CHAPTER 1

TWO-LEVEL FACTORIAL AND FRACTIONAL FACTORIAL DESIGNS IN BLOCKS OF SIZE TWO. PART 2

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Two-level factorial and fractional factorial designs can be blocked in a variety of ways, depending on block size, on which factorial estimates are required to be estimated clear of blocks, and on the permissible number of runs to be performed. A previous (2003) paper by the same authors discussed blocks of size two for 2^k designs when k = 2, 3, 4 and 5. "Best" combination designs were given which provided the most pure (within block) estimates of main effects and two-factor interactions from choices that combined various confounding patterns. Extensions were also given for 2^{k-p} fractional factorial designs with the same number of runs (that is, when the k-p value being considered is the same as the k value previously considered for the 2^k designs). An example illustrated the use of follow-up inter-block analysis. The present article discusses larger designs of the same type, when k = 6, 7 and 8. In addition, some "superior" sequential combinations of 2^k designs with various choices of initial arrangements are given. The popular mirror-image pairing design is not the best initial arrangement in a sequence, but is a valuable one because it provides pure estimates of all the main effects.

1. Introduction

In many experimental situations, it is desirable to group sets of experimental runs together in blocks. The block size is governed by many considerations and represents, in most experiments, the number of runs that can be made without worrying (much) about variation caused by factors not being studied specifically in the experiment. Often a block is some natural interval of time (e.g., a week, a day, a work shift) or of space (an oven, a greenhouse, a work bench, a reactor) or of personnel (a research worker, a research team), and so on. An excellent discussion of some of the practical considerations that dictate the need to block experimental designs is given by Rosenbaum[1] (1999, p. 127), and a specific illustration is given by Yang and Draper[2] (2003, p. 294).

When only a single factor is being examined, we have the case of a simple comparative experiment as described by Box, Hunter and Hunter[3] (1978, pp. 97-102). When more than two factors are involved, any allocation of the runs of a 2^k or 2^{k-p} design into pairs will involve confounding some effects of possible interest with the blocks. Thus more than one replicate of the design, blocked differently, will be necessary. Box, Hunter and Hunter[3] (1978, p. 341) discuss an example of this type in which a 2^3 design is blocked into pairs in four separate ways to give a total of $4(2^3) = 32$ runs in 16 blocks. In each of the four portions of the design, different effects are confounded with blocks, so that an overall balance is achieved, and all main effects and interactions are estimable. Draper and Guttman[4] (1997) showed that for a 2^k design, $k2^k$ runs are needed to estimate all main effects and all interactions. In this article, however, we shall confine interest to estimating only main effects and two-factor interactions, while tentatively assuming that all interactions between three or more factors are zero. Using the notation xfi for an interaction between x factors for $x \geq 2$, we can write this assumption as "$\geq$ 3fi = 0." As might be anticipated, this can be done with more economical designs containing fewer blocks. We assume here that block effects are additive, representing changes in overall level only, and that there are no block-factor interactions. (In our 2003 paper[2], we also discussed the inter-block analysis of such designs, in which it is assumed that block effects are random variables. Such an analysis would also apply here in similar circumstances.)

Because blocks of size two are very restrictive, any 2^k design must be run several times over in various blocking configurations to estimate all main effects and 2fis. Consider, for example, the so-called "mirror image" or "foldover" pairs of runs, in which the levels of the factors are completely reversed in the second run of the pair. Only main effects (and not 2fis) can be estimated from such pairings. Other pairs of blocked runs would thus be needed to estimate the 2fis, if we began with a foldover design. Moreover, in certain applications where 2^k designs are already subject to some initial fractionation, it becomes impossible to form foldover pairs at all, as pointed out by Rosenbaum[1] (1999, p. 131). Thus we need to seek alternative blocking methods for blocks of size two.

In blocking entire 2^k or 2^{k-p} designs in blocks of size two in this article, we shall pair up runs using conventional ideas of blocking generators. Note that this is not the most general situation. A much wider problem would be to form all the possible pairs that could be chosen from 2^k runs (there are $2^k(2^k - 1)/2$ such pairs) and then to consider how to add pairs one at a time sequentially to form a useful design. (For k = 3, we would select from 28 pairs, for example.) We believe that a design chosen in this more general way would not be an improvement over the designs we shall choose by using blocking generators, simply because it is essential to build certain symmetries in order to estimate the effects. Moreover, such designs might not be resolvable, that is, might not permit division into sets of blocks, each set of which contains an entire 2^3 design within it. However, we have not investigated these wider issues.

For the designs we derive, estimates of the main effects and 2fis are made by least squares estimation. The model fitted will include a general mean, terms for all main effects and 2fis, and terms for B–1 blocking variables, where B is the number of blocks of size two in the design. There are thus $1 + k + k(k-1)/2 + B - 1 = B + k(k+1)/2$ terms in the model. The total number of runs is n = 2B. In order for the model to be estimable, all main effects and 2fis must be estimable internally *within* one or more blocks (pairs of runs) somewhere, and usually several times, in the design. (See Yang and Draper[2], 2003, pp. 295-297 for a detailed discussion of this point.)

2. The Six Factor, 64 Runs, 2^6 Design

To divide the 64 runs of a 2^6 design into 32 pairs of runs requires choice of five generators, which we select from the set of six main effects and

15 two-factor interactions. In all there are $2^6 - 1 = 63$ possible divisions of this type, that is, there are 63 possible "blocking arrangements", listed in Table 1. (It can be shown that Table 1 provides all the possible blocking arrangements (in pairs) for a 2^6 design, although they can also be re-described in other ways, using alternative generators. Choices of other generators will simply reproduce one of the 63 arrangements shown.) These 63 are divided into six "types" depending on the numbers of main effects and 2fis blocking generators used. Arrangement No 1, for example is defined by

$$\begin{aligned} \mathbf{I} &= \mathbf{1} = \mathbf{2} = \mathbf{3} = \mathbf{4} = \mathbf{5} \\ &= \mathbf{12} = \mathbf{13} = \mathbf{14} = \mathbf{15} = \mathbf{23} = \mathbf{24} = \mathbf{25} = \mathbf{34} = \mathbf{35} = \mathbf{45}. \end{aligned} \tag{1}$$

The pairs of runs in this specific arrangement do not contain any pure comparisons for the five defining main effects and the ten 2fi shown in (1), but pure (within block) estimates can be made for the main effect 6 and the remaining 2fi, namely, 16, 26, 36, 46, and 56. Thus combinations of various of the 63 arrangements in Table 1 will permit estimation of particular effects in the overall design. We specifically seek combinations that will estimate *all* main effects and 2fis. These ideas are discussed more fully for smaller designs in Yang and Draper[2] (2003). Note that the foldover (mirror image) design is No. 63, type 6, in Table 1.

We consider the choice of d arrangements from the arrangements available. For $k \leq 5$, it was feasible to examine *all* combinations of arrangements, but limits on available space prevented this for larger d. For example, for $k = 6$, all combinations were examined for $d \leq 5$ but, for larger d, we discuss only the "best" sequential combinations given a selected initial arrangement. Previously used selection criteria still apply, as described below.

3. Definitions and Notation

We need some additional notation in addition to that already mentioned. Specifically:

Table 1. Block generators for all possible blocking arrangements for k = 6.

Type	No.	Generators					Type	No.	Generators				
		B_1	B_2	B_3	B_4	B_5			B_1	B_2	B_3	B_4	B_5
1	1	1	2	3	4	5	4	42	1	2	36	46	56
	2	1	2	3	4	6		43	1	3	26	46	56
	3	1	2	3	5	6		44	1	4	26	36	56
	4	1	2	4	5	6		45	1	5	26	36	46
	5	1	3	4	5	6		46	1	6	25	35	45
	6	2	3	4	5	6		47	2	3	16	46	56
2	7	1	2	3	4	56		48	2	4	16	36	56
	8	1	2	3	5	46		49	2	5	16	36	46
	9	1	2	3	6	45		50	2	6	15	35	45
	10	1	2	4	5	36		51	3	4	16	26	56
	11	1	2	4	6	35		52	3	5	16	26	46
	12	1	2	5	6	34		53	3	6	15	25	45
	13	1	3	4	5	26		54	4	5	16	26	36
	14	1	3	4	6	25		55	4	6	15	25	35
	15	1	3	5	6	24		56	5	6	14	24	34
	16	1	4	5	6	23	5	57	1	26	36	46	56
	17	2	3	4	5	16		58	2	16	36	46	56
	18	2	3	4	6	15		59	3	16	26	46	56
	19	2	3	5	6	14		60	4	16	26	36	56
	20	2	4	5	6	13		61	5	16	26	36	46
	21	3	4	5	6	12		62	6	15	25	35	45
3	22	1	2	3	46	56	6	63	16	26	36	46	56
	23	1	2	4	36	56							
	24	1	2	5	36	46							
	25	1	2	6	35	45							
	26	1	3	4	26	56							
	27	1	3	5	26	46							
	28	1	3	6	25	45							
	29	1	4	5	26	36							
	30	1	4	6	25	35							
	31	1	5	6	24	34							
	32	2	3	4	16	56							
	33	2	3	5	16	46							
	34	2	3	6	15	45							
	35	2	4	5	16	36							
	36	2	4	6	15	35							
	37	2	5	6	14	34							
	38	3	4	5	16	26							
	39	3	4	6	15	25							
	40	3	5	6	14	24							
	41	4	5	6	13	23							

t = blocking type, as in the first column of Table 1; when a vector **t** = (t_1, t_2, …, t_d) is used, it defines a combination of d blocking types used to form a design.

xfi already means an x-factor interaction, and we shall use 1fi to mean a main effect.

u = 0, 1, …, d is the number of times an effect is estimated in a combination design.

f_u is the non-negative integer frequency of u. For example if u = 0 occurs twice in a combination design, $f_0 = 2$.

f = (f_0, f_1, f_2, …, f_d) is a collection of the frequency values f_u for a combination design. This defines the *overall confounding pattern of a combination design.* It is sometimes abbreviated as *overall pattern* or simply OP.

$\mathbf{f}_x = (f_0, f_1, f_2, \ldots, f_d)^{(xfi)}$ is a collection of the frequency values f_u for the group of x-factor interactions. Here, x could be 1, 2, …, or k and, for the combination designs within each overall pattern, $\mathbf{f}_1 + \mathbf{f}_2 + \ldots + \mathbf{f}_k = \mathbf{f}$. This is called the *x-factor interaction group confounding pattern of a combination design*, or more simply, the *xfi group pattern.*

Each design has a set of $\mathbf{f}_i$ values, ($\mathbf{f}_1$, $\mathbf{f}_2$, …, $\mathbf{f}_k$) which is called its *group pattern*, abbreviated in tables as GP. Because we focus only on main effects (1fi) and 2fi for this article, we mostly list only ($\mathbf{f}_1$, $\mathbf{f}_2$). When this is done, statements of "different group patterns" refer to sets of ($\mathbf{f}_1$, $\mathbf{f}_2$) only.

The following notation is also used:

(xfi:f_u) = an f_u value which belongs to one of the $\mathbf{f}_x = (f_0, f_1, f_2, \ldots, f_d)^{(xfi)}$. Where restricted as above, this refers to the frequency values for 1fi and 2fi only.

$\text{Sum}_x(u)$, x = 1 or 2. If x = 1, $\text{Sum}_1(u)$, is the sum of the u values in the main effects, and if x = 2, it is the sum of the u values in the two-factor interactions. Thus, for example,

$$\text{Sum}_1(u) = \sum_{u=1}^{d} u f_u \text{ , where } f_u \in (f_0, f_1, f_2, \ldots, f_d)^{(1fi)}.$$

$Std_x(u)$, x = 1or 2. If x = 1, it is the standard deviation of the u values in the main effects, and if x = 2, it is the standard deviation of the u values in the two-factor interactions.

Although our study examines u values, we do not save them all individually, in order to conserve computer storage space. The **f** and the $\mathbf{f}_i$'s summarize the u patterns. The only information lost is the order of the u values, which is irrelevant because the order changes if the effects are renamed, but the **f** and the $\mathbf{f}_i$'s do not. It is also unnecessary to store all the combination designs. Keeping track of the blocking arrangements (the row numbers in $\mathbf{C}_k$ that constitute each combination design) and the combinations of blocking types, namely the t values, is sufficient to identify the types of arrangements that constitute the various combination designs.

4. Combination Design Selection Process

We compare the u value patterns of all possible combination designs. Cases that do not provide estimates of all 1fi and 2fi are discarded immediately if alternatives exist that *do* estimate these. The larger the u values, the better the design, in general. The best situation is to have a design whose u values are uniformly better than those for other competing designs. This can be examined directly if there are not many patterns of u values to examine. Otherwise, we instead examine the sums and standard deviations of the u values, that is, $Sum_1(u)$, $Sum_2(u)$, $Std_1(u)$, and $Std_2(u)$, by plotting them, leading to the selection of just a few designs for further detailed examination. Desirable are designs with high Sums and small Stds.

5. Case k = 6, d = 2

There are, in total, $\binom{63}{2} = 1953$ combination designs to examine. The overall pattern **f** = (15, 32, 16) applies to all of them. This means that, in each possible combination design, 15 effects have no internal (that is, within block or intra-block) estimates, 32 have one internal estimate, and 16 effects have two internal estimates. The numbers total to 63, which accounts for all possible main effects and interactions of all orders. In what follows, however, we consider only 1fi and 2fi estimation in evaluating these designs in more detail. Within this (15, 32, 16) pattern lie 37 different group patterns, in none of which can all 1fis and 2fis be estimated. Table 2 shows all 14 group patterns that have (1fi: $f_0 = 0$), that is, the 14 group patterns that provide estimates of all main effects. The best of these is the group pattern GP6 with (2fi: $f_0 = 3$) which estimates all but three 2fis. It has the following u values for 1fis and 2fis:

GP6: (1, 1, 1, 1, 2, 2) and (0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2).

This means it provides one internal estimate of four main effects, two internal estimates of two main effects, does not provide estimates of three 2fi, provides one internal estimate of eight 2fi and two internal estimates of four 2fi. This is the best result possible for combining two (d = 2) of the designs in Table 1. The 45 arrangements that form the optimal group pattern GP6 are given in Yang[5] (2002, Appendix, List 1).

Table 2. k = 6, d = 2; the 14 group patterns that have (1fi: $f_0 = 0$).

	1fi (f_0, f_1, f_2)	2fi (f_0, f_1, f_2)	(t_1, t_2)	Number of combination designs
GP1	0 1 5	10 5 0	(5, 6)	6
GP2	0 2 4	6 8 1	(5, 5)	15
GP3	0 2 4	7 8 0	(4, 6)	15
GP4	0 3 3	4 9 2	(4, 5)	60
GP5	0 3 3	6 9 0	(3, 6)	20
GP6	0 4 2	3 8 4	(4, 4)	45
GP7	0 4 2	4 8 3	(3, 5)	60
GP8	0 4 2	7 8 0	(2, 6)	15
GP9	0 5 1	4 5 6	(3, 4)	60
GP10	0 5 1	6 5 4	(2, 5)	30
GP11	0 5 1	10 5 0	(1, 6)	6
GP12	0 6 0	6 0 9	(3, 3)	10
GP13	0 6 0	7 0 8	(2, 4)	15
GP14	0 6 0	10 0 5	(1, 5)	6

The basic rule for this design type is to select two type 4 arrangements, i.e., $(t_1, t_2) = (4, 4)$, that do *not* have common 1fi block generators. For example, $(\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{B_5})$ = **(1, 2, 36, 46, 56)** and **(3, 4, 16, 26, 56)** together will give such a design.

Table 3 shows an example of an optimal design of type GP6, consisting of 128 runs in 64 blocks of size 2. Blocks 1-32 are defined by the blocking generators $(\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{B_5})$ = **(5, 6, 14, 24, 34)**, and blocks 33-64 are defined by $(\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{B_5})$ = **(3, 4, 16, 26, 56)**.

Table 3. An example of a best combination design for k = 6, d = 2.

	Factor No.							Factor No.					
Block	1	2	3	4	5	6	Block	1	2	3	4	5	6
1	+	+	+	−	−	−	33	+	+	−	−	+	−
	−	−	−	+	−	−		−	−	−	−	−	+
2	+	+	+	−	+	−	34	+	+	+	−	+	−
	−	−	−	+	+	−		−	−	+	−	−	+
3	+	+	+	−	−	+	35	+	+	−	+	+	−
	−	−	−	+	−	+		−	−	−	+	−	+
4	+	+	+	−	+	+	36	+	+	+	+	+	−
	−	−	−	+	+	+		−	−	+	+	−	+
5	−	+	+	−	−	−	37	−	+	−	−	+	−
	+	−	−	+	−	−		+	−	−	−	−	+
6	−	+	+	−	+	−	38	−	+	+	−	+	−
	+	−	−	+	+	−		+	−	+	−	−	+
7	−	+	+	−	−	+	39	−	+	−	+	+	−
	+	−	−	+	−	+		+	−	−	+	−	+
8	−	+	+	−	+	+	40	−	+	+	+	+	−
	+	−	−	+	+	+		+	−	+	+	−	+
9	+	−	+	−	−	−	41	+	−	−	−	+	−
	−	+	−	+	−	−		−	+	−	−	−	+
10	+	−	+	−	+	−	42	+	−	+	−	+	−
	−	+	−	+	+	−		−	+	+	−	−	+
11	+	−	+	−	−	+	43	+	−	−	+	+	−
	−	+	−	+	−	+		−	+	−	+	−	+
12	+	−	+	−	+	+	44	+	−	+	+	+	−
	−	+	−	+	+	+		−	+	+	+	−	+
13	−	−	+	−	−	−	45	−	−	−	−	+	−
	+	+	−	+	−	−		+	+	−	−	−	+

Table 3. (*Continued*)

Block	Factor No. 1	2	3	4	5	6	Block	Factor No. 1	2	3	4	5	6
14	–	–	+	–	+	–	46	–	–	+	–	+	–
	+	+	–	+	+	–		+	+	+	–	–	+
15	–	–	+	–	–	+	47	–	–	–	+	+	–
	+	+	–	+	–	+		+	+	–	+	–	+
16	–	–	+	–	+	+	48	–	–	+	+	+	–
	+	+	–	+	+	+		+	+	+	+	–	+
17	+	+	–	–	–	–	49	+	+	–	–	–	–
	–	–	+	+	–	–		–	–	–	–	+	+
18	+	+	–	–	+	–	50	+	+	+	–	–	–
	–	–	+	+	+	–		–	–	+	–	+	+
19	+	+	–	–	–	+	51	+	+	–	+	–	–
	–	–	+	+	–	+		–	–	–	+	+	+
20	+	+	–	–	+	+	52	+	+	+	+	–	–
	–	–	+	+	+	+		–	–	+	+	+	+
21	–	+	–	–	–	–	53	–	+	–	–	–	–
	+	–	+	+	–	–		+	–	–	–	+	+
22	+	+	–	–	+	–	54	–	+	+	–	–	–
	+	–	+	+	+	–		+	–	+	–	+	+
23	–	+	–	–	–	+	55	–	+	–	+	–	–
	+	–	+	+	–	+		+	–	–	+	+	+
24	–	+	–	–	+	+	56	–	+	+	+	–	–
	+	–	+	+	+	+		+	–	+	+	+	+
25	+	–	–	–	–	–	57	+	–	–	–	–	–
	–	+	+	+	–	–		–	+	–	–	+	+
26	+	–	–	–	+	–	58	+	–	+	–	–	–
	–	–	+	+	+	–		–	+	+	–	+	+
27	+	+	–	–	–	+	59	+	–	–	+	–	–
	–	+	+	+	–	+		–	+	–	+	+	+
28	+	–	–	–	+	+	60	+	–	+	+	–	–
	–	+	+	+	+	+		–	+	+	+	+	+
29	–	–	–	–	–	–	61	–	–	–	–	–	–
	+	+	+	+	–	–		+	+	–	–	+	+
30	–	–	–	–	+	–	62	–	–	+	–	–	–
	+	+	+	+	+	–		+	+	+	–	+	+
31	–	–	–	–	–	+	63	–	–	–	+	–	–
	+	+	+	+	–	+		+	+	–	+	+	+
32	–	–	–	–	+	+	64	–	–	+	+	–	–
	+	+	+	+	+	+		+	+	+	+	+	+

6. Case k = 6, d=3

The $\binom{63}{3}$ =39711 combination designs lead to two overall confounding patterns, namely, OP1: **f** = (7, 24, 24, 8) and OP2: **f** = (15, 0, 48, 0). None of the designs in the latter estimate all the 1fi and 2fi, and so they are dropped from further consideration. OP1 contains 39060 combination designs and 228 group patterns, but only three of these 228 provide a full set of 1fi and 2fi estimates. These three group patterns are displayed in Table 4.

Table 4. k = 6, d = 3; the group patterns that can estimate 1fi's and 2fi's (A indicates the number of combination designs available).

		u values for 1fi's and 2fi's	t	A
OP1	GP47	(1,1,2,2,2,3) & (1,1,1,1,1,1,1,2,2,2,2,2,2,3,3)	(3,4,4)	360
OP1	GP76	(1,1,1,2,2,3) & (1,1,1,1,1,1,2,2,2,2,2,2,2,3,3)	(3,3,4)	360
OP1	GP86	(1,1,1,2,2,2) & (1,1,1,1,1,1,2,2,2,2,2,2,3,3,3)	(3,3,3)	120

We see that GP47 (just) provides the best choice, provided we rate 1fi estimation more important than 2fi estimation. Such a design consists of two type 4 arrangements and a type 3 arrangement, i.e., **t** = (3, 4, 4) in Table 4. Thus an example of this best design type would be to choose a suitable type 3 arrangement to add to Table 3. The basic rule for such a choice is as follows. Choose a type 3 arrangement that has a 1fi blocking variable taken from each of the type 4 blocking variable sets and then add a third 1fi blocking variable that does not appear in either type 4 arrangement. Then, complete the blocking generators with two 2fi chosen from Table 1 to obtain a type 3 arrangement. To the arrangements (**5**, **6**, **14**, **24**, **34**) and (**3**, **4**, **16**, **26**, **56**) of Table 3, we can add, for example, the blocking generators (**5** or **6**, **3** or **4**, **1** or **2**). We choose here (**2**, **4**, **6**, *, *) and can then see from an inspection of type 3 arrangements of Table 1, that blocking arrangement No. 36, namely, (**2**, **4**, **6**, **15**, **35**) will be suitable. The runs of this arrangement are given in Table 5. The 96 run design obtained from the combination of Tables 3 and 5 turns out to be not only a best combination design for k = 6, d = 3, but also a best sequential combination (for the addition of 64 runs) to follow up an initial type 4 arrangement consisting of blocks 1-32 of Table 3. (This would be true whatever the choice of the type 3 arrangement, according to the indicated rule, was made.)

Table 5. An example of a best arrangement to add for k = 6, d = 3.

Block	Factor No. 1	2	3	4	5	6	Block	Factor No. 1	2	3	4	5	6
1	+	−	+	−	−	−	17	+	−	−	−	−	−
	−	−	−	−	+	−		−	−	+	−	+	−
2	+	+	+	−	−	−	18	+	+	−	−	−	−
	−	+	−	−	+	−		−	+	+	−	+	−
3	+	−	+	+	−	−	19	+	−	−	+	−	−
	−	−	−	+	+	−		−	−	+	+	+	−
4	+	+	+	+	−	−	20	+	+	−	+	−	−
	−	+	−	+	+	−		−	+	+	+	+	−
5	+	−	+	−	−	+	21	+	−	−	−	−	+
	−	−	−	−	+	+		−	−	+	−	+	+
6	+	+	+	−	−	+	22	+	+	−	−	−	+
	−	+	−	−	+	+		−	+	+	−	+	+
7	+	−	+	+	−	+	23	+	−	−	+	−	+
	−	−	−	+	+	+		−	−	+	+	+	+
8	+	+	+	+	−	+	24	+	+	−	+	−	+
	−	+	−	+	+	+		−	+	+	+	+	+
9	−	−	+	−	−	−	25	−	−	−	−	−	−
	+	−	−	−	+	−		+	−	+	−	+	−
10	−	+	+	−	−	−	26	−	+	−	−	−	−
	+	+	−	−	+	−		+	+	+	−	+	−
11	−	−	+	+	−	−	27	−	−	−	+	−	−
	+	−	−	+	+	−		+	−	+	+	+	−
12	−	+	+	+	−	−	28	−	+	−	+	−	−
	+	+	−	+	+	−		+	+	+	+	+	−
13	−	−	+	−	−	+	29	−	−	−	−	−	+
	+	−	−	−	+	+		+	−	+	−	+	+
14	−	+	+	−	−	+	30	−	+	−	−	−	+
	+	+	−	−	+	+		+	+	+	−	+	+
15	−	−	+	+	−	+	31	−	−	−	+	−	+
	+	−	−	+	+	+		+	−	+	+	+	+
16	−	+	+	+	−	+	32	−	+	−	+	−	+
	+	+	−	+	+	+		+	+	+	+	+	+

7. Case k = 6, d = 4

There are $\binom{63}{4}$ = 595665 combination designs, which split into the overall patterns of Table 6. The examination of the 178 group patterns that estimate all 1fis and 2fis is tedious, but a plot of $Sum_2(u)$ versus $Sum_1(u)$, defined in Section 3, reduces the number of group patterns to 45. A subsequent plot of $Std_2(u)$ versus $Std_1(u)$ quickly isolates, as superior choices, the three group pattern types shown in Table 7.

Table 6. Overall patterns for case k = 6, d = 4.

	f	A	B	C
OP1	(3, 16, 24, 16, 4)	546840	1254	173
OP2	(7, 0, 48, 0, 8)	9765	74	2
OP3	(7, 8, 24, 24, 0)	39060	228	3
Total		595665		178

A = No. of combination designs
B = No. of group patterns
C = No. of group patterns that estimate all 1fi and 2fi

Table 7. Superior group patterns for case k = 6, d = 4.

		u values for 1fis and 2fis [$Sum_1(u)$, $Sum_2(u)$] & [$Std_1(u)$, $Std_2(u)$]	t	A
OP1	GP172	(2,2,2,2,2,3) & (1,1,2,2,2,2,2,2,2,2,3,3,3,4,4) [13, 35] & [0.408, 0.900]	(3, 3, 3, 4)	360
OP2	GP 24	(2,2,2,2,2,2) & (2,2,2,2,2,2,2,2,2,2,2,2,4,4,4) [12, 36] & [0.000, 0.828]	(3, 3, 3, 3)	30
OP3	GP 23	(2,2,2,3,3,3) & (1,1,1,2,2,2,2,2,2,3,3,3,3,3,3)	(3, 4, 4, 4)	120

A = No. of combination designs

There is clearly some preference leeway here for saying which choice is "best", depending on how one weights the criteria that have been applied to choose these group patterns. (OP3, GP23) provides the most (15) 1fi internal estimates for example. However the estimation pattern for (OP2, GP24) shows less variability. We note that (OP3, GP23) has a **t**-pattern of (3, 4, 4, 4); this means that it provides a sequential possibility to our previous example, as follows. We can add a type 4 arrangement to any of the (3, 4, 4) patterns that arise in the k = 6, d = 3 case dealt with earlier. The added arrangement must be one that uses 1fi block generators that have *not* been used in type 4 arrangements already chosen. For example, suppose we had already used these arrangements of Table 1:

Arrangement 27	**1**	**3**	**5**	**26**	**46**	(Type 3)
Arrangement 42	**1**	**2**	**36**	**46**	**56**	(Type 4)
Arrangement 51	**3**	**4**	**16**	**26**	**56**	(Type 4).

The only possible addition is thus a Type 4 using (**5**, **6**, *, *, *), and we see from Table 1 that the only possibility is

Arrangement 56 **5 6 14 24 34** (Type 4).

Another example is derived from the specific design represented in Tables 3 and 5. This is made up of

Arrangement 56	**5**	**6**	**14**	**24**	**34**	(Type 4)
Arrangement 51	**3**	**4**	**16**	**26**	**56**	(Type 4)
Arrangement 36	**2**	**4**	**6**	**15**	**35**	(Type 3).

The only possible addition is thus a Type 4 using (**1**, **2**, *, *, *), and we see from Table 1 that the only possibility is

Arrangement 42 **1 2 36 46 56** (Type 4).

We recall that we can actually estimate all 1fi and 2fi with only three arrangements, as described in the k = 6, d = 3 case. However, *if we begin sequentially with a foldover type design with mirror image pairs of runs*, four arrangements are needed.

8. Case k = 6, d = 5

We omit the parallel detailed discussion for this case and simply present Table 8, similar in format to Table 7, showing some superior choices. An example of the design types shown in the first and second items in Table 8 is further explained in the k = 6, d = 6 sequential case which follows.

Table 8. Selected superior group patterns for case k = 6, d = 5.

		u values for 1fis and 2fis [Sum_1(u), Sum_2(u)] & [Std_1(u), Std_2(u)]	t	A
OP1	GP179	(3,3,3,3,3,3) & (2,2,2,2,2,2,2,2,2,4,4,4,4,4,4) [18, 42] & [0.000, 1.014]	(3,3,4,4,4)	360
OP4	GP105	(3,3,3,3,3,3) & (2,2,2,2,2,2,2,2,2,4,4,4,4,4,4) [18, 42] & [0.000, 1.014]	(3,3,4,4,4)	60
OP4	GP285	(2,3,3,3,3,3) & (2,2,2,2,2,2,3,3,3,3,3,4,4,4,4) [17, 43] & [0.408, 0.834]	(3,3,3,4,4)	360
OP5	GP 63	(2,2,3,3,3,3) & (2,2,2,2,3,3,3,3,3,3,3,3,4,4,4) [16, 44] & [0.516, 0.704]	(3,3,3,3,4)	90

A = No. of combination designs

9. Case k = 6, d = 6

We do not give a full enumeration of the $\binom{63}{6}$ = 67945521 possible combinations for this case. Instead we discuss the way in which it would be sensible to build up to a superior pattern sequentially, proceeding through the values d = 1, 2, 3, 4, 5 and 6, and making use of sensible choices as the design is enlarged stage by stage. We make use of the information already established in foregoing sections. A review of superior choices for successive d-values shows that it makes sense to consider the following pattern of **t**-values for d = 2 through 5: (4, 4), (3, 4, 4), (3, 4, 4, 4), and (3, 3, 4, 4, 4). When d = 3, we already can estimate all 1fi's and 2fi's. We would need to begin the pattern with an initial type 4 arrangement to achieve this sequence. The following example illustrates the types of choices to achieve such a pattern.

Suppose we make the initial choice of a type 4 arrangement as No. 56 in Table 1 with blocking generators (**5**, **6**, **14**, **24**, **34**). The second arrangement should be another type 4 that does not use the same 1fi blocking generators; for example No. 51, (**3**, **4**, **16**, **26**, **56**). The third, of type 3, should share one 1fi with each of the previous choices and have a third, different 1fi. There are eight possible choices, from which we select No. 36, (**2**, **4**, **6**, **15**, **35**). The fourth arrangement selected from Table 1 should be of type 4 and such that the 1fi generators were not used by the previously chosen type 4 estimators. The only possible choice is No. 42, (**1**, **2**, **36**, **46**, **56**). The 5th arrangement should be a type 3 arrangement that has the three 1fi block variables *not* used in the previous type 3 arrangement. This is (**1**, **3**, **5**, **26**, **46**), No. 27. (This is a superior design type, as indicated in Section 8.) The 6th arrangement can be the mirror-image pairing, No. 63, (**16**, **26**, **36**, **46**, **56**), which will enhance the estimation of main effects and provide a well balanced design overall. Note that d = 3 sets are sufficient to estimate all 1fi's and 2fi's. Table 9 summarizes the internal estimation patterns that result from all possible choices of the 6th arrangement, including No. 63, while holding the first five choices fixed as explained above.

10. Sequential Designs for k = 7 Factors

We comment only briefly on this case because, with $2^7 = 128$ runs in 64 blocks of size two, designs are very large. Moreover, three (d = 3) such sets are required to obtain estimates of all 1fi's and 2fi's, and such a design is *not* an extension of the best choices of two-set designs. Only one type of group pattern is suitable. This is obtained as follows: Pick a type 4 arrangement that has three 1fi generators, for example, ($\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_5$, $\mathbf{B}_6$) = (**1**, **2**, **3**, **47**, **57**, **67**). Add a second type 4 arrangement with one 1fi generator in common and two not, for example, (**1**, **4**, **5**, **27**, **37**, **67**). The third arrangement should have: (a) a 1fi generator that has appeared only once from the first arrangement; (b) a (different) 1fi generator that has appeared only once from the second arrangement; and (c) a 1fi generator not used before. An example is (**2** or **3**, **4** or **5**, **6** or **7**, *, *) which provides eight choices, with the asterisked generators completely defined by the specific choices made earlier. A table similar to Table 1, but for k = 7, shows that if (**2**, **4**, **6**, **7**, *, *) is selected, the appropriate design is generated by (**2**, **4**, **6**, **7**, **15**, **35**). The internal estimation pattern for this k = 7, d = 3 design is

$$(1,1,1,2,2,2,3) \; \& \; (1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,3,3,3).$$

Table 9. Summary information for sequential choice of a sixth arrangement added to numbers (56, 51, 36, 42, 27) for the case k = 6, d=6.

Add	Type	u values for 1fis and 2fis [$Sum_1(u)$, $Sum_2(u)$] & [$Std_1(u)$, $Std_2(u)$]
1–6	1	(3,3,3,3,3,4) & (2,2,2,2,2,2,3,3,3,4,4,4,4,5,5) [19, 47] & [0.408, 1.125]
7–8, 11–13, 15, 18, 20, 21	2	(3,3,3,3,4,4) & (2,2,2,2,2,3,3,3,3,4,4,5,5,5,5) [20, 50] & [0.516, 1.234]
9–10, 14, 16–17, 19	2	(3,3,3,3,4,4) & (2,2,2,3,3,3,3,3,3,4,4,4,4,5,5) [20, 50] & [0.516, 0.976]
22–26, 31–32, 37–41	3	(3,3,3,4,4,4) & (2,2,2,2,3,3,3,3,3,4,4,5,5,5,5) [21, 51] & [0.548, 1.183]
28–30, 33–35	3	(3,3,3,4,4,4) & (2,2,3,3,3,3,3,3,3,4,4,4,4,5,5) [21, 51] & [0.548, 0.910]
43, 45, 48, 50, 52, 55	4	(3,3,4,4,4,4) & (2,2,2,2,2,3,3,3,3,4,4,5,5,5,5) [22, 50] & [0.516, 1.234]
44, 46–47, 49, 53–54	4	(3,3,4,4,4,4) & (2,2,2,3,3,3,3,3,3,4,4,4,4,5,5) [22, 50] & [0.516, 0.976]
57–62	5	(3,4,4,4,4,4) & (2,2,2,2,2,2,3,3,3,4,4,4,4,5,5) [23, 47] & [0.408, 1.125]
63	6	(4,4,4,4,4,4) & (2,2,2,2,2,2,2,2,2,4,4,4,4,4,4) [24, 42] & [0.000, 1.014]

11. Sequential Designs for k = 8 Factors

Four sets of 256 runs in 128 blocks of two are needed to estimate all 1fi's and 2fi's, although all 1fi and all but one 2fi can be estimated with three sets of 256 runs in 128 blocks of two. An example is the following. We first combine two type 5 combinations with one common 1fi blocking generator, for example, ($\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_5$, $\mathbf{B}_6$, $\mathbf{B}_7$) = **(1, 2, 3, 48, 58, 68, 78)** and **(1, 4, 5, 28, 38, 68, 78)**. The third choice is a type 5 arrangement selected from possibilities (**2** or **3**, **4** or **5**, **6** or **7** or **8**, *, *, *, *) using, in other words, one of the non-common 1fi's from each of the first two selections, one 1fi not used before, and the asterisked values determined from the first three choices. A suitable choice is, for example, (**2**, **4**, **6**, **18**, **38**, **58**, **78**). This triple of arrangements leaves one 2fi unestimated. There are many possibilities for the choice of a fourth arrangement. The ones that provide the most (22) internal 1fi estimates are those of type 7, for example, (**8**, **17**, **27**, **37**, **47**, **57**, **67**). The pattern of internal estimates for this quadruple combination is then:

(2,2,2,3,3,3,3,4) & (1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3,3,3)

and $Sum_1(u) = 22$, $Sum_2(u) = 52$, $Std_1(u) = 0.707$ and $Std_2(u) = 0.756$. Further examples of, and additional details about, designs of the type discussed in this article are given in Yang[5] (2002).

Acknowledgements

We gratefully acknowledge the help of Professor Gregory Reinsel in providing additional hard disc capacity for our computing at the University of Wisconsin, and we mourn his death at age 56 on May 5, 2004. We also thank a referee for a number of helpful suggestions.

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