MODELING TRANSITION: NEW SCENARIOS, SYSTEM SENSITIVITY AND FEEDBACK CONTROL

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The problem of controlling or delaying transition to turbulence in shear flows has been the subject of numerous papers over the past twenty years. Although there is no single mathematical framework that describes transition for all possible flows, new approaches to (non-classical) linear hydrodynamic stability theory have provided tremendous improvements in the fundamental understanding of this process. In particular, ideas from robust control theory have been used to develop new linear theories in an attempt to explain some of the failures of classical linear hydrodynamic stability theory. This mostly linear theory has produced some new scenarios that may be exploited in control design and analysis. In addition, these theories have been tested on low-dimensional model problems with mixed success. In this paper, we review some of these linear theories and discuss the roles of uncertainty, system sensitivity and modern feedback control in the transition problem. A boundary control problem defined by Burgers' equation is employed to illustrate how the distributed parameter control theory can be used as a framework for computing feedback functional gains that provide practical guidance in sensor/actuator placement. Low-dimensional models are employed to explain the basic ideas and to illustrate how one can employ bifurcation analysis to predict transition. These examples are also used to show how feedback controllers can delay transition and alter the global dynamics of such systems.

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1. Introduction and Motivating Problem

Designing feedback controllers for active control of fluid flows has received considerable attention from the research community (Gad-el-Hak, 1996, 2000; Gunzburger, 1995; Gunzburger *et al.*, 1999; Sritharan, 1995, 1998 and the references therein). Although the basic problem has been the subject of many experimental and computational studies, much work remains to be done on the development of a "practical theory" (and the corresponding computational tools) that can be used to attack realistic 3D problems at high Reynolds numbers. However, recent advances in hydrodynamic stability theory combined with new mathematical and computational tools offers the potential for breakthroughs on this problem. In this paper, we discuss a portion of this work and indicate some of the mathematical and computational challenges that remain to be addressed.

The problem of controlling an incompressible viscous fluid in a given domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) by forcing on the boundary $\partial\Omega = \Gamma$ may be modeled by the Navier–Stokes equations. In many important flow control problems, the domain Ω is unbounded (channel and external flows). However, in order to keep the discussion as simple as possible, we focus on regular bounded domains and only briefly indicate how the unbounded domain problem may differ. In particular, let $\tilde{\mathbf{w}}(t,x)$, $p(t,x)$ and $\nu = \frac{1}{Re} > 0$ denote the velocity field, pressure field and kinematic viscosity, respectively. We consider the controlled Navier–Stokes equations given by

$$\tilde{\mathbf{w}}_t(t,x) + (\tilde{\mathbf{w}}(t,x) \cdot \nabla)\tilde{\mathbf{w}}(t,x) = \nu\Delta\tilde{\mathbf{w}}(t,x) - \nabla p(t,x), x \in \Omega, \quad t > 0, \tag{1.1}$$

$$\nabla \cdot \tilde{\mathbf{w}}(t,x) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.2}$$

with initial data

$$\tilde{\mathbf{w}}(0,x) = \tilde{\mathbf{w}}_0(x), \quad x \in \Omega, \tag{1.3}$$

and "controlled" boundary condition

$$\tilde{\mathbf{w}}(t,x) = \tilde{\mathbf{u}}(t,x), \quad x \in \partial\Omega, \quad t > 0. \tag{1.4}$$

Conservation of mass requires that the boundary control $\tilde{\mathbf{u}}(t,x)$ satisfy

$$\int_\Gamma \tilde{\mathbf{u}}(t,x) \cdot \tilde{\mathbf{n}}(x)\, d\Gamma = 0, \tag{1.5}$$

where $\tilde{\mathbf{n}}(x)$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$. In many practical situations, the control has the form

$$\tilde{\mathbf{u}}(t,x) = \sum_{i=1}^{m} u_i(t)\tilde{\mathbf{g}}_i(x), \tag{1.6}$$

where for each $i = 1, 2, \ldots, m$,

$$\int_\Gamma \tilde{\mathbf{g}}_i(x) \cdot \tilde{\mathbf{n}}(x)\, d\Gamma = 0$$

and hence the control constraint (1.5) is satisfied automatically.

Let $\tilde{\mathbf{W}}(x)$ and $P(x)$ denote a steady state (equilibrium) solution to the uncontrolled problem

$$\begin{aligned} (\tilde{\mathbf{w}}(x) \cdot \nabla)\tilde{\mathbf{w}}(x) &= \nu\Delta\tilde{\mathbf{w}}(x) - \nabla p(x), \quad x \in \Omega, \\ \nabla \cdot \tilde{\mathbf{w}}(x) &= 0, \quad x \in \Omega, \\ \tilde{\mathbf{w}}(x) &= \mathbf{0}, \quad x \in \partial\Omega \end{aligned} \tag{1.7}$$

and define the velocity and pressure perturbation fields by $\tilde{\mathbf{v}}(t,x) = \tilde{\mathbf{W}}(x) - \tilde{\mathbf{w}}(t,x)$ and $q(t,x) = P(x) - p(t,x)$, respectively. The controlled perturbation equations become

$$\tilde{\mathbf{v}}_t + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} = \nu\Delta\tilde{\mathbf{v}} - ((\tilde{\mathbf{W}} \cdot \nabla)\tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{W}}) - \nabla q, \quad x \in \Omega, \quad t > 0, \tag{1.8}$$

where

$$\nabla \cdot \tilde{\mathbf{v}}(t,x) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.9}$$

$$\tilde{\mathbf{v}}(0,x) = \tilde{\mathbf{v}}_0(x), \quad x \in \Omega, \tag{1.10}$$

and

$$\tilde{\mathbf{v}}(t,x) = \tilde{\mathbf{u}}(t,x), \quad x \in \partial\Omega, \quad t > 0. \tag{1.11}$$

1.1. *A state space formulation*

Here we briefly outline the steps one takes to construct a (rigorous) state space model for the system of partial differential equations from above. Details may be found in the references Barbu and Triggiani (2004), Barbu *et al.* (2005), Fursikov (2004), Sritharan (1990, 1998), Temam (1984, 1988). The important function spaces are defined by

$$\mathbf{H} = \left\{\tilde{\mathbf{v}} \in L^2(\Omega;\mathbb{R}^n) : \nabla \cdot \tilde{\mathbf{v}} = 0,\ \tilde{\mathbf{v}} \cdot \tilde{\mathbf{n}} = 0,\ x \in \Omega\right\}$$

and

$$\mathbf{V} = \left\{\tilde{\mathbf{v}} \in H^1(\Omega;\mathbb{R}^n) : \nabla \cdot \tilde{\mathbf{v}} = 0,\ \tilde{\mathbf{v}} = 0,\ x \in \Omega\right\}.$$

Here,

$$L^2(\Omega;\mathbb{R}^n) = \left\{\tilde{\mathbf{v}} : \Omega \to \mathbb{R}^n : \int_\Omega \|\tilde{\mathbf{v}}(\mathbf{x})\|^2_{\mathbb{R}^n}\, dx < +\infty\right\}$$

is the usual Lebesgue space of square integrable vector functions and for $m > 0$, $H^m(\Omega;\mathbb{R}^n)$ is the Sobolev space of vector functions whose distributional derivatives of order up to m belong to $L^2(\Omega;\mathbb{R}^n)$. If

$$\mathcal{V} = \{\tilde{\mathbf{v}} : \Omega \to \mathbb{R}^n : \tilde{\mathbf{v}} \in C_0^\infty(\Omega), \nabla \cdot \tilde{\mathbf{v}} = 0\},$$

then one can show (Sritharan, 1990; Temam, 1984) that $\mathbf{H}$ is the closure of $\mathcal{V}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\mathbf{V}$ is the closure of $\mathcal{V}$ in $H_0^1(\Omega;\mathbb{R}^n)$.

Let $P_H : L^2(\Omega;\mathbb{R}^n) \to \mathbf{H}$ be the orthogonal projection onto $\mathbf{H}$ (the Leray projection) and define the Stokes operator $A_S : D(A_S) \subset \mathbf{H} \to \mathbf{H}$ on the domain

$$D(A_S) = H^2(\Omega;\mathbb{R}^n) \cap \mathbf{V}$$

by

$$A_S\tilde{\mathbf{v}} = P_H \Delta \tilde{\mathbf{v}}, \quad \tilde{\mathbf{v}} \in D(A_S).$$

It follows (see p. 14 in Sritharan, 1990) that the dense and continuous embeddings

$$D(A_S) \subset \mathbf{V} \subset \mathbf{H} = \mathbf{H}' \subset \mathbf{V}' \subset [D(A_S)]'$$

are compact, where the prime denotes the dual space. Let $A_0 : \mathbf{V} \to \mathbf{V}'$ be the lifting of $A_S : D(A_S) \subset \mathbf{H} \to \mathbf{H}$ to $\mathbf{V}$ and define the linear operator $\mathcal{R} : \mathbf{V} \to \mathbf{V}'$ by

$$\mathcal{R}\tilde{\mathbf{v}} = -P_H((\tilde{\mathbf{W}} \cdot \nabla)\tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{W}})$$

and hence the Oseen operator $A(\nu) : \mathbf{V} \to \mathbf{V}'$ given by

$$A = A(\nu) = \nu A_0 + \mathcal{R}$$

is the linearized operator corresponding to the steady state solution given by (1.7). The nonlinear operator $F : \mathbf{V} \to \mathbf{V}'$ is defined by

$$[F(\tilde{\mathbf{v}})](\tilde{\varphi}) = \sum_{i,j=1}^{n} \int_{\Omega} v_i \frac{\partial v_j}{\partial x_i} \varphi_j.$$

In order to complete the model, one must define the control input operator. The basic idea is to use the Lions structure (Lions, 1969) (see also Barbu *et al.*, 2005; Bensoussan *et al.*, 1992a). In particular, one "lifts" $A : \mathbf{V} \to \mathbf{V}'$ to $A_1 : \mathbf{H} \to [D(A^*)]'$ by

$$A_1 \tilde{\mathbf{v}} = \tilde{\mathbf{w}} \quad \text{if and only if} \quad \tilde{\mathbf{w}}(\tilde{\varphi}) = \langle \tilde{\mathbf{v}}, A^* \tilde{\varphi} \rangle_{\mathbf{H}}$$

and defines $B : L^2(\Gamma; R^n) \to [D(A^*)]'$ by

$$B\tilde{\mathbf{g}} = -A_1 D \tilde{\mathbf{g}}$$

where D is the Dirichlét map $D : L^2(\Gamma; \mathbb{R}^n) \to H^{1/2}(\Omega; \mathbb{R}^n) \cap \mathbf{H}$. The state space model for the boundary control problem (1.8)–(1.11) now has the (very) weak formulation

$$\frac{d}{dt}\tilde{\mathbf{v}}(t) = (\nu A_0 + \mathcal{R})\tilde{\mathbf{v}}(t) + F(\tilde{\mathbf{v}}(t)) + B\tilde{\mathbf{u}}(t) \in [D(A^*)]'. \tag{1.12}$$

The system (1.12) has some important features that are typical of many flow control problems. Note that although the Stokes operator νA_0 is self-adjoint, the Oseen operator $A = \nu A_0 + \mathcal{R}$ can be highly non-normal if $\mathcal{R} \neq 0$. Therefore, the linearized control system

$$\frac{d}{dt}\tilde{\mathbf{v}}(t) = (\nu A_0 + \mathcal{R})\tilde{\mathbf{v}}(t) + B\tilde{\mathbf{u}}(t), \quad \tilde{\mathbf{v}}(0) = \tilde{\mathbf{v}}_0 \tag{1.13}$$

can be highly sensitive to parameter variations, initial data and inputs $u(\cdot)$. Computational algorithms for linear control problems with a non-normal A operator require special effort. In addition, the nonlinear operator is conservative in the sense that

$$\langle F(\tilde{\mathbf{v}}), \tilde{\mathbf{v}} \rangle_{\mathbf{V}} = 0 \tag{1.14}$$

for all $\tilde{\mathbf{v}} \in \mathbf{V}$ and this special structure tends to be highly sensitive to disturbances (Burns, 2003). Sensitivity and non-normality of the linear system lead to many difficulties in the corresponding control problem.

1.2. *A feedback control problem*

We consider the feedback control problems for the Navier–Stokes system

$$\frac{d}{dt}\tilde{\mathbf{v}}(t) = (\nu A_0 + \mathcal{R})\tilde{\mathbf{v}}(t) + F(\tilde{\mathbf{v}}(t)) + B\tilde{\mathbf{u}}(t) \tag{1.15}$$

and the corresponding linearized system

$$\frac{d}{dt}\tilde{\mathbf{v}}(t) = (\nu A_0 + \mathcal{R})\tilde{\mathbf{v}}(t) + B\tilde{\mathbf{u}}(t), \tag{1.16}$$

where again $\nu = \frac{1}{Re}$. It is important to note that there are many "feedback control problems" (linear feedback, nonlinear feedback, LQR, LQG, H^∞, Min-Max, etc.) and several technical approaches to each of these problems. Obviously, one cannot cover all these areas in a single paper, so we focus on a feedback stabilization problem that has a direct connection to transition control.

The basic scenario is that there is a critical Reynolds number, Re_{crit}, such that if $Re < Re_{\text{crit}}$, the open-loop linearized operator $A(Re) = (\nu A_0 + \mathcal{R})$ generates an exponentially stable $\mathcal{C}_0$-semigroup $S(t, Re) = e^{A(Re)t}$ satisfying

$$\|S(t, Re)\|_{\mathbf{H}} \leq Me^{-\gamma t}$$

where $M = M(Re) \geq 1$ and $\gamma = \gamma(Re) > 0$. If $Re > Re_{\text{crit}}$, then there is an initial condition $\tilde{\mathbf{v}}(0) = \tilde{\mathbf{v}}_0$ such that

$$\lim_{t \to +\infty} \|S(t, Re)\tilde{\mathbf{v}}_0\|_{\mathbf{H}} = +\infty,$$

i.e. $S(t, Re)$ is unstable. In this case, this initial condition may produce transition in the full nonlinear equation. It is well known that the situation is much more complex and transition can occur for values much lower than Re_{crit}. In fact, if one considers the $A(Re) = (\nu A_0 + \mathcal{R})$ linearization about the plane Couette flow, then $Re_{\text{crit}} = +\infty$ and the linearized operator is always stable. Attempts to understand and explain this phenomena has motivated research in classical linear hydrodynamic stability theory for more than a century (see the excellent summary in Drazin, 2002).

Modern hydrodynamic stability theory based on robustness, pseudospectrum and sensitivity analysis has provided a much better understanding of this process and generated new scenarios to explain transition (Drazin, 2002; Schmid and Henningson, 2001). A fundamental new idea in these approaches is that because $A(Re) = (\nu A_0 + \mathcal{R})$ is non-normal and $M = M(Re)$ grows like $[Re]^\theta$ where $\theta > 1$, a small initial data can produce large transient growth due entirely to the linear part of the equation.

Therefore, even when $A(Re) = (\nu A_0 + \mathcal{R})$ is stable, once this transient growth becomes "large enough" the nonlinear terms become important and nonlinear "mixing" leads to transition.

Clearly, there are still gaps in building a complete theory for such scenarios, but the basic idea helps understand why and how feedback control might be used to delay or prevent transition. For example, assume that one can find a linear *feedback gain operator*

$$K : D(K) \subseteq \mathbf{H} \to \mathbf{U} = L^2(\Gamma; R^n) \tag{1.17}$$

such that the closed-loop operator

$$A_{CL}(Re) \triangleq A(Re) - BK = (\nu A_0 + \mathcal{R} - BK) \tag{1.18}$$

generates a closed-loop semigroup $S_{CL}(t, Re) = e^{A_{CL}(Re)t}$ satisfying

$$\|S_{CL}(t, Re)\|_{\mathbf{H}} \leq e^{-\hat{\gamma}t}$$

with $\hat{\gamma} > 0$. In this case, the corresponding closed-loop nonlinear Navier–Stokes equations

$$\frac{d}{dt}\tilde{\mathbf{v}}(t) = (\nu A_0 + \mathcal{R} - BK)\tilde{\mathbf{v}}(t) + F(\tilde{\mathbf{v}}(t)) \tag{1.19}$$

would be "monotonically stable" as defined on p. 5 in Schmid and Henningson (2001). It is important to note that this does not automatically imply asymptotic stability of the base flow. One must also establish that the zero equilibrium $\tilde{\mathbf{v}}_0 = 0$ for the nonlinear closed-loop system (1.19) is stable. In particular, one needs to show that for $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\tilde{\mathbf{v}}_0\|_{\mathbf{H}} < \delta$ implies that:

1. there exists a unique solution $\tilde{\mathbf{v}}(t, \tilde{\mathbf{v}}_0)$ to (1.19) with $\tilde{\mathbf{v}}(0, \tilde{\mathbf{v}}_0) = \tilde{\mathbf{v}}_0$ defined for all $t > 0$,
2. $\|\tilde{\mathbf{v}}(t, \tilde{\mathbf{v}}_0)\| \leq \varepsilon$ for all $t > 0$.

Producing a monotonically stable closed-loop system may prove difficult, but it is not impossible for certain flows (Kang and Ito, 1994) and there are recent results that imply one can stabilize a 3D nonlinear flow with linear feedback even when $Re > Re_{\text{crit}}$. Fursikov (2004) has some very interesting results along this line. Also, Barbu, Lasiecka and Triggiani (see Theorem 6.1 in Barbu *et al.*, 2005) have proven the existence of a linear feedback operator $K : H^{1/2+\varepsilon}(\Omega; R^n) \cap \mathbf{H} \to \mathbf{U} = L^2(\Gamma; R^n)$ such that the closed-loop Navier–Stokes equation (1.19) is exponentially asymptotically stable. Moreover, this feedback gain operator can be "computed" by solving an abstract Riccati operator equation. Although the results in Barbu *et al.*

(2005) only apply to problems on bounded domains, these results provide insight and some promise that one can deal with certain exterior flows. In addition, there is considerable numerical and experimental evidence that the same is true for channel flows (Choi *et al.*, 1994; Cortelezzi, *et al.*, 2001; Wang *et al.*, 1992; Wiltse and Glezer, 1993).

One benefit of this form of a feedback law is that it can generate useful spatial information about sensor and actuator placement (Burns and King, 1994, Burns *et al.*, 1995, 1998). Under suitable conditions, it is possible to represent the gain operator as an integral of the form

$$\tilde{\mathbf{u}}(t,x) = -K\tilde{\mathbf{v}}(t,\cdot) = -\int_\Omega k(x,y)\tilde{\mathbf{v}}(t,y)\,dy, \tag{1.20}$$

where the kernel is called the *functional gain*. Moreover, there are many practical cases where the functional gain has highly localized support. In some cases (LQR boundary control), the functional gain may have local support so that practical information comes from being able to compute $k(x,y)$.

This will be illustrated by the simple Burgers' equation below and has been applied to a wide variety of distributed parameter control problems in Bewley (2001), Burns and King (1994), and Burns *et al.* (1995, 1998). In view of these results, we focus on the use the linearized control system (1.16) to design feedback controllers and then apply this linear control to the full nonlinear system (1.15). This is a standard approach and a good "first step" in any design process.

It is desirable to have some basic knowledge about the open-loop uncontrolled dynamics for both the nonlinear and linear systems. When Ω is smooth and bounded, much is known about the spectrum of the linearized Oseen operator $A = \nu A_0 + \mathcal{R}$ and the stability of the linearized problem. In particular, one has the following lemma (see p. 1448 in Barbu and Triggiani, 2004 and Theorem 3.6 in Sritharan, 1990).

Lemma 1: *If Ω is bounded with smooth boundary, then the Oseen operator $A = \nu A_0 + \mathcal{R}$ has compact resolvent and generates an analytic semigroup on $\mathbf{H}$. The spectrum of A is only point spectrum (i.e. $\sigma(A) = \sigma_p(A)$), all the eigenvalues have finite geometric multiplicity, accumulate only at $-\infty$ and there are at most a finite number of eigenvalues $\lambda_i, i = 1, 2, \ldots, m$ satisfying $\mathrm{Re}(\lambda_i) \geq 0$.*

Lemma 1 implies that in order to stabilize the linearized flow, one needs only to "move" a finite number of eigenvalues to the left hand complex plane. This fact is the basis of much of the work in the paper of Barbu and Triggiani (2004) and has been extended to boundary control in Barbu *et al.* (2005). However, for unbounded domains, $A = \nu A_S + \mathcal{R}$ can have a non-empty essential spectrum. For example, for certain exterior domains the spectrum of $A = \nu A_S + \mathcal{R}$ has the form $\sigma(A) = \sigma_p(A) \cup \Lambda(Re)$ where again there are at most a finite number of eigenvalues λ_i, $i = 1, 2, \ldots, m$ satisfying $\mathrm{Re}(\lambda_i) \geq 0$ and the essential spectrum lies inside the parabolic region

$$\Lambda(Re) = \left\{\lambda = \alpha + \beta i : \alpha \leq 0,\ \beta^2 \leq -Re\alpha\right\}$$

(see Theorem 3.11 in Sritharan, 1990). Observe that the parabolic region $\Lambda(Re)$ opens up as $Re \to +\infty$ and hence the non-zero essential spectrum can move closer to the imaginary axis. This can impact sensitivity and control design. The case of channel flows is again different because the boundary is not compact.

1.3. *Hydrodynamic stability and feedback control*

We close this section by noting some recent work that has considerable impact on the control problem. Also, we point out some important technical issues that need to be addressed when one considers control problems governed by highly sensitive nonlinear systems with non-normal linearizations. A detailed discussion of these issues along with several illustrative examples may be found in Burns (2003).

In 1880, Lord Rayleigh (Rayleigh, 1880) wrote a fundamental paper on the stability of fluid motions and since then the field of linear hydrodynamic stability theory has been a centerpiece of classical fluid dynamics. One hundred years later, beginning in the late 1980's and early 1990's, Henningson, Reddy, Schmid, Trefethen and co-workers begin to develop a new approach to hydrodynamic stability. This modern approach is still based on a linear theory, but differs from classical linear hydrodynamic stability in that singular values and pseudo-spectrum play the key role in their work. The observation that linearization about a non-trivial laminar flow leads to a non-normal linear problem is the key to this theory. The references Baggett *et al.* (1995), Henningson (1987), Henningson *et al.* (1993), Henningson and Reddy (1994), Reddy and Henningson (1993), Reddy *et al.* (1993), Schmid and Henningson (1994), Schmid *et al.* (1996), Schmid (2000), Trefethen *et al.* (1993) provide the foundations for this work and the recent book by Schmid and Henningson (2001) provides an excellent and modern treatment of this area. Much of this work (certainly not all) focuses on the idea that small (but very specific) "initial" data can produce large transient growth due to the non-normality of linear operator until the nonlinear terms become "important" and produce transition. Considerable effort has been devoted to the problem of identifying the specific initial data (Tollmien–Schlichting waves, oblique waves, etc.) and the corresponding threshold amplitudes that generate this initial large transient growth.

In the mid 1990's, a group of researchers including Bamieh, Dahleh, Farrell, Ioannou and co-workers developed a similar linear theory based on ideas from robust control theory. In addition to identifying amplitude thresholds for specific initial data, this effort focused on possible input disturbances that also get magnified due to the non-normality of the operator $A = \nu A_0 + \mathcal{R}$ (Bamieh and Dahleh, 1998, 2001; Butler and Farrell, 1992; Farrell and Ioannou, 1993; Trefethen, 1997). Bamieh and Dahleh suggested that for channel flows, an unmodeled disturbance could come from extremely small wall-roughness or forced boundary conditions and this disturbance could be amplified by the non-normal linear system leading to transition. Moreover, this observation also suggests that boundary control has the potential to significantly delay or eliminate transition in a wide variety of shear flows (Choi *et al.*, 1994; Cortelezzi and Speyer, 1998; Cortelezzi *et al.*, 2001; Joshi *et al.*, 1997). Almost all of this work focuses on linear input-output theory and the role that the nonlinear term plays in this scenario is not fully understood.

Although the basis for both scenarios is linear stability analysis, the ideas put forth by these groups have proven to be very useful and have been used to more accurately predict critical transition numbers. However, as noted in Schmid and Henningson (2001), in order to provide a complete description of the total transition process, one must develop a framework that can be used to analyze the precise role that the nonlinear term plays in the transition mechanism. At this time, it is probably not possible to develop a theory directly applicable to the 3D Navier–Stokes equations. However, even simple 1D partial differential equations can exhibit extreme sensitivity to boundary disturbances. For example, Burgers' equation is known to be supersensitive to changes in Dirichlét boundary conditions (Garbey and Kaper, 2000; LaForgue and O'Malley, 1993) and hyper-sensitive to changes in the Neumann boundary conditions (Allen *et al.*, 2002). In Allen *et al.*, (2002), it is shown that the specific nonlinearity, combined with hyper-sensitive boundary conditions can produce an unexpected transition from a small initial state to a large steady state solution of a "nearby" problem. This problem can be analyzed completely and it is possible to see the exact cause of the breakdown. This extreme sensitivity to boundary disturbances makes Burgers' equation a good infinite-dimensional model for testing transition scenarios. In addition, we shall use Burgers' equation to illustrate the computation of functional gains.

We also present some bifurcation analysis and control results for low order finite-dimensional models found in the literature (Henningson *et al.*, 1993; Henningson and Reddy, 1994; Henningson, 1996; Schmid and Henningson, 2001; Trefethen *et al.*, 1993; Waleffe, 1995a). These "simple" models can provide insight into possible transition scenarios and provide useful cases to illustrate the power of linear feedback control. There are numerous control-related issues that need to be addressed before one can confidently attack realistic flow control problems. Two issues are specific to problems with the structure described above.

1. Small variations in the non-normal operator $(\nu A_0 + \mathcal{R})$ can produce dramatic changes in the stability of the linear system and radical changes in the (global) dynamics can be produced by small perturbations of the nonlinear term (Allen *et al.*, 2002; Burns, 2003; Burns and Singler, 2005). This high sensitivity and lack of robustness can be used to explain certain routes to transition (Henningson *et al.*, 1993; Henningson and Reddy, 1994; Henningson, 1996; Waleffe *et al.*, 1993; Waleffe, 1995a). In terms of control, the open-loop system is not robust with respect to uncertainties and even a stable system can proceed to transition when subjected to a small change. Therefore, one objective of the feedback control might be to robustly stabilize the system and it is not always clear how best to approach this problem even for the simple finite-dimensional models.
2. Since A is not normal, then one must be careful when developing approximations for feedback control or optimization of such systems in order to ensure convergence of the design. In addition, there are other important computations (e.g., power density functions and pseudo-spectra) that require careful approximations of the adjoint A^* and this is a non-trivial problem which is often ignored (Banks and Burns, 1978; Banks and Kunisch, 1982; Banks and Ito, 1997; Burns *et al.*, 1988; Burns, 2003; Burns *et al.*, 2003). It is important to develop accurate and rigorous algorithms so that the numerical control laws can be used to address practical design questions such as where to locate sensor/actuator pairs (Banks and Ito, 1997; Burns and King, 1994; Burns *et al.*, 1995, 1998).

The uncontrolled system (1.12) with conservative nonlinearity satisfying (1.14) falls into a general class of distributed parameter systems considered by Ghidaglia (1984) and Temam (1988). This framework provides a powerful tool for investigating existence, uniqueness and regularity results for the nonlinear Navier–Stokes equations. In addition, this framework can be used to investigate control design and sensitivity for such systems (Burns, 2003). However, several theoretical questions remain open and certain "gaps" exist when the boundary is unbounded or the problem is 3D (i.e. when $n = 3$). We turn now to a general distributed parameter control problem that is motivated by the structure of the flow control problem above.

2. A Mathematical Framework

Navier–Stokes equations, Burgers' equation and most of the proposed low-order finite-dimensional models of transition fall into a general framework first developed by Ghidaglia (1984). The basic structure is summarized as follows. Let W, V and Z be separable Hilbert spaces satisfying

$$W \subset V \subset Z = Z' \subset V' \subset W',$$

where W', V' and Z' are the dual spaces of W, V and Z, respectively. We assume that the injections are continuous and each space is dense in the following one. Let $a(\cdot,\cdot)$ be a symmetric bilinear form on V satisfying

$$a(v,v) \geq \gamma \left\| v \right\|_V \tag{2.1}$$

for some $\gamma > 0$. Let $\nu = \frac{1}{Re} > 0$ and define the associated isomorphism $A_0 : V \to V'$ by $[A_0 z]v = -a(z,v)$. The (unbounded) self-adjoint restriction operator $A_S : D(A_S) \subseteq V \subset Z \to Z$ is defined on $D(A_S) = \{z \in V : A_0 z \in Z\}$ by $A_S z = A_0 z$ for all $z \in D(A_S)$. Note that for $z \in D(A_S)$ and $v \in Z$, $a(z,v) = -\langle A_S z, v\rangle$. We assume that the self-adjoint linear operator νA_S generates a $\mathcal{C}_0$-semigroup on Z. The input operator $B : U \to W'$ is a bounded linear operator mapping the control space U into a space W' containing Z. Although there are considerably technical issues to be addressed, roughly speaking the Dirichlét boundary control problem requires that $W = D(A_S^*)$ and for the Neumann boundary control problem, one sets $W = V$. When control is applied through internal "body forces", then W is not required and $B : U \to Z$ is a bounded linear operator with range in Z.

Let $\mathcal{R} : V \to V'$ be a bounded linear operator which maps $D(A_S)$ into Z. Thus, we can define the (possibly unbounded) linear operator $\mathcal{R}_S : D(\mathcal{R}_S) = D(A_S) \to Z$ to be the restriction of $\mathcal{R}$ to $D(A_S)$. In addition, we assume that $[\nu A_S + \mathcal{R}_S]$ generates a $\mathcal{C}_0$-semigroup $S(t) : Z \to Z$ on Z and note that $A = [\nu A_0 + \mathcal{R}]$ is the standard extension of $[\nu A_S + \mathcal{R}_S]$ to V. Thus, we follow the standard abuse of notation and say that $A = [\nu A_0 + \mathcal{R}]$ generates $S(t)$ on Z. The nonlinear operator is defined by a trilinear form. Therefore, let $f : V \times V \to V'$ be a continuous bilinear operator with the property that f maps $D(A_S) \times D(A_S)$ into Z. The nonlinear operator $F : V \to V'$ is defined by $F(v) = f(v, v)$. Given the framework above, we consider the abstract control system

$$\dot{z}(t) = [\nu A_0 + \mathcal{R}]z(t) + F(z(t)) + Bu(t), \tag{2.2}$$

with initial data

$$z(0) = z_0 \in Z. \tag{2.3}$$

Although this system is quite general, there are reasonable control systems that are not covered by this framework and much remains to be done to complete the theory. The important point for this paper is that nonlinear term F is often conservative. In particular, if $\langle\langle \cdot, \cdot \rangle\rangle : V' \times V \to C$ denotes the duality map defined by

$$\langle\langle u, v \rangle\rangle = u(v),$$

and

$$\langle\langle F(v), v \rangle\rangle = [F(v)]v = [f(v, v)]v = \langle\langle f(v, v), v \rangle\rangle = 0, \tag{2.4}$$

for all $v \in V$, then we say that F is *conservative*. When the linear part of the system is stable but near an unstable operator and the corresponding nonlinear term is conservative, then the nonlinear system can be hypersensitive to small disturbances. This plays a central role in a recent scenario presented in Burns and Singler (2005). Before discussing possible transition mechanisms, we present some basic results on linear feedback control.

2.1. *The LQR control problem*

We now consider specific feedback control problems for the distributed parameter system

$$\dot{z}(t) = [\nu A_0 + \mathcal{R}]z(t) + F(z(t)) + Bu(t), \quad z(0) = z_0, \tag{2.5}$$

and the corresponding linearized system

$$\dot{z}(t) = [\nu A_0 + \mathcal{R}]z(t) + Bu(t), \quad z(0) = z_0. \tag{2.6}$$

In order to keep the discussion as simple as possible, we limit ourselves to the case with bounded input operator $B : U \to Z$. We use the linearized system (2.6) to design a feedback controller and then apply this linear control to the full nonlinear system (2.5).

The linear system (2.6) is *exponentially stabilizable* if there is a bounded linear operator $K : Z \to U$ such that the closed-loop operator $[\nu A_0 + \mathcal{R}] - BK$ generates an exponentially stable $\mathcal{C}_0$-semigroup $S_{CL}(t)$ (for the basic definitions, see Curtain and Zwart, 1995; Kato, 1976; Lions, 1969). Let $Q : Z \to Z$ be a self-adjoint bounded linear operator, $r > 0$ and define the cost function

$$J(u, z_0) = \int_0^{+\infty} \{\langle Qz(t), z(t)\rangle_Z + r\,\langle u(t), u(t)\rangle\}\,dt, \tag{2.7}$$

where $z(t)$ is the mild solution of (2.6) defined by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,ds, \tag{2.8}$$

and $S(t)$ is the semigroup generated by $\nu A_0 + \mathcal{R}$. The *LQR problem* is to find a $u(\cdot) \in L_2(0, +\infty; U)$ that minimizes J.

It follows from standard distributed parameter control theory (Bensoussan *et al.*, 1992a; Curtain and Zwart, 1995; Lions, 1969) that if (2.6) is exponentially stabilizable, then the minimizer of (2.7) exists and is given by state feedback

$$u(t) = -K_{lqr}z(t).$$

The feedback gain operator $K_{lqr} : Z \to U$ is given by

$$K_{lqr} = \frac{1}{r}B^*\Pi$$

where $\Pi = \Pi^*$ is the self-adjoint linear operator $\Pi : Z \to Z$ that solves the operator Riccati equation

$$\tilde{A}^*\Pi + \Pi\tilde{A} - r\Pi BB^*\Pi + Q = 0, \tag{2.9}$$

where $\tilde{A} = [\nu A_S + \mathcal{R}_S]$. Moreover, the gain operator produces an exponentially stable linear closed-loop system. We use this result to compute the functional gains that define the gain operator K_{lqr}. This controller is then applied to the nonlinear system which yields the full closed-loop system

$$\dot{z}(t) = [\nu A_0 + \mathcal{R} - BK]z(t) + F(z(t)), \quad z(0) = z_0 \in Z. \tag{2.10}$$

In order to make this approach practical, we need computational algorithms to solve the operator Riccati equation (2.9) and mathematical tools to analyze the resulting closed-loop nonlinear system. Again, we emphasize the point that it is not necessary to have a complete existence theory for the nonlinear open-loop system in order to proceed with this method. In fact, there is no such theory for the 3D Navier–Stokes equations yet the results in Barbu and Triggiani (2004) and Barbu *et al.*, (2005) are valid for this 3D problem. As the simple example below illustrates, the closed-loop system (2.10) can be stabilized by linear feedback even when the open-loop problem has the property that every neighborhood of zero contains a solution with finite blowup time.

Example 1: Consider the controlled ordinary differential equation

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ [y(t)]^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{2.11}$$

with initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{2.12}$$

First note that (2.11) and (2.12) is well-posed on $\mathbb{R}^2$, i.e. given any initial data $\mathbf{z}_0 = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T$ there exist a unique solution

$$z(t, z_0) = \begin{bmatrix} x(t, \mathbf{z}_0) \\ y(t, \mathbf{z}_0) \end{bmatrix}$$

defined on a finite interval $(0, T)$ where $T = T(\mathbf{z}_0) > 0$. If $y_0 > 0$ and $\mathbf{z}_0 = \begin{bmatrix} 0 & y_0 \end{bmatrix}^T$, then $y(t, \mathbf{z}_0) = \frac{y_0}{1-ty_0}$ so that the solution always has finite blowup time $T = T(\mathbf{z}_0) = \frac{1}{y_0}$. Note that the open-loop system has only one equilibrium $\mathbf{z}_e = \begin{bmatrix} 0 & 0 \end{bmatrix}^T = 0$ and $\mathbf{z}_e$ is certainly not stable. We fix $r = 1$ and solve two LQR problems with different Q matrices. In particular, if

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

then $K_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $K_2 = \begin{bmatrix} 0.3985 & 0.7841 \end{bmatrix}$, respectively. The corresponding closed-loop linear operators are given by

$$A_{CL1} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A_{CL2} = \begin{bmatrix} -0.8985 & 0.2159 \\ -0.3985 & -0.7841 \end{bmatrix},$$

and when the feedback controllers are applied to the nonlinear system both controllers exponentially stabilize $\mathbf{z}_0 = 0$. Moreover, $\|K_1\| = 1$ and $\|K_2\| = 0.8796$, so the control energies required to stabilize the system are about the same. However, there are major differences between the controllers. First, the controller defined by K_1 is "local" in that it makes use of only one state $y(t)$. Although the control defined by K_2 requires both states (not local), it is more robust in the sense that the closed-loop operator A_{CL2} has a stability radius of 0.7879 while the closed-loop operator A_{CL1} has a stability radius of 0.5. The radius of stability for the closed-loop nonlinear system with feedback gain K_1 is equal to 1. If $y_0 < 1$, then any initial data of the form $\mathbf{z}_0 = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T$ produces a stable response. The half-plane $\Sigma_1 = \{[x_0 \quad y_0]^T : y_0 < 1\}$ is an invariant set with attractor $\mathcal{M} = \{0\}$. The radius of stability for the closed-loop nonlinear system with feedback gain K_2 is less than 1. Moreover, there is an invariant set Σ_2 (the shaded region in Fig. 1) defined by the set of all initial conditions below the stable manifold Υ of the unstable hyperbolic equilibrium. Observe that this closed-loop system allows for large values of y_0 if x_0 is also large.

As noted above, it has recently been shown in Barbu *et al.* (2005) that the linear LQR boundary feedback controller designed by the Riccati equation will (locally) stabilize the full 3D Navier–Stokes equation. On one hand this is a very strong theoretical result, but this control law has two possible drawbacks. The LQR controller is infinite dimensional in that it must be applied to the entire boundary and nothing is known about the support of the functional gain. It is not obvious that LQR type design will prove practical and it is not clear that the LQR controllers will work in channel or other shear flows. This requires additional work. Also, in order to use this

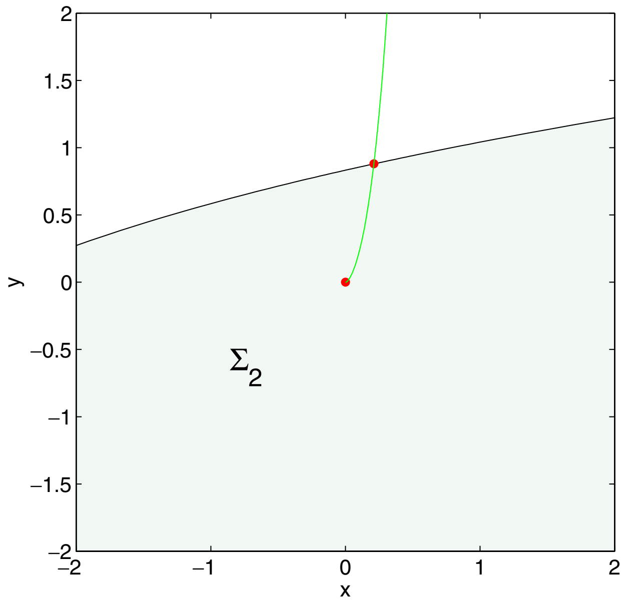

Fig. 1. Invariant set for the nonlinear closed-loop system, K_2.

type of design, one must be able to compute the feedback gains. Considerable progress has been made in the area, but new computational algorithms need to be developed to handle fully 3D flows.

The simple example above illustrates several important issues. However, in order to illustrate the potential practical application of the distributed parameter control theory we turn to a control problem for Burgers' equation.

2.2. *Control of Burgers' equation*

Now we turn to an infinite-dimensional system that falls into the above framework so that we can illustrate some practical benefits of distributed parameter control theory. We use Burgers' equation as an example because it is known to be highly sensitive to "small disturbances" in the boundary conditions and it provides an infinite-dimensional problem where we can illustrate the calculations of functional gains. Therefore, we consider Burgers' equation with a convective term

$$z_t(t,x) + z(t,x)z_x(t,x) = \nu z_{xx}(t,x) + \kappa z_x(t,x), \quad 0 < x < 1 \tag{2.13}$$

with initial condition

$$z(0,x) = \varphi(x).$$

The boundary condition at $x = 0$ is $z(t,0) = 0$ and we apply a Dirichlét boundary control at the right boundary

$$z(t,1) = u(t). \tag{2.14}$$

Here, we assume ν, $\kappa > 0$, $u(t) \in L^2(0,+\infty)$ and $\varphi(\cdot) \in L^2(0,1)$. Let $Z = L^2(0,1)$, $D(A_S) = H_0^1(0,1) \cap H^2(0,1)$, and define the differential operator $A_S : D(A_S) \to Z$ by

$$A_S w(\cdot) = w_{xx}(\cdot). \tag{2.15}$$

Let $V = H_0^1(0,1)$ and define $a(\cdot,\cdot)$ to be the symmetric bilinear form on V given by

$$a(w,v) = \int_0^1 w_x(x)v_x(x)\,dx. \tag{2.16}$$

It follows that

$$V \subset Z = Z' \subset V', \tag{2.17}$$

where the injections are continuous and each space is dense in the following one. Moreover, $a(v(\cdot), v(\cdot))$ satisfies

$$a(v(\cdot), v(\cdot)) \geq \alpha \|v(\cdot)\|_V \tag{2.18}$$

for some $\alpha > 0$ and $a(v(\cdot), v(\cdot))$ defines an associated isomorphism $A_0 : V \to V'$ by

$$[A_0 z(\cdot)]v(\cdot) = -a(z(\cdot), v(\cdot)). \tag{2.19}$$

Let $\mathcal{R} : V \to V'$ be defined by

$$[\mathcal{R}w(\cdot)]v(\cdot) = \kappa \int_0^1 w_x(x)v(x)\, dx \tag{2.20}$$

and note that $\mathcal{R}$ maps $D(A_S)$ into $V \subseteq Z$ so that $\mathcal{R}_S$ is well defined by restricting $\mathcal{R}$ to $D(A_S)$. In addition, it is easy to show that $[\nu A_S + \mathcal{R}_S]$ generates a $\mathcal{C}_0$-semigroup $S(t) : Z \to Z$ on Z. Define the continuous bilinear mapping $f : V \times V \to V'$ by

$$[f(z(\cdot), w(\cdot))]v(\cdot) = -\int_0^1 z(x)w_x(x)v(x)\, dx \tag{2.21}$$

and observe that f maps $D(A_S) \times D(A_S)$ into $H^1(0,1) \subset Z$. In particular, if $z(\cdot)$ and $w(\cdot)$ belong to $D(A_S)$, then $f(z(\cdot), w(\cdot)) = -z(x)w_x(x) \in H^1(0,1)$. Let $F : V \to V'$ denote the operator given by

$$F(z(\cdot)) = f(z(\cdot), z(\cdot)) \tag{2.22}$$

and note that

$$[F(z(\cdot))]v(\cdot) \triangleq -\int_0^1 z(x)z_x(x)v(x)\, dx \tag{2.23}$$

maps $D(A_S)$ into Z. For each $z(\cdot) \in V$, F satisfies

$$\langle F(z(\cdot)), z(\cdot)\rangle = -\int_0^1 z(x)z_x(x)z(x)\, dx = 0, \tag{2.24}$$

and hence the nonlinear term in conservative.

As noted above, the B operator will not map into $Z \subset V'$. One approach that works for this simple (essentially self-adjoint) problem is to define the space W by the domain of A_S, extend the state space and consider a very weak form of the problem. Define W to be the space $D(A_S) = D([A_S]^*)$ with graph norm

$$\|z(\cdot)\|_W = \|[A_S]^* z(\cdot)\|_Z + \|z(\cdot)\|_Z . \tag{2.25}$$

It follows that the injections

$$W \subset V \subset Z = Z' \subset V' \subset W' \tag{2.26}$$

are all continuous and dense. One now lifts the operator $A_0 : V \to V'$ defined by $[A_0 z(\cdot)]v(\cdot) = -a(z(\cdot), v(\cdot))$ to an operator $A_1 : Z \to W'$. The basic idea is to integrate by parts **twice** and define $A_1 : Z \to W'$ by

$$[A_1 z]\, w = \left\langle z, [A_S]^* w \right\rangle_Z = \langle z, [A_S] w \rangle_Z = \int_0^1 z(x) w_{xx}(x)\, dx \tag{2.27}$$

for all $w(\cdot) \in W' = \left[H_0^1(0,1) \cap H_2(0,1)\right]'$. Let $D : \mathbb{R}^1 \to L_2(0,1) = Z$ be the Dirichlét map

$$[Du](x) = xu, \tag{2.28}$$

and define $B : \mathbb{R}^1 \to W'$ by

$$B = -A_1 D. \tag{2.29}$$

It is easy to see that for $w(\cdot) \in W$

$$\begin{aligned}
[Bu]w(\cdot) &= \int_0^1 [xu] w_{xx}(x)\, dx \\
&= [xu] w_x(x)|_{x=0}^{x=1} - \int_0^1 [u] w_x(x)\, dx \\
&= u w_x(1) - [u] w(x)|_{x=0}^{x=1} \\
&= u[\delta_1'](w(\cdot)),
\end{aligned}$$

where δ_1' is the (distributional) derivative of the delta function at $x = 1$. It follows (Bensoussan *et al.*, 1992a; Curtain and Zwart, 1995; Lions, 1969) that the linearized system may be formulated as the well-posed control system in W'

$$\dot{z}(t) = [\nu A_S + \mathcal{R}_S] z(t) + Bu(t) \in W'. \tag{2.30}$$

This is the linear system we use for control design. Note that we do not consider the nonlinear system until we close the loop. The nonlinear closed-loop system will have the form

$$\dot{z}(t) = [\nu A_0 + \mathcal{R} - BK]z(t) + F(z(t)), \quad z(0) = z_0 \tag{2.31}$$

and one makes use of regularity results to show that this nonlinear closed-loop system is stable (Bensoussan *et al.*, 1992a; Burns and Kang, 1991; Burns *et al.*, 1998).

If one applies LQR theory (or LQG, Min-Max, etc.), then the optimal controllers have the form

$$u_{opt}(t) = -Kw(t,\cdot) = -\int_0^1 k(x)w(t,x)\,dx,$$

where $k(x)$ is the *functional gain*. For LQR problems where the control is applied through a Dirichlét boundary term, these gains tend to become singular near the controlled boundary (Burns *et al.*, 2002a). We focus on the accurate computation of these functional gains and discuss how the choice of a LQR problem impacts the support and singularity of the functional gains. We do make use of the nonlinear system (2.31) to develop finite element approximation schemes. For convection dominated flows, one needs to use upwinding or some form of stabilized finite element scheme. The approximation theory is complete for the linear equation (2.30). However, the implementation of these schemes in more than one space dimension can be complicated.

2.3. *Numerical examples*

Let $0 < \nu < 1$ and $b_L = 1 - \sqrt{\nu}$. Define $q(\cdot) : [0,1] \to \mathbb{R}$ by

$$q(x) = \begin{cases} q_L, & b_L \le x \le 1 \\ q_S, & 0 \le x < b_L \end{cases}, \tag{2.32}$$

where q_S and q_L are positive numbers. For $r > 0$ and $\alpha \ge 0$, define the cost function by

$$J_\alpha(u(\cdot)) = \int_0^\infty e^{\alpha t} \left\{ \left(\int_0^1 q(x)|w(t,x)|^2\,dx \right) + r\,|u(t)|^2 \right\} dt. \tag{2.33}$$

Note that if $0 < q_S \ll q_L$, then the cost function places a large penalty on the solution in the "boundary layer", $b_L \le x \le 1$. Also, when $\alpha > 0$ there is an additional performance requirement. The boundary control problem is to minimize $J_\alpha(u(\cdot))$ defined by (2.33) subject to (2.30). Standard representation theory implies that the optimal controller has the form

$$u_{opt}(t) = -K_{\mu,\alpha} w(t,\cdot) = -\int_0^1 k_{\mu,\alpha}(x) w(t,x)\,dx, \tag{2.34}$$

where the functional gain $k_{\mu,\alpha}(\cdot) \in L^2(0,1)$ (Burns and Kang, 1991; Burns and King, 1994). Note that this functional gain depends on the choice of $\mu,\ \alpha$ and the weight $q(x)$.

We compute approximations $[k_{\mu,\alpha}]^N(\cdot)$ of $k_{\mu,\alpha}(\cdot)$ for $\alpha \geq 0$ and $0 < \mu \ll 1$ by using finite element methods with mesh refinement schemes given in references Burns *et al.* (2002) and Burns *et al.* (2002a). In Run 1, we show the functional gains for $\alpha = 0$, $\alpha = 0.1$ and $\alpha = 0.2$ to illustrate how the choice of weighting can impact the support of the optimal functional gains. In Run 2, we consider a convection dominated problem to illustrate some of the numerical and convergence issues connected to solving the Riccati equation.

Run 1. For the first run we set $\nu = \frac{1}{120}$, $\kappa = 0$, $r = 0.25$ $q_S = 1$ and $q_L = \frac{50}{\sqrt{\nu}} = 50\sqrt{120} = 547.723$. Here, the selection of the weights places a heavy penalty on the boundary layer near the control boundary. In particular, we focus on the region $b_L < x < 1$ which has thickness $\sqrt{\nu} = \frac{1}{\sqrt{120}} = 0.0913$. To emphasize the role that α plays in the problem, we consider three cases corresponding to $\alpha = 0,\ \alpha = 0.1$ and $\alpha = 0.2$ and use a uniform mesh to compute the functional gains. In Fig. 2 we see that the functional gains for $\alpha > 0$ has global support over the entire interval $[0,1]$ and the gains become more significant on the interior of the domain as α increases. However, in all the cases above, the gains are singular near the boundary.

Run 2. This run illustrates the importance of developing good approximation schemes for convection dominated flow when the Peclet number $Pe \triangleq \kappa Re$ is large. This points to the need for the development of special numerical methods to solve the forward problem and the Riccati equation. We consider the case $\alpha = 0$, $\nu = \frac{1}{10{,}000}$, $\kappa = 1$, $r = 0.25$, but fix the weighting function to be $q(x) \equiv 1$ for all x. Note that Peclet number is given by $Pe = \kappa Re = 10{,}000$. Also, we use a distributed control so that

$$[Bu](x) = b(x)u,$$

where $b(x) = x^{10}$. This simple problem is sufficient to demonstrate the need for good algorithms. We solve the Riccati equation by using two finite element schemes. Scheme one is the standard Galerkin finite element method

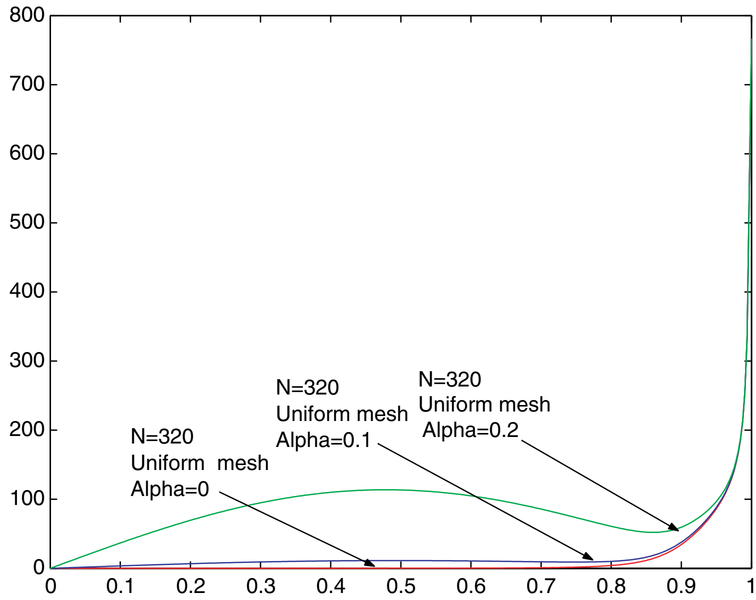

Fig. 2. Functional gains for $\alpha = 0$, $\alpha = 0.1$ and $\alpha = 0.2$.

and scheme two is a stabilized finite element (upwind) scheme. It can be shown that both methods produce strongly convergent functional gains. Moreover, the optimal functional gain is bounded and smooth on the interior of $(0, 1)$. Again we see that the LQR controller produces functional gains with support near the control boundary at $x = 1$. However, Fig. 3 illustrates that the functional gains computed by standard Galerkin finite elements become oscillatory near the $x = 1$. These numerical oscillations can be reduced if one uses a stabilized finite element scheme. Also, Figs. 4 and 5 show that both schemes converge, but the upwind scheme produced almost no oscillations and had converged by $N = 64$.

The finite-dimensional model problem in Example 1 with finite blowup time and the controlled Burgers' equation serve to illustrate the following important issues.

1. Linear feedback can stabilize highly nonlinear systems. Even if the open-loop model is not known to be well-posed, it may still be possible to use the linearization to compute a feedback controller. Therefore, one does not need to prove the existence of global solutions to the open-loop

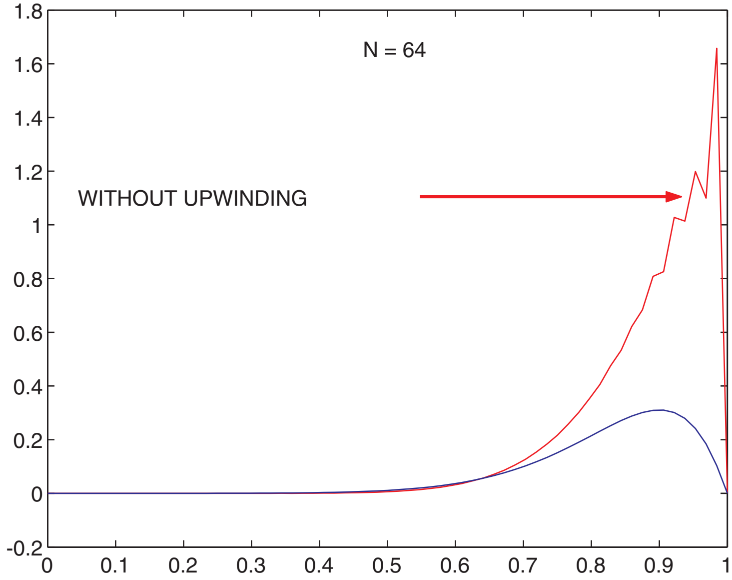

Fig. 3. Functional gains when $Pe = 10,000$ and $N = 64$ elements.

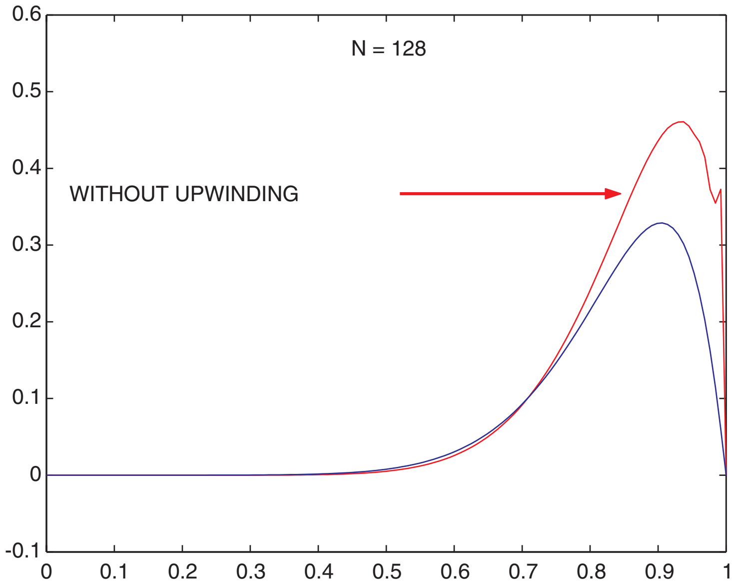

Fig. 4. Functional gains when $Pe = 10,000$ and $N = 128$ elements.

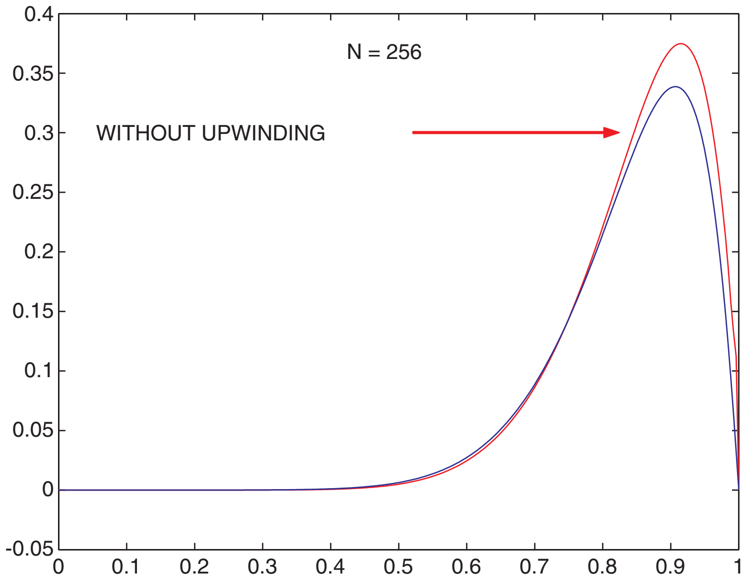

Fig. 5. Functional gains when $Pe = 10,000$ and $N = 256$ elements.

equations to design a control law. This is important since the 3D Navier–Stokes are not known to be globally well-posed.

2. Linear feedback is sufficient to dramatically alter the closed-loop dynamical system. In addition, different design methodologies generally produce distinct closed-loop systems.
3. Although many methods such as LQR produce feedback laws that depend on knowledge of the entire state, some designs lead to controllers with local support. Therefore, this information helps give practical guidance about where to place sensors and what spatial information one may need to implement the feedback law. This type of information can help build robust low order controllers.
4. In order to take advantage of these observations one must be able to accurately compute the functional gains. This is a difficult problem and requires special numerical methods that address dual convergence and other approximation issues. However, considerable progress has been made in understanding the important issues and computational tools are under development.
5. All "general" methods should be viewed as a first step in the total design process. In order to build a practical controller, one must have

a mathematical framework to analyze and compare various controllers. The framework above has been helpful in this respect. However, a better understanding of the nonlinear open-loop dynamics is always helpful. For example, if one understands the mechanism that triggers transition, then the control problem should be formulated to yield a closed-loop system that avoids this mechanism.

In the next section, we present some numerical results that offer some insight into a possible mechanism for transition. We make use of the low-dimensional models found in Baggett and Trefethen (1997).

3. Low Order Model Problems

As noted above, during the past few years, several low-dimensional model problems have been suggested in an attempt to describe specific aspects of transition. We consider 2D and 3D systems that are typical of the those found in the papers Baggett and Trefethen (1997), Henningson (1996), Waleffe *et al.* (1993) and Waleffe (1995a). However, we focus on the role that small constant uncertainties play in transition and illustrate how feedback can delay or eliminate transition in these cases.

Both systems have the form

$$\dot{z}(t) = A(R)z(t) + \|z(t)\|Sz(t) + Bu(t) + G\varepsilon, \tag{3.1}$$

where $A(R) = [\frac{1}{R}A_0 + \mathcal{R}]$, $A_0 < 0$ is diagonal and $S = -S^*$ is skew-adjoint. The system (3.1) is sensitive to initial data and inputs. The impact of the nonlinear term can be very complex and extremely sensitive to small perturbations. These low order models are constructed to mimic the main features of the Navier–Stokes equations and they are typical of many special flow problems (Baggett and Trefethen 1997; Schmid and Henningson, 2001; Waleffe, 1995). It is interesting to note that similar model problems have been used by the control theory community to test various approaches to robustness, to evaluate approximation schemes and as examples to illustrate ill-conditioning, sensitivity and non-convergence of computational control methods (Burns *et al.*, 1988; Burns, 2003; Burns *et al.*, 2003; Datta, 2004). Although there are several models of this type, Baggett and Trefethen (1997) have shown that all these low-dimensional models have many common features.

We will discuss two such models and show how bifurcation analysis under uncertainty can describe a possible route to transition. The idea is to view the disturbance as a perturbation of the conservative nonlinearity.

If in addition the linear operator is highly non-normal, then the dynamical system can become extremely sensitive to small disturbances and transition occurs even when the linearized system is stable. In particular, the two dimensional system is defined by

$$A(R) = \begin{bmatrix} -\alpha/R & 1 \\ 0 & -\beta/R \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

and

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.3}$$

The three-dimensional system is defined by

$$A(R) = \begin{bmatrix} -\alpha/R & 1 & 0 \\ 0 & -\beta/R & 1 \\ 0 & 0 & -\gamma/R \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 & -1/2 \\ 1 & 0 & 1/4 \\ 1/2 & -1/4 & 0 \end{bmatrix} \tag{3.4}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.5}$$

where all constants are positive. Both models have the property that the linear operator $A(R)$ is stable for all $R > 0$ and the two-dimensional nonlinear model is also dissipative. In particular, the nonlinear two-dimensional system defined by (3.2) and (3.3) has a compact global attractor. The nonlinear three-dimensional system defined by (3.4) and (3.5) is more complex, but exhibits features very similar to those one finds in plane Couette flows.

As noted above, the problem with classical linear analysis is that it fails to predict the correct critical Reynolds number that yields transition. For plane Couette flows, the linearized equations are always stable and theoretically, one should not see transition if the initial flow state is sufficiently close to the plane Couette flow. However, if one views a “small” constant disturbance as a perturbation of the **conservative nonlinear term**, then standard bifurcation theory under uncertainty yields a transition scenario which matches many flow cases. Understanding this mechanism is crucial to the development of feedback control laws. The following simple models are sufficient to illustrate the basic ideas and to demonstrate how feedback can be useful in the delaying of transition.

3.1. *The two-dimensional model*

In this case, we set $\alpha = 1.2$ and $\beta = 1.4$. We call the eigenvector $z_{TS} = [1 \quad 0]^T$ corresponding to the smallest eigenvalue $-\alpha/R$ the Tollmien–Schlichting initial state because of the similarity to the Tollmien–Schlichting waves in plane Poiseuille flows. We refer to the vector $z_{OB} = [1 \quad 1]^T$ as the oblique state. Observe that $A(R)$ is stable for all $R > 0$. In addition, one can show that this two-dimensional system has a compact global attractor. In Fig. 6, the light lines are the stable manifold and the dark lines are the unstable manifolds for the hyperbolic critical points. The union of the five equilibrium and unstable manifolds is the global attractor. The basin of attraction for the zero equilibrium lies between the stable manifolds. If $\varepsilon = 0$, then the zero ($z_0 = 0$) equilibrium is locally asymptotically stable for all R. However, the radius $\delta(R)$ of the largest ball about z_0 that lies in the domain of attraction converges to 0 and is approximately given by $\delta(R) = O(R^{-2})$. When one adds a small "uncertainty" such as an $\varepsilon = 0.0001$ perturbation to the nonlinear term, there is a subcritical bifurcation near $R = 6$ as illustrated in Fig. 7. The light lines are the stable manifold and the dark lines are the unstable manifolds for the single hyperbolic critical point. The union of the three equilibrium and the unstable manifolds is the global attractor. In this case, all initial states near $z_0 = 0$ transition.

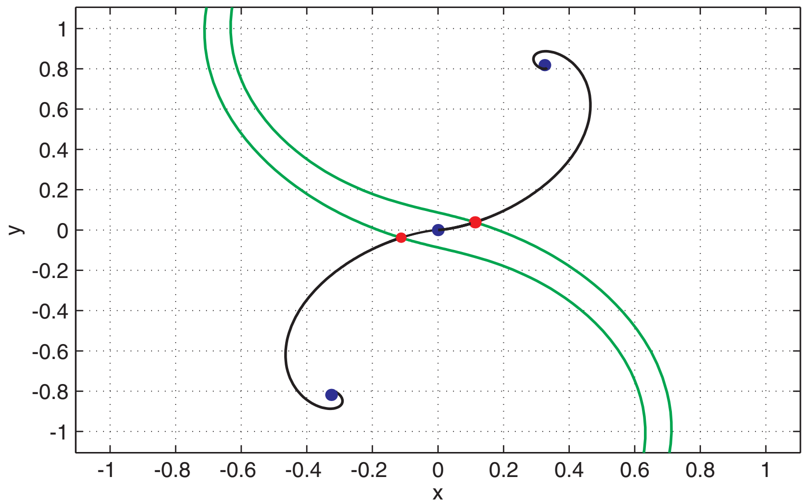

Fig. 6. Phase portrait without disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 4$, $\varepsilon = 0$).

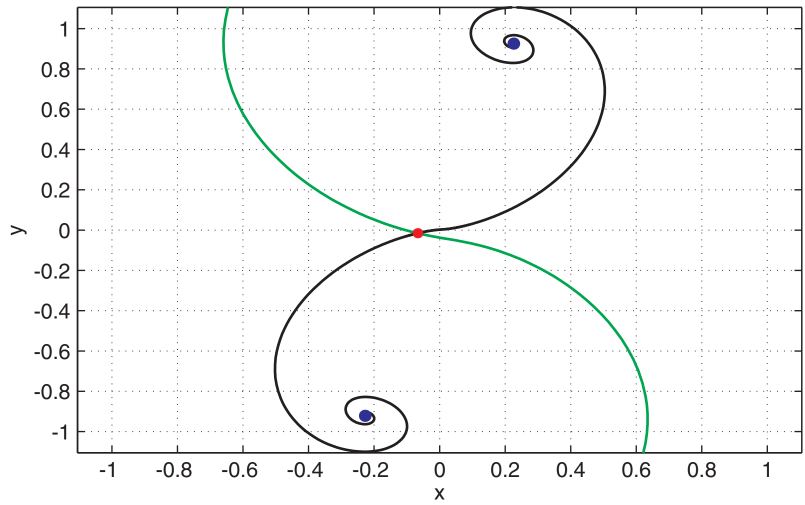

Fig. 7. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$).

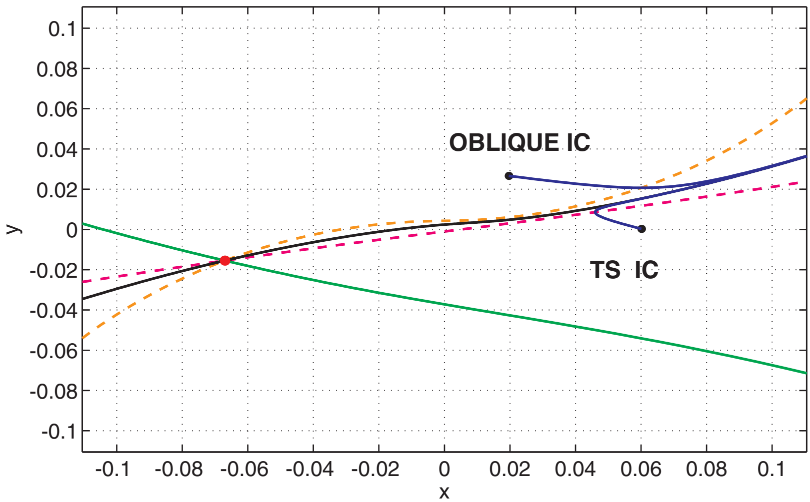

Fig. 8. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$).

In Fig. 8, we zoom in near zero. The dashed lines are the nullclines and the disturbance of size $\varepsilon = 0.0001$ produces a subcritical bifurcation. There are only three critical points. The light lines are the stable manifold and the dark lines are the unstable manifolds for the single hyperbolic critical

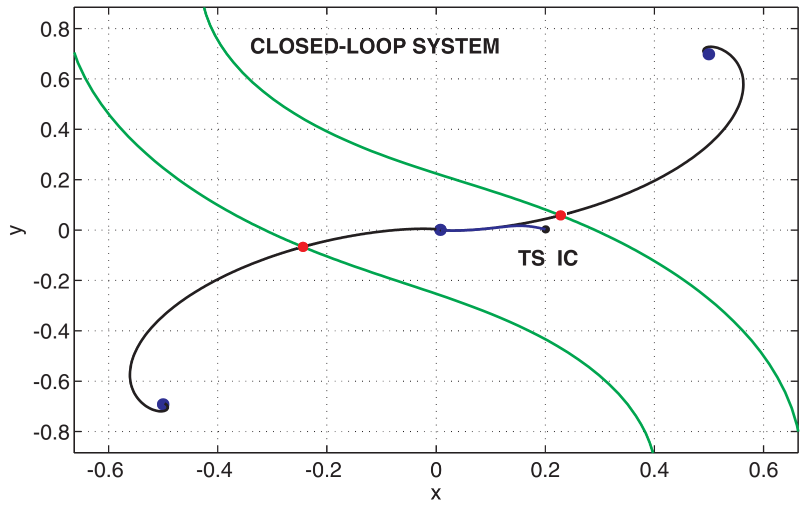

Fig. 9. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$) with a LQR feedback controller.

point. The union of the three equilibrium and the unstable manifolds is the global attractor. However, all initial states above the stable manifold go into transition to the distant stable equilibrium. However, this simple example illustrates how and why the oblique initial state goes into transition before the Tollmien–Schlichting initial state as observed in Schmid and Henningson (2001).

Finally, Fig. 9 shows that if one applies a LQR feedback control to this system, then the closed-loop system looks much like the $R = 4$ open-loop system and hence feedback delays transition. The LQR control was computed with weighting matrices $Q = I_2$ and $r = 25$. The disturbance of size $\varepsilon = 0.0001$ no longer produces a subcritical bifurcation and there are five critical points. Again, the basin of attraction for the zero equilibrium lies between the stable manifolds and is much greater than the open-loop system with no disturbance.

3.2. *The three-dimensional model*

We present this three-dimensional system to illustrate how one might use feedback in a fully developed chaotic flow. Because we are no longer restricted by dimension, this system is more complex and, for various values of the parameter $R > 1$, it exhibits periodic, quasi-periodic and chaotic

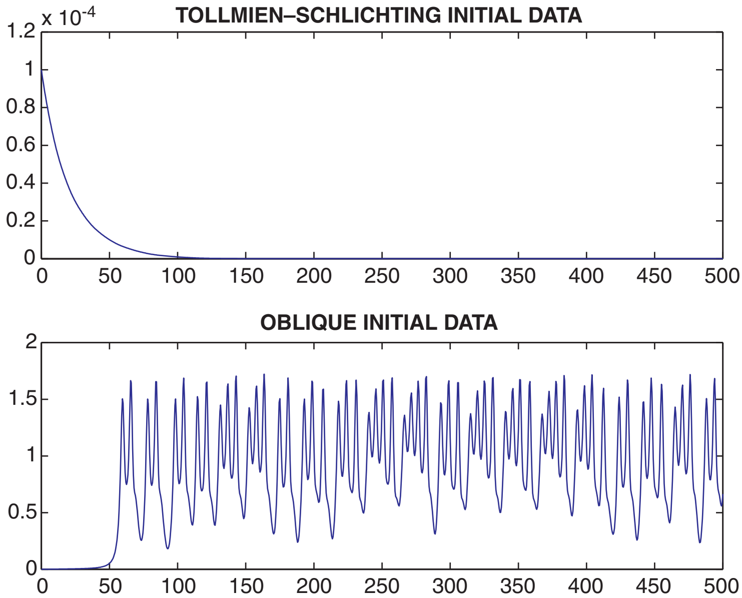

Fig. 10. Open-loop energies with initial data of norm $\|z_0\| = 10^{-4}$.

attractors. For the runs presented below, we set $\alpha = 0.5$, $\beta = 0.75$ and $\gamma = 1.0$. Again, the eigenvector $z_{TS} = [1 \quad 0 \quad 0]^T$ corresponding to the smallest eigenvalue $-\alpha/R$ is called the Tollmien–Schlichting state. The vector $z_{OB} = [1 \quad 1 \quad 1]^T$ is called the oblique state. If $9.5 < R < 23$, then there is a chaotic (local) attractor. The results presented below are based on $R = 10$ and initial states $\bar{z}$ satisfying $\|\bar{z}\| = 10^{-4}$. In Fig. 10, we plot the energies of the open-loop responses to the Tollmien–Schlichting and oblique initial states, respectively. The Tollmien–Schlichting initial state returns to zero state, but the oblique initial state goes into transition to the chaotic attractor with a transition time of approximately 50 s. As observed in the two-dimensional example, if one sets $\varepsilon = 10^{-6}$, then the Tollmien–Schlichting initial state also goes into transition to the chaotic attractor and the transition time increases to approximately 100 s.

In order to test the feedback control, we computed a LQR controller and used a "capturing" algorithm that turns on the control only if $t > 150$ and the trajectory "wanders" into the domain of attraction for the closed-loop system. A version of this method was suggested by Yorke and co-workers in the papers of Shinbrot *et al.* (1992; 1992a). For the case here, we wait

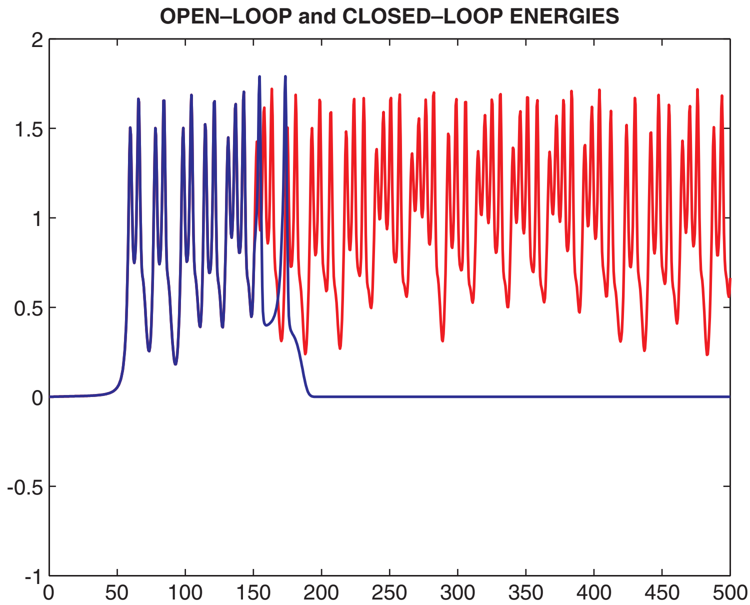

Fig. 11. Energies of the open-loop and closed-loop 3D system.

until the flow is fully chaotic ($t > 150$ for both initial states) and then only turn on the feedback control law when $\|z(t)\| < 1$. The weighting matrices for the LQR problem were $Q = I_3$ and $r = 1$. Figure 11 shows the open-loop and closed-loop energies for the oblique initial state. The capturing feedback control law is turned on at $t = 150$ and the fully developed flow is stabilized by $t = 190\,\mathrm{s}$. This example illustrates the power of feedback to change the global nature of the nonlinear dynamics.

4. Summary and Conclusions

The two models considered in the previous section have many of the mathematical features common to flow control problems. Also, all the examples above clearly show that it might be possible to develop a rigorous theoretical framework to explain some transition scenarios as a subcritical "bifurcation under uncertainty". The linear part of such non-normal systems is extremely important in understanding sensitivity and control design. However, it is the perturbation of the conservative nonlinear term that might explain a transition mechanism. Even a small perturbation to the condition

$$\langle F(z), z\rangle = 0$$

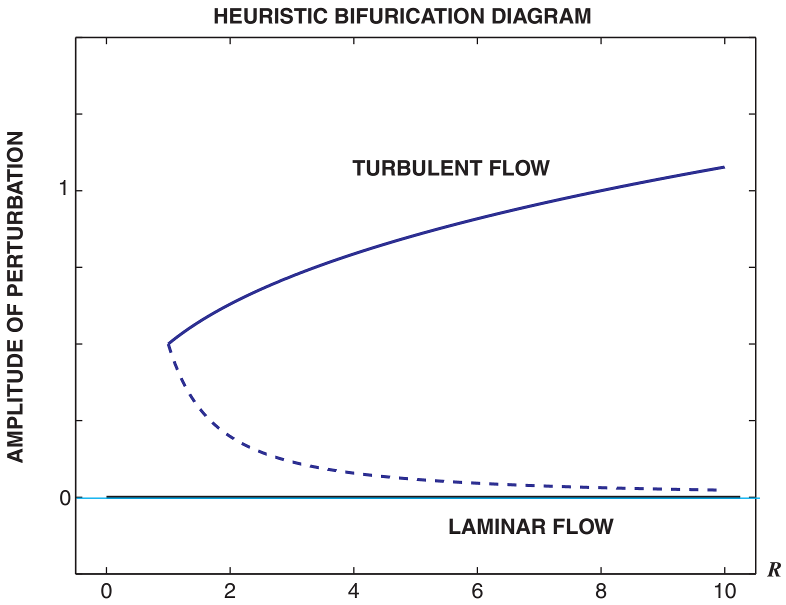

Fig. 12. Heuristic bifurcation diagram for low-dimensional models. The laminar flow is stable for all $R > 0$ but the stability radius decays to 0 as $R \to +\infty$. An initial state must be above the dashed blue line to transition.

can produce bifurcation diagrams such as shown in Figs. 12 and 13. In particular, if $\varepsilon \neq 0$ then the perturbed nonlinear term becomes

$$F_\varepsilon(z) = F(z) + G\varepsilon,$$

so that

$$\langle F_\varepsilon(z), z\rangle = \langle F(z), z\rangle + \langle G\varepsilon, z\rangle = \langle G\varepsilon, z\rangle = \varepsilon(u + v + w)$$

is no longer conservative and not definite. It is the perturbation of the conservative nonlinear term that provides the transition mechanism.

In addition to providing a framework to help with the fundamental understanding of transition, the abstract formulation above can be used to quantify sensitivity and uncertainty. Moreover, distributed parameter control theory combined with numerical analysis provides a basis for developing control laws and computational algorithms. However, much remains to be done in all of these areas.

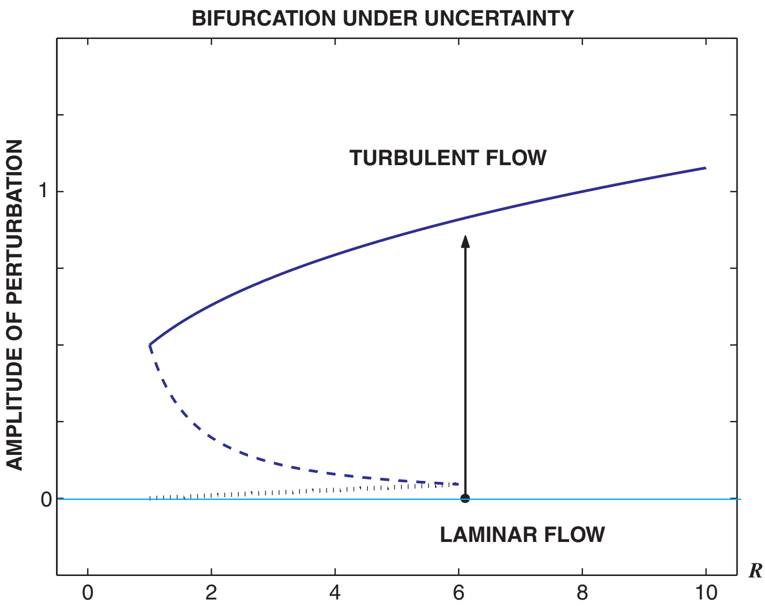

Fig. 13. A bifurcation under uncertainty. The small constant disturbance produces a non-conservative nonlinear term which leads to a subcritical bifurcation. The laminar flow state is no longer an equilibrium for $R > R_{\text{crit}}$ and transition occurs.

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