additional variables) and therefore the mathematical description of observable quantities is related to the *theory of functions* and classical analysis.

When in the XIX century a major problem of theoretical physics was the description of complex systems, with 10^{23} degrees of freedom, as required for the foundations of thermodynamics and statistical mechanics, it became clear that new mathematical ideas were needed; one could not reasonably think to consider a Cauchy problem for 10^{23} initial data. This led to abandon the idea that a physical state is described by a point in phase space and to rather describe a state as a probability measure on the phase space. In this way *probability theory and random variables* entered in a crucial and philosophically important way into the framework of theoretical physics, at the basis of Classical Statistical Mechanics.

The quantum mechanical revolution, which took place in the twenties and early thirties, realized that at the microscopic level it is no longer correct to pretend that the physical observable quantities are described by an abelian algebra of functions or of random variables. The Heisenberg analysis of physically realizable experiments on microscopic systems indicated that the measurement of an observable in general limits the precision by which another observable can be subsequently measured. The mathematical abstraction of this deep physical fact is the realization that the algebra of observables is not described by an algebra of functions, but rather by an *algebra of operators in a Hilbert space.* As mentioned before, the passage from the commutative structure of classical mechanics and/or of classical statistical mechanics to the non-commutative structure of quantum mechanics is the deep and crucial feature shared by the modern non-commutative extension of calculus, probability, geometry etc.

Last but not least, quantum mechanics had a dramatic impact on the development of mathematical logic, giving rise to the so-called *Quantum Logic* : whereas the lattice of propositions of classical logic has the structure of a Boolean algebra (equivalently that of a lattice of commutative projections), the lattice of quantum propositions is non-boolean and it corresponds to a lattice of non-commutative projections [Birkhoff and Von Neumann (1936); Beltrametti and Cassinelli (1981); Cohen (1989); Garden (1984); Hooker (1975); Pitowski (1989); Règdei (1998)].

The aim of these lectures is to provide at least the flavor of the philosophical revolution induced by quantum mechanics concerning the mathematical description of physical systems. The lectures are primarily addressed to people interested in questions of principle and in the mathematical foundations of physical theories, also in view of the fertile mutual influence between theoretical physics and mathematics.

In order to make the ideas at the basis of quantum mechanics understandable also to people with a mathematical education but with no great familiarity with physics, we will reduce the detailed description of the many experimental facts which led to the crisis of classical mechanics to the minimum and will rather extract and emphasize the overall simple and profound message for the mathematical description of quantum systems.

Once the Heisenberg revolutionary discovery has been accepted, namely that there are intrinsic limitations to the precise measurements of physical quantities (Heisenberg's uncertainty relations) leading to the non abelianity of the algebra of observables, the whole mathematical structure of Quantum Mechanics follows as a theorem (Gelfand-Naimark): the states of a physical system are described by vectors of a Hilbert space and the observables by Hilbert space operators. Also the Schroedinger formulation of Quantum Mechanics in terms of wave functions follows from Von Neumann uniqueness theorem on the regular (irreducible) representations of the Weyl algebra.

Those who will hopefully find the subject sufficiently interesting and stimulating are warmly referred to standard textbooks to deepen the mathematical and logical structure of quantum mechanics and to appreciate its impact on the description of the physical world [Dirac (1958); Feynman et al.(1963); Heisenberg (1930); Jauch (1968); Von Neumann (1955); Mackey (1963); Piron (1976); Segal (1963)].

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