Preface

The main justification for this book is that there have been significant advances in continued fractions over the past decade, but these remain for the most part scattered across the literature, and under the heading of topics from algebraic number theory to theoretical plasma physics.

We now have a better understanding of the rate at which assorted continued fraction or greatest common denominator (gcd) algorithms complete their tasks. The number of steps required to complete a gcd calculation, for instance, has a Gaussian normal distribution.

We know a lot more about *badly approximable* numbers. There are several related threads here. A badly approximable number is a number x such that $\{q|p - qx|\colon\ p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ is bounded below by a positive constant; badly approximable numbers have continued fraction expansions with bounded partial quotients, and so we are led to consider a kind of Cantor set E_M consisting of all $x \in [0,1]$ such that the partial quotients of x are bounded above by M. The notion of a badly approximable *rational* number has the ring of crank mathematics, but it is quite natural to study the set of rationals r with partial quotients bounded by M. The number of such rationals with denominators up to n, say, turns out to be closely related to the Hausdorff dimension of E_M, (comparable to $n^{2\dim E_M}$) which is in turn related to the spectral radius of linear operators $L_{M,s}$, acting on some suitably chosen space of functions f, and given by $L_{M,s}f(t) = \sum_{k=1}^{m}(k+t)^{-s}f(1/(k+t))$. Similar operators have been studied by, among others, David Ruelle, in connection with theoretical one-dimensional plasmas, and they are related to entropy.

Alongside these developments there has been a dramatic increase in the computational power available to investigators. This has been helpful on the theoretical side, as one is more likely to seek a proof for a result when, following computations and graphical rendering of the output, that result leaps off the screen.

Consider, for instance, the venerable Hurwitz complex continued fraction algorithm. This algorithm takes as input a complex number ξ (say, inside the unit square centered on 0), and returns a sequence $\langle a_n \rangle$ of Gaussian integers $a_1, a_2, \ldots$, all outside the unit disk, such that

$$\xi = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

The algorithm uses an auxiliary sequence $\langle \xi_n \rangle$, with $\xi_1 = \xi$ and $\xi_{n+1} = 1/\xi_n - a_n$. Is there any particular pattern to the distribution of the ξ_k's? What sorts of numbers have atypical expansions?

These questions are analogs of questions for which, in the case of the real numbers and the classical continued fraction expansion, answers are known or suspected. Almost always, the expansion of a randomly chosen real input $\xi \in (0,1)$ will have the property that if $\xi = 1/(a_1 + 1/(a_2 + \cdots))$, then the ξ_n given by the same recurrence relation as mentioned above are distributed according to the *Gauss density* $1/((1+x)\log 2)$. Quadratic irrationals have ultimately periodic continued fraction expansions, and therefore, their ξ_n are not so distributed, but in the case of real inputs these seem to be the only algebraic exceptions. Back in the complex case, to assemble some tens of thousands of data points (a bare minimum considering that a 1-megapixel image is hardly high resolution) can require extensive computations. But once this is done, it turns out there are some surprises-there are algebraic numbers with expansions atypical of randomly chosen inputs, yet not of degree 2. This is discussed in Chapter 5.

Passing from the complex numbers, at once one-dimensional and two-dimensional, we turn our attention to simultaneous diophantine approximation of real n-tuples $\xi = (\xi_1, \ldots, \xi_n)$. Here, we are looking for a positive integer q, and further integers $(p_1, \ldots, p_n)$, such that $e(q, \xi) := \max\{|p_j - q\xi_j|, 1 \leq j \leq n\}$ will be 'small'. The Dirichlet principle guarantees that there are infinitely many choices of q such that, in combination with the unique sensible choice of the p_j's, gives $e(q, \xi) \ll q^{-1/n}$. (If ξ contains only rational entries, these q are eventually just multiples of a common-denominator representation of ξ, and the errors are zero.)

Computing good choices of q by head-on search is computationally prohibitively expensive, as the sequence of good q tends to grow exponentially. We discuss two algorithms for this task. Both rely upon the insight that approximation of ξ is related to the task of finding *reduced* bases of $(n+1) \times (n+1)$ lattices of the form

$$\begin{pmatrix} 1 & 0 & \ldots 0 \\ 0 & 1 & \ldots 0 \\ \vdots & 0 & \ddots \vdots \\ -\xi_1 & -\xi_2 & \ldots \epsilon \end{pmatrix}.$$

There are different ways to give exact meaning to the notion of a reduced lattice basis. The general idea is that the vectors should form an integral basis of the lattice, and they should be short.

The Gauss lattice reduction algorithm treats the two-dimensional case, and has recently been analyzed by Daudé, Flajolet, and Vallée. It is discussed in Chapter 2.

The Lagarias geodesic multidimensional continued fraction algorithm uses a form of Minkowski reduction, which is computationally feasible for modest dimensions and gives in a sense best-possible answers, while the Lenstra-Lenstra-Lovasz algorithm gives much quicker answers when the dimension is large, but at the risk that the results obtained may not be quite so good. These are discussed in Chapter 6.

Some numbers, for instance, e, have continued fraction expansions featuring fairly frequent, ever-larger partial quotients. (Liouville numbers take this to an extreme!) Others, for instance, $\sqrt{2}$, have continued fraction expansions with bounded partial quotients. Chapter 3 is dedicated to this latter type of number, further broken down as the union $\cup_{M=2}^{\infty} E_M$ of continued fraction Cantor sets. Of course, even E_2 is uncountable, and quadratic irrationals are but the tip of this iceberg. The size of the E_M is best understood in the context of Hausdorff dimension, and we discuss this. The *discrepancy* of the sequence $\langle n\alpha \rangle$, as well as the behavior of related sums, is discussed as well. Chapters 4 and 9 also treat the topic of E_M.

In Chapter 4, we look at the ergodic theory of continued fractions. (There is a recent book by Dajani and Kraaikamp which treats the topic more extensively.) Portions of this chapter first appeared in *New York J. of Math.* 4, pp. 249-258.

Chapter 5 is devoted to the complex continued fraction algorithms of Asmus Schmidt and of Adolf Hurwitz. Interest in the former has perhaps suffered from the lack of a convenient algorithms for computer implementation, while it seems not to have been recognized that the latter enjoys many good properties beyond those initially established by Hurwitz. In particular, there is an analog to the Gauss density for the Hurwitz algorithm; it even makes a pretty picture, and is featured on the cover.

Chapter 6 is devoted to multidimensional Diophantine approximation, and in particular, to the so-called *Hermite approximations* to numbers and vectors, and the Lagarias geodesic multidimensional continued fraction algorithm.

Chapter 7 discusses an interesting generalization of the approximation properties of quadratic irrationals. The field $\mathbb{Q}(\sqrt{2})$, seen as a vector space over $\mathbb{Q}$, has the canonical basis $\{1, \sqrt{2}\}$. The field $\mathbb{Q}(2^{1/3})$, seen also as a vector space over $\mathbb{Q}$, has canonical basis $\{1, 2^{1/3}, 2^{2/3}\}$. Thus from a certain point of view, we should expect theorems about quadratic irrationals to have analogues not in the context of rational approximations to a single number such as $2^{1/3}$, but rather in the context of simultaneous diophantine approximation to $(2^{1/3}, 2^{2/3})$. And so it is.

Chapter 8 discusses Marshall Hall's theorem concerning sums of continued fraction Cantor sets. This theorem has undergone various iterations and the current strongest version seems to be due to Astels. We give a taste of his approach.

Chapter 9 discusses the functional-analytic techniques arising out of work by K. I. Babenko, E. Wirsing, D. Ruelle, D. Mayer, and others, or, if one goes back all the way, out of the conjecture by Gauss concerning the frequency of the partial quotients in continued fraction expansions of typical numbers. Combined with modern computing power, it becomes possible to evaluate, say, the Wirsing constant, to many digits. Portions of this chapter first appeared in Number Theory for the Millennium, Vol II, pp. 175-194.

Chapter 10 discusses a dynamical-systems perspective, related to Chapter 9 but bringing new tools to the analysis. This approach has scored a real triumph recently, with the result by V. Baladi and B. Vallée that all the standard variants of the Euclidean algorithm have Gaussian normal distribution statistics for a wide variety of measures of the work they must do on typical inputs.

Chapter 11 discusses so-called *conformal iterated function systems.* Much of the material of continued fractions can be seen as an instance of such systems. In this topic, the names Mauldin and Urbański are prominent.

Finally, Chapter 12 discusses convergence of continued fractions, in the spirit of Perron's classic book, and the later classic by Jones and Thron.

Little in this chapter is new, but it would be a pity to omit all mention of these wonderful results. One tidbit that emerges from an extensive theory going back to Jacobi and Laplace is a (standard) continued fraction expansion for $e^{2/k}$. Also discussed are analytical continued fraction expansions, such as those for $\log(1+z)$, $\tanh(z)$, and e^z. This also gives us the classical Euler expansion of e itself as a continued fraction, so that at the end of the book we have come full circle to the beginnings of the subject matter.

What's New Quite a bit, actually. Theorems 3.2, Theorems 4.2 and 4.3, Theorems 5.1-5.5, Theorem 6.2, Theorems 7.1-7.3, the estimate for the Wirsing constant in Chapter 9, Theorem 9.5, the proof of Theorem 12.1, and Theorem 12.5, are, so far as the author is aware, new.

What's Wrong Nothing that I know of. But there is scant chance that the text is free of errors. Any errors remaining after proofreading are the author's. Where the works of others have been restated, if there is a mistake, it is in my restatement of their work. As to algorithms, they are presented here solely for purposes of illustration. No warranty, express or implied, is made that these algorithms are free of all defect. (And with algorithms, even a typographical error invalidates the algorithm.)

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In a sense, it should go without saying that no book is written from scratch. Everything has roots, and the content presented here is a distillation and compilation of the work of hundreds of researchers, over and above the author's own contribution. Many of their names have been left out, but only because citation chains must be pruned, or through an oversight.

It should also go without saying that authors do not work in a vacuum. For creating and explaining LaTeX, (particularly as it applies to books,) my thanks to the Knuths and to George Grätzer, respectively. For editorial assitance and help with LaTeX, my thanks to Mary Chapman and Robin Campbell, respectively.

It should further go without saying that authors are far from disembodied scholarly entities. The kilo-hours that are devoted to preparing a manuscript are a gift from their families. As leaf litter forms the floor of a forest, on occasion, yellow legal pad litter formed a floor of sorts across one room and another. My thanks to Pam for the gifts of time to work, and patience with the worker.

My thanks finally to Lucille, who always held that scholars should write books, but would have had to wait until her 114th birthday or so to see the result. These things should go without saying, but they should not go unsaid.