

# Contents

<b>Foreword</b>	<b>vii</b>
<b>1 Linear Systems: Elimination Method</b>	<b>1</b>
1.1 Examples of Linear Systems . . . . .	1
1.1.1 A Review Example . . . . .	1
1.1.2 Covering a Sphere with Hexagons and Pentagons . . . . .	2
1.1.3 A Literal Example . . . . .	7
1.2 Homogeneous Systems . . . . .	11
1.2.1 A Chemical Reaction . . . . .	11
1.2.2 Reduced Forms . . . . .	12
1.3 Elimination Algorithm . . . . .	17
1.3.1 Elementary Row Operations . . . . .	18
1.3.2 Comparison of the Systems ( $S$ ) and ( $HS$ ) . . . . .	21
1.4 Appendix . . . . .	22
1.4.1 Potentials on a Grid . . . . .	22
1.4.2 Another Illustration of the Fundamental Principle . . . . .	23
1.4.3 The Euler Theorem $f + v = e + 2$ . . . . .	25
1.4.4 Fullerenes, Radiolarians . . . . .	25
1.5 Exercises . . . . .	26
<b>2 Vector Spaces</b>	<b>31</b>
2.1 The Language . . . . .	31
2.1.1 Axiomatic Properties . . . . .	31
2.1.2 An Important Principle . . . . .	32
2.1.3 Examples . . . . .	33
2.1.4 Vector Subspaces . . . . .	35
2.2 Finitely Generated Vector Spaces . . . . .	36
2.2.1 Generators . . . . .	36
2.2.2 Linear Independence . . . . .	39
2.2.3 The Dimension . . . . .	41
2.3 Infinite-Dimensional Vector Spaces . . . . .	44
2.3.1 The Space of Polynomials . . . . .	45
2.3.2 Existence of Bases: The Mathematical Credo . . . . .	47
2.3.3 Infinite-Dimensional Examples . . . . .	49

2.4	Appendix . . . . .	52
2.4.1	Set Theory, Notation . . . . .	52
2.4.2	Axioms for Fields of Scalars . . . . .	56
2.5	Exercises . . . . .	56
<b>3</b>	<b>Matrix Multiplication</b> . . . . .	<b>60</b>
3.1	Row by Column Multiplication . . . . .	60
3.1.1	Linear Fractional Transformations . . . . .	60
3.1.2	Linear Changes of Variables . . . . .	61
3.1.3	Definition of the Matrix Product . . . . .	62
3.1.4	The Map Produced by Matrix Multiplication . . . . .	66
3.2	Row Operations and Matrix Multiplication . . . . .	67
3.2.1	Elementary Matrices . . . . .	68
3.2.2	An Inversion Algorithm . . . . .	70
3.2.3	LU Factorizations . . . . .	72
3.2.4	Simultaneous Resolution of Linear Systems . . . . .	76
3.3	Matrix Multiplication by Blocks . . . . .	76
3.3.1	Explanation of the Method . . . . .	76
3.3.2	The Field of Complex Numbers . . . . .	79
3.4	Appendix . . . . .	80
3.4.1	Affine Maps . . . . .	80
3.4.2	The Field of Quaternions . . . . .	81
3.4.3	The Strassen Algorithm . . . . .	82
3.5	Exercises . . . . .	82
<b>4</b>	<b>Linear Maps</b> . . . . .	<b>88</b>
4.1	Linearity . . . . .	88
4.1.1	Preliminary Considerations . . . . .	88
4.1.2	Definition and First Properties . . . . .	90
4.1.3	Examples of Linear Maps . . . . .	91
4.2	General Results . . . . .	92
4.2.1	Image and Kernel of a Linear Map . . . . .	92
4.2.2	How to Construct Linear Maps . . . . .	94
4.2.3	Matrix Description of Linear Maps . . . . .	95
4.3	The Dimension Theorem for Linear Maps . . . . .	98
4.3.1	The Rank-Nullity Theorem . . . . .	98
4.3.2	Row-Rank versus Column-Rank . . . . .	99
4.3.3	Application: Invertible Matrices . . . . .	101
4.4	Isomorphisms . . . . .	102
4.4.1	Generalities . . . . .	102
4.4.2	Models of Finite-Dimensional Vector Spaces . . . . .	104
4.4.3	Change of Basis: Components of Vectors . . . . .	105
4.4.4	Change of Basis: Matrices of Linear Maps . . . . .	107
4.4.5	The Trace of Square Matrices . . . . .	107
4.5	Appendix . . . . .	108
4.5.1	Inverting Maps Between Sets . . . . .	108

4.5.2	Another Proof of Invertibility . . . . .	109
4.6	Exercises . . . . .	112
<b>5</b>	<b>The Rank Theorem</b>	<b>116</b>
5.1	More on Row- versus Column-Rank . . . . .	116
5.1.1	Factorizations of a Matrix . . . . .	116
5.1.2	Low Rank Examples . . . . .	117
5.1.3	A Basis for the Column Space . . . . .	118
5.2	Direct Sum of Vector Spaces . . . . .	119
5.2.1	Sum of Two Subspaces . . . . .	119
5.2.2	Supplementary Subspaces . . . . .	121
5.2.3	Direct Sum of Two Subspaces . . . . .	123
5.2.4	Independent Subspaces (General Case) . . . . .	125
5.2.5	Finite Direct Sums of Vector Spaces . . . . .	126
5.3	Projectors . . . . .	128
5.3.1	An Example and General Definition . . . . .	128
5.3.2	Geometrical Meaning of $P^2 = P$ . . . . .	129
5.3.3	Tricks of the Trade . . . . .	132
5.4	Appendix . . . . .	133
5.4.1	Pyramid of Ages . . . . .	133
5.4.2	Color Theory . . . . .	134
5.4.3	Genetics . . . . .	138
5.4.4	Einstein Summation Convention . . . . .	139
5.5	Exercises . . . . .	140
<b>6</b>	<b>Eigenvectors and Eigenvalues</b>	<b>144</b>
6.1	Introduction . . . . .	144
6.2	Definitions and Examples . . . . .	145
6.2.1	Definitions . . . . .	145
6.2.2	Simple $2 \times 2$ Examples . . . . .	146
6.2.3	A $4 \times 4$ Example . . . . .	148
6.2.4	Abstract Examples . . . . .	150
6.3	General Results . . . . .	153
6.3.1	Estimation of the Number of Eigenvalues . . . . .	153
6.3.2	Localization of Eigenvalues . . . . .	154
6.3.3	A Method for Finding Eigenvectors . . . . .	155
6.3.4	Eigenvectors and Commutation . . . . .	156
6.4	Applications of Eigenvectors . . . . .	157
6.4.1	The Fibonacci Numbers . . . . .	157
6.4.2	Diagonalization . . . . .	160
6.5	Appendix . . . . .	162
6.5.1	Eigenvectors of $AB$ and of $BA$ . . . . .	162
6.5.2	Complements on the Fibonacci Numbers . . . . .	163
6.6	Exercises . . . . .	163

<b>7</b>	<b>Inner-Product Spaces</b>	<b>167</b>
7.1	About Multiplication and Products . . . . .	167
7.1.1	The Dot Product in Plane Geometry . . . . .	168
7.1.2	The Dot Product in $\mathbf{R}^n$ . . . . .	171
7.2	Abstract Inner Products and Norms . . . . .	172
7.2.1	Definition and Examples . . . . .	172
7.2.2	The Cauchy–Schwarz–Bunyakovskii Inequality . . . . .	174
7.2.3	The Pythagorean Theorem . . . . .	175
7.2.4	More Identities . . . . .	176
7.3	Orthonormal Bases . . . . .	179
7.3.1	Euclidean Spaces . . . . .	179
7.3.2	The Best Approximation Theorem . . . . .	181
7.3.3	First Application: Periodic Functions . . . . .	183
7.3.4	Second Application: Least Squares Method . . . . .	184
7.4	Orthogonal Subspaces . . . . .	187
7.4.1	Orthogonal of a Subset . . . . .	188
7.4.2	The Support of a Linear Map . . . . .	189
7.4.3	Least Squares Revisited . . . . .	192
7.5	Appendix: Finite Probability Spaces . . . . .	194
7.5.1	Random Variables . . . . .	194
7.5.2	Algebras of Random Variables . . . . .	197
7.5.3	Independence of Random Variables . . . . .	199
7.6	Exercises . . . . .	200
<b>8</b>	<b>Symmetric Operators</b>	<b>205</b>
8.1	Definition and First Properties . . . . .	205
8.1.1	Intrinsic Characterization of Symmetry . . . . .	206
8.1.2	General Properties of Symmetric Operators . . . . .	207
8.2	Diagonalization . . . . .	208
8.2.1	Statement of the Result . . . . .	208
8.2.2	Existence of Eigenvectors . . . . .	209
8.2.3	Inductive Construction . . . . .	211
8.3	Applications . . . . .	212
8.3.1	Quadratic Forms . . . . .	212
8.3.2	Classification of Quadrics . . . . .	213
8.3.3	Positive Definite Operators . . . . .	216
8.4	Appendix . . . . .	219
8.4.1	Principal Axes and Statistics . . . . .	219
8.4.2	Functions of a Symmetric Operator . . . . .	220
8.4.3	Special Configurations . . . . .	222
8.5	Exercises . . . . .	225

<b>9</b>	<b>Duality</b>	<b>227</b>
9.1	Geometric Introduction . . . . .	227
9.1.1	Duality for Platonic Solids . . . . .	227
9.1.2	The Pappus Theorem and its Dual . . . . .	229
9.2	Dual of a Vector Space . . . . .	231
9.2.1	Definition and First Properties . . . . .	231
9.2.2	Dual Bases . . . . .	233
9.2.3	Bidual of a Vector Space . . . . .	234
9.3	Dual of a Normed Space . . . . .	235
9.3.1	Dual Norm . . . . .	235
9.3.2	Dual of a Euclidean Space . . . . .	236
9.3.3	Dual of Important Norms in $\mathbf{R}^n$ . . . . .	238
9.4	Transposition of Linear Maps . . . . .	240
9.4.1	Transposition of Operators in Euclidean Spaces . . . . .	240
9.4.2	Abstract Formulation of Transposition . . . . .	241
9.5	Exercises . . . . .	243
<b>10</b>	<b>Determinants</b>	<b>246</b>
10.1	From Space Geometry to Determinants . . . . .	247
10.1.1	Areas in $\mathbf{R}^3$ . . . . .	247
10.1.2	The Cross Product in $\mathbf{R}^3$ . . . . .	249
10.1.3	The Scalar Triple Product . . . . .	251
10.2	Volume Forms in Vector Spaces . . . . .	254
10.2.1	Properties of Volume Forms: Uniqueness . . . . .	255
10.2.2	Construction of Volume Forms in $\mathbf{R}^n$ . . . . .	258
10.3	Determinant of an Operator . . . . .	260
10.3.1	Volume-Amplification Factor . . . . .	260
10.3.2	Determinants and Row Operations . . . . .	263
10.4	Examples of Determinants . . . . .	266
10.4.1	Geometric Examples . . . . .	267
10.4.2	Arithmetic and Algebraic Examples . . . . .	268
10.4.3	Examples in Calculus . . . . .	270
10.4.4	Symbolic Determinants . . . . .	272
10.5	Appendix . . . . .	274
10.5.1	Permutations and Signs . . . . .	274
10.5.2	More Examples . . . . .	275
10.6	Exercises . . . . .	277
<b>11</b>	<b>Applications</b>	<b>285</b>
11.1	The Characteristic Polynomial . . . . .	285
11.1.1	Definition and Basic Properties . . . . .	285
11.1.2	Examples . . . . .	287
11.2	The Spectrum of an Operator . . . . .	288
11.2.1	Changing the Field of Scalars . . . . .	288
11.2.2	Roots of the Characteristic Polynomial . . . . .	289
11.2.3	Existence of a Complex Eigenvalue . . . . .	293

11.3	Cramer's Rule . . . . .	294
11.3.1	Solution of Regular Linear Systems . . . . .	294
11.3.2	Inversion of a Matrix . . . . .	297
11.3.3	LU Factorizations: Necessary Condition . . . . .	298
11.4	Construction of Orthonormal Bases . . . . .	299
11.5	A Selection of Important Results . . . . .	301
11.5.1	The Frobenius and Cayley–Hamilton Theorems . . . . .	301
11.5.2	Restricting Scalars from $\mathbf{C}$ to $\mathbf{R}$ . . . . .	304
11.6	Appendix . . . . .	306
11.6.1	Back to $AB$ and $BA$ . . . . .	306
11.6.2	Covariant Components . . . . .	307
11.6.3	Series of Matrices . . . . .	308
11.7	Exercises . . . . .	310
<b>12</b>	<b>Normal Operators</b>	<b>315</b>
12.1	Orthogonal Matrices . . . . .	315
12.1.1	General Properties . . . . .	315
12.1.2	Geometric Properties . . . . .	318
12.1.3	Spectral Properties . . . . .	319
12.2	Transposition and Normal Operators . . . . .	321
12.2.1	Skew-Symmetric Operators . . . . .	322
12.2.2	Back to Orthogonal Operators . . . . .	323
12.2.3	Normal Operators, Spectral Properties . . . . .	324
12.3	Hermitian Inner Products . . . . .	325
12.3.1	Hermitian Inner Product in $\mathbf{C}^n$ . . . . .	325
12.3.2	The Adjoint of an Operator . . . . .	326
12.3.3	Special Classes of Complex Operators . . . . .	327
12.3.4	The Spectral Theorem for Normal Operators . . . . .	329
12.4	Appendix . . . . .	330
12.4.1	General Properties of Isometries . . . . .	330
12.4.2	The Polar Decomposition . . . . .	331
12.4.3	The Singular Value Decomposition . . . . .	333
12.4.4	Anti-Commutation Relations . . . . .	337
12.5	Exercises . . . . .	339
<b>A</b>	<b>Helpful Supplements</b>	<b>345</b>
A.1	Some Hints for the Exercises . . . . .	345
A.2	Answers to Some Exercises . . . . .	350
A.3	Review Exercises . . . . .	354
A.4	Axioms for Fields and Vector Spaces . . . . .	363
A.5	Summary for the Cross Product in $\mathbf{R}^3$ . . . . .	364
A.6	Inner Products, Norms, and Distances . . . . .	366
A.7	The Greek Alphabet . . . . .	367
A.8	References . . . . .	368
	<b>Index</b>	<b>369</b>