# CHAPTER I

# The Schwarz Lemma and Its Generalizations

## 1 The Schwarz–Pick Lemma

Let $D$ be the open unit disk in the complex plane $\mathbf{C}$, i.e.,

$$D = \{z \in \mathbf{C}; |z| < 1\}.$$

Let $f : D \to D$ be a holomorphic mapping such that $f(0) = 0$. Then the classical Schwarz lemma states

$$|f(z)| \leqq |z| \quad \text{for } z \in D$$

and

$$|f'(0)| \leqq 1,$$

and the equality $|f'(0)| = 1$ or the equality $|f(z)| = |z|$ at a single point $z \neq 0$ implies

$$f(z) = \varepsilon z \quad \text{with } |\varepsilon| = 1.$$

Now we shall drop the assumption $f(0) = 0$. If $f : D \to D$ is an arbitrary holomorphic mapping, we fix an arbitrarily chosen point $z \in D$ and consider the automorphisms $g$ and $h$ of $D$ defined by

$$g(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta} \quad \text{for } \zeta \in D,$$
$$h(\zeta) = \frac{\zeta - f(z)}{1 - \overline{f(z)}\zeta} \quad \text{for } \zeta \in D.$$

Then the composed mapping $F = h \cdot f \cdot g$ is a holomorphic mapping of $D$ into itself which sends 0 into itself. Since $F(0) = 0$ and $F'(0) = h'(f(z))f'(z)g'(0)$, we obtain

$$F'(0) = \frac{1 - |z|^2}{1 - |f(z)|^2} f'(z).$$

Hence,

$$\frac{1-|z|^2}{1-|f(z)|^2}|f'(z)| \leqq 1,$$

or

$$\frac{|f'(z)|}{1-|f(z)|^2} \leqq \frac{1}{1-|z|^2} \quad \text{for } z \in D.$$

We may conclude the following:

**Theorem 1.1** *Let $f$ be a holomorphic mapping of the unit disk $D$ into itself. Then*

$$\frac{|df|}{1-|f|^2} \leqq \frac{|dz|}{1-|z|^2} \quad \textit{for } z \in D,$$

*and the equality at a single point $z$ implies that $f$ is an automorphism of $D$.*

This result, which was derived from the Schwarz lemma, is actually equivalent to the Schwarz lemma. In fact, if $f : D \to D$ is a holomorphic mapping such that $f(0) = 0$, then by setting $z = 0$ in the inequality above, we obtain

$$|f'(0)| \leqq 1,$$

and if $|f'(0)| = 1$, then $f$ is an automorphism of $D$. Moreover,

$$\int_0^{|f(\zeta)|} \frac{|df|}{1-|f|^2} \leqq \int_0^{|\zeta|} \frac{|dz|}{1-|z|^2}.$$

Hence,

$$\log \frac{1+|f(\zeta)|}{1-|f(\zeta)|} \leqq \log \frac{1+|\zeta|}{1-|\zeta|},$$

which implies

$$\frac{2}{1-|f(\zeta)|} - 1 = \frac{1+|f(\zeta)|}{1-|f(\zeta)|} \leqq \frac{1+|\zeta|}{1-|\zeta|} = \frac{2}{1-|\zeta|} - 1.$$

It follows that $|f(\zeta)| \leqq |\zeta|$. The equality $|f(\zeta)| = |\zeta|$ at a single point $\zeta \neq 0$ implies the equality

$$\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$$

for all $z$ lying on the line segment joining 0 and $\zeta$. By Theorem 1.1, $f$ is an automorphism of $D$, and hence, $f(z) = \varepsilon z$ for some $\varepsilon$ with $|\varepsilon| = 1$. This proves that Theorem 1.1 implies the classical Schwarz lemma.

We shall now consider Theorem 1.1 from a differential geometric viewpoint. If we consider the Kaehler metric $ds_D^2$ on $D$ given by

$$ds_D^2 = \frac{dz\, d\bar{z}}{(1-|z|^2)^2},$$

then the inequality in Theorem 1.1 may be written as follows:

$$f^*(ds_D^2) \leqq ds_D^2.$$

The metric $ds_D^2$ is called the *Poincaré metric* or the *Poincaré–Bergman metric* of $D$. Now, Theorem 1.1 may be stated as follows:

**Theorem 1.2** *Let $D$ be the open unit disk in $\mathbf{C}$ with the Poincaré–Bergman metric $ds_D^2$. Then every holomorphic mapping $f : D \to D$ is distance-decreasing, i.e., satisfies*

$$f^*(ds_D^2) \leqq ds_D^2,$$

*and the equality at a single point of $D$ implies that $f$ is an automorphism of $D$.*

If $f$ is an automorphism of $D$, then the Schwarz lemma applied to both $f$ and $f^{-1}$ implies that $f$ is an isometry.

We note that the Guassian curvature of the metric $ds_D^2$ is equal to −4 everywhere. (In general, the Gaussian curvature of a metric $2hdzd\bar{z}$ is given by $-(1/h)(\partial^2 \log h/\partial z\partial\bar{z})$.)

## 2 A Generalization by Ahlfors

Let $D_a$ be the open disk of radius $a$ in $\mathbf{C}$,

$$D_a = \{z \in \mathbf{C}; |z| < a\}.$$

Then the metric

$$ds_a^2 = \frac{4a^2\, dz\, d\bar{z}}{A(a^2 - |z|^2)^2} \quad (A > 0)$$

on $D_a$ has Gaussian curvature $-A$. The following theorem of Ahlfors [1] generalizes the Schwarz lemma.

**Theorem 2.1** *Let $M$ be a one-dimensional Kaehler manifold with metric $ds_M^2$ whose Gaussian curvature is bounded above by a negative constant $-B$. Then every holomorphic map $f : D_a \to M$ satisfies*

$$f^*\, ds_M^2 \leqq \frac{A}{B} ds_a^2.$$

**Proof** Let $u$ be the nonnegative function on $D_a$ defined by

$$f^*(ds_M^2) = u\, ds_a^2.$$

We want to prove that $u \leqq A/B$ on $D_a$. Although $u$ may not attain its maximum in (the interior of) $D_a$ in general, we shall show that we have only to consider the case where $u$ attains its maximum in $D_a$. Let $r$ be a positive number less than $a$. Let $z_0$ be an arbitrary point of $D_a$. Taking $r$ sufficiently close to $a$, we may assume that $z_0$ is in $D_r$. We denote by $ds_r^2$ the metric on $D_r$ obtained from $ds_a^2$ by replacing $a$ by $r$ but keeping the same constant $A$. From the explicit expression for $ds_a^2$, it is clear that $(ds_r^2)_{z_0} \to (ds_a^2)_{z_0}$ as $r \to a$. If we define a nonnegative function $u_r$ on $D_r$ by $f^*(ds_M^2) = u_r ds_r^2$, then $u_r(z_0) \to u(z_0)$ as $r \to a$. Hence it suffices to prove that $u_r \leqq A|B$ on $D_r$, If we write $f^*(ds_M^2) = h\, dz\, d\bar{z}$ on $D_a$, then $h$ is bounded on $\bar{D}_r$ (the closure of $D_r$). On the other hand, the coefficient $4r^2|A(r^2 - |z|^2)^2$ of $ds_r^2$ approaches infinity at the boundary of $D_r$. Hence, the function $u_r$ defined on $D_r$ goes to zero at the boundary of $D_r$. In particular, $u_r$ attains its maximum in $D_r$. The problem is thus reduced to the case where $u$ attains its maximum in $D_a$.

We shall now impose the additional assumption that $u$ attains its maximum in $D_a$, say at $z_0 \in D_a$. If $u(z_0) = 0$, then $u \equiv 0$ and there is nothing to prove. Assume that $u(z_0) > 0$. Then the mapping $f : D_a \to M$ is nondegenerate in a neighborhood of $z_0$ so that $f$ is a biholomorphic map from an open neighborhood $U$ of $z_0$ onto the open set $f(U)$ of $M$. Identifying $U$ with $f(U)$ by the map $f$, we use the coordinate system $z$ of $D_a \subset \mathbf{C}$ also as a local coordinate system in $f(U)$. If we write $ds_M^2 = 2h\, dz\, d\bar{z}$ on $f(U)$, then $f^*(ds_M^2) = 2h\, dz\, d\bar{z}$ on $U$. If we write $ds_a^2 = 2g\, dz\, d\bar{z}$, then $u = h/g$. The Gaussian curvature $k$ of the metric $ds_M^2 = 2h\, dz\, d\bar{z}$ is given by

$$k = -\frac{1}{h}\frac{\partial^2 \log h}{\partial z\, \partial \bar{z}}.$$

The Gaussian curvature $-A$ of the metric $ds_a^2 = 2g\, dz\, d\bar{z}$ is given by

$$-A = \frac{1}{g}\frac{\partial^2 \log g}{\partial z\, \partial \bar{z}}.$$

Since $k \leqq -B$ by our assumption, we have

$$\frac{\partial^2 \log u}{\partial z\, \partial \bar{z}} = \frac{\partial^2 \log h}{\partial z\, \partial \bar{z}} - \frac{\partial^2 \log g}{\partial z\, \partial \bar{z}} = -kh - Ag \geqq Bh - Ag.$$

Since $\log u$ attains its maximum at $z_0$, the left-hand side in the inequality above is nonpositive at $z_0$ and so is the right-hand side. Hence, $A/B \geqq$

$h/g = u$ at $z_0$. Since $u$ attains its maximum at $z_0$, it follows that $A/B \geqq u$ everywhere. □

## 3 The Gaussian Plane Minus Two Points

In view of Theorem 2.1 we are naturally interested in finding a one-dimensional Kaehler manifold whose Gaussian curvature is bounded above by a negative constant. As we shall see later (see Sec. 4 of Chapter IV), the Gaussian plane $\mathbf{C}$ cannot carry such a metric. The metric

$$ds^2 = 2(1 + |z|^2)\, dz\, d\bar{z}$$

on $\mathbf{C}$ has curvature $k = -1(1+|z|^2)^3$, which is strictly negative everywhere but is not bounded above by any negative constant.

If a one-dimensional complex manifold $M$ carries a Kaehler metric whose Gaussian curvature is bounded above by a negative constant, so does any covering manifold of $M$. If $M$ is the Gaussian plane minus a point, say the origin, that is, $M = \mathbf{C} - \{0\}$, then the universal covering manifold of $M$ is $\mathbf{C}$, the covering projection being given by

$$z \in \mathbf{C} \to e^{2\pi i z} \in \mathbf{C} - \{0\}.$$

This shows that $\mathbf{C} - \{0\}$ does not admit a Kaehler metric whose Gaussian curvature is bounded above by a negative constant.

Consider now the Gaussian plane minus two points, say $M = \mathbf{C} - \{0, 1\}$. If we use the modular function $\lambda(z)$, we can show that $M$ carries a complete Kaehler metric with negative constant curvature. Let $H$ denote the upper half-plane in $\mathbf{C}$, i.e., $H = \{z = x + iy \in \mathbf{C};\ y > 0\}$. Then the modular function $\lambda$ gives a covering projection $\lambda : H \to C - \{0, 1\}$. If we digress a little, the group of deck-transformations is given by

$$z \in H \to \frac{az + b}{cz + d} \in H$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbf{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{mod } 2.$$

This group, known as the congruence subgroup mod 2, is a normal subgroup of index 6 in the modular group $SL(2; \mathbf{Z})$, and its fundamental domain is given by $F$ in Fig. 1. The boundary of $F$ and the imaginary axis are mapped into the real axis by $\lambda$. In Fig. 2, one sees roughly how the fundamental domain $F$ is mapped onto $\mathbf{C} - \{0, 1\}$ by $\lambda$. On the upper half-plane $H$, the

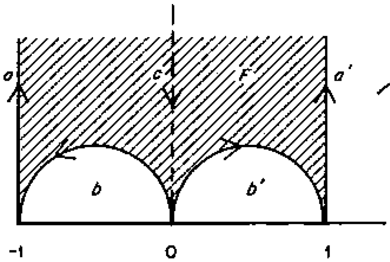


Fig. 1.

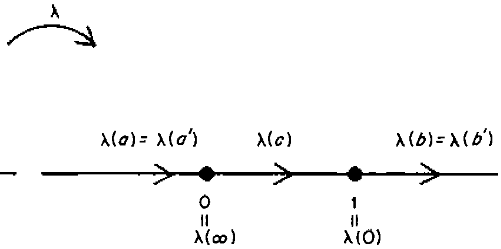


Fig. 2.

metric $ds^2 = 2dz\,d\bar{z}/Ay^2$ has curvature $-A$ and is invariant by the group of holomorphic transformations of $H$. It follows immediately that this metric induces a metric of curvature $-A$ on $\mathbf{C}-\{0,1\}$. Unfortunately, the modular function $\lambda$ is so complicated that the induced metric on $\mathbf{C}-\{0,1\}$ cannot be expressed in a simple form in terms of the natural coordinate of $\mathbf{C}-\{0,1\}$. (For the definition and basic properties of the modular function $\lambda$, see for instance Ahlfors [2] and Ford [1].)

We shall now give a more elementary construction of a metric with Gaussian curvature $\leqq -4$ on $\mathbf{C}-\{0,1\}$. The construction is due to Grauert and Reckziegel [1]. Given a positive $C^\infty$ function $g(z,\bar{z})$ defined in an open set in $\mathbf{C}$, we define a real-valued function $K(g)$ defined in the same open set as follows:

$$K(g) = -\frac{1}{g}\frac{\partial^2 \log g}{\partial z\,\partial \bar{z}}.$$

The definition is motivated by the fact that $K(g)$ is the Gaussian curvature of the metric $ds^2 = 2g\,dz\,d\bar{z}$. We first prove the following:

**Proposition 3.1** *For positive functions $f$ and $g$, we have*

(a) $cK(cg) = K(g)$ *for all positive numbers* $c$;
(b) $fgK(fg) = fK(f) + gK(g)$;
(c) $(f+g)^2K(f+g) \leqq f^2K(f) + g^2K(g)$;
(d) *If* $K(f) \leqq -k_1 < 0$ *and* $K(g) \leqq -k_2 < 0$,

*then*

$$K(f+g) \leqq -k_1k_2/(k_1+k_2).$$

**Proof** (a) and (b) are immediate from the definition of $K(g)$. (c) follows from the following, which can be easily verified from the definition of $K(g)$.

$$\begin{aligned} &fg(f+g)[f^2K(f)+g^2K(g)-(f+g)^2K(f+g)] \\ &= \left| f\frac{\partial g}{\partial z} - g\frac{\partial f}{\partial z}\right|^2 \geqq 0. \end{aligned}$$

(d) is a consequence of (c) and

$$[f^2K(f)+g^2K(g)]/(f+g)^2 \leqq -k_1k_2/(k_1+k_2).$$

This latter inequality follows from

$$-(f^2k_1+g^2k_2)/(f+g)^2 \leqq -k_1k_2/(k_1+k_2). \qquad \square$$

We set

$$p(z,\bar{z}) = |2z|^{2a-2}(1+|2z|^{2a}),$$

where $a$ is a constant, $0 < a \leqq \frac{1}{5}$. Then $p$ is a positive $C^\infty$ function on $\mathbf{C}-\{0\}$. We have

$$K(p) = -4a^2/(1+|2z|^{2a})^3.$$

We set

$$f(z,\bar{z}) = p(z,\bar{z})p(z-1,\bar{z}-1).$$

Then $f$ is a positive $C^\infty$ function on $\mathbf{C}-\{0,1\}$. By (b) of Proposition 3.1, we have

$$K[f(z,\bar{z})] = \frac{K[p(z,\bar{z})]}{p(z-1,\bar{z}-1)} + \frac{K[p(z-1,\bar{z}-1)]}{p(z,\bar{z})}.$$

Since

$$\frac{K[p(z,\bar{z})]}{p(z-1,\bar{z}-1)} \leqq \frac{a^2}{2(1+3^{2a})} \quad \text{for } 0<|z|\leq\frac{1}{2}$$

and

$$\frac{K[p(z-1,\bar{z}-1)]}{p(z,\bar{z})} \leqq -\frac{a^2}{2(1+3^{2a})} \quad \text{for } 0<|z-1|\leqq\frac{1}{2},$$

we obtain

$$K[f(z,\bar{z})] \leqq -\frac{a^2}{2(1+3^{2a})} \quad \text{if } 0<|z|\leqq\frac{1}{2} \quad \text{or} \quad 0<|z-1|\leqq\frac{1}{2}.$$

On the other hand, for $|z-1| \geqq \frac{1}{2}$, we have

$$\begin{aligned}
&|2z-2|^{2a-2}(1+|2z-2|^{2a})(1+|2z|^{2a})^3 \\
&\quad = |2z-2|^{2a-2/5}(|2z-2|^{-2/5}+|2z-2|^{2a-2/5}) \\
&\qquad \times (|2z-2|^{-2/5}+|2z|^{2a}|2z-2|^{-2/5})^3 \\
&\quad \leqq 1 \cdot (1+1)(1+|2z|^{2a}|2z-2|^{-2a})^3 \\
&\quad \leqq 2(1+3^{2a})^3.
\end{aligned}$$

Similarly, for $|z| \geqq \frac{1}{2}$, we have

$$|2z|^{2a-2}(1+|2z|^{2a})(1+|2z-2|^{2a})^3 \leqq 2(1+3^{2a})^3.$$

It follows that if $|z| \geqq \frac{1}{2}$ and $|z-1| \geqq \frac{1}{2}$, then

$$\begin{aligned}
K[f(z,\bar{z})] &= \frac{K[p(z,\bar{z})]}{p(z-1,\bar{z}-1)} + \frac{K[p(z-1,\bar{z}-1)]}{p(z,\bar{z})} \\
&\leqq -\frac{2a^2}{(1+3^{2a})^3} - \frac{2a^2}{(1+3^{2a})^3} \\
&= -\frac{4a^2}{(1+3^{2a})^3}.
\end{aligned}$$

Since $-4a^2/(1+3^{2a})^3 \leqq -a^2/2(1+3^{2a})$ for $0 < a \leqq \frac{1}{5}$, we have finally

$$K[f(z,\bar{z})] \leqq -\frac{a^2}{2(1+3^{2a})} \quad \text{for } z \in \mathbf{C} - \{0,1\}.$$

We have shown that the metric $ds^2 = 2f(z,\bar{z})\,dz\,d\bar{z}$ on $\mathbf{C} - \{0,1\}$ has Gaussian curvature $K(f) \leqq -a^2/2(1+3^{2a}) < 0$. But this metric is not complete. Since $ds = \sqrt{2f}|dz|$ has a singularity of order $|z|^{a-1}$ at 0, the point 0 is at a finite distance from any point of $\mathbf{C} - \{0,1\}$. Similarly, for the point 1. It is also easy to verify that the point at infinity of $\mathbf{C}$ is at a finite distance from any point of $\mathbf{C} - \{0,1\}$ with respect to the metric $ds^2$. We shall make an adjustment to obtain a complete metric whose Gaussian curvature is bounded above by a negative constant.

Choose a real-valued $C^\infty$ function $s(z,\bar{z})$ on $\mathbf{C}$ such that

$$\begin{aligned}
s(z,\bar{z}) &= 1 \quad \text{for } |z| \leqq \frac{1}{4}, \\
0 \leqq s(z,\bar{z}) &\leqq 1 \quad \text{for } \frac{1}{4} \leqq |z| \leqq \frac{1}{3}, \\
s(z,\bar{z}) &= 0 \quad \text{for } |z| \geqq \frac{1}{3}.
\end{aligned}$$

We set

$$q(z,\bar z) = s(z,\bar z)/|z|^2(\log|z|^2)^2 \quad \text{for } 0<|z|\leqq \frac{1}{3},$$
$$= 0 \quad \text{for } |z|\geqq \frac{1}{3}.$$

We set

$$h(z,\bar z) = q(z,\bar z) + q(z-1,\bar z-1) + \frac{1}{|z|^4} q\left(\frac{1}{z},\frac{1}{\bar z}\right).$$

Since $K[q(z,\bar z)] = -2$ for $0<|z|<\frac{1}{4}$, we obtain

$$K[h(z,\bar z)] = -2 \quad \text{on } \left\{0<|z|<\frac{1}{4}\right\}\cup\left\{0<|z-1|<\frac{1}{4}\right\}\cup\{|z|>4\}.$$

We set

$$g = f + ch, \quad 0<c<1.$$

If we denote $a^2/2(1+3^{2a})$ by $k'$ so that $K(f)\leqq -k'$ on $\mathbf{C}-\{0,1\}$, then by (d) of Proposition 3.1 we have

$$K[g(z,\bar z)] \leqq -\frac{2k'}{2+ck'} < -\frac{2k'}{2+k'} < 0$$

on the domain $\{0<|z|<\frac{1}{4}\}\cup\{0<|z-1|<\frac{1}{4}\}\cup\{|z|>4\}$. In the complementary set which is compact, we use the following estimate (cf. (c) of Proposition 3.1):

$$\begin{aligned} K(g) &= K[c(f+h)+(1-c)f] \\ &\leqq \frac{c^2(f+h)^2K[c(f+h)] + (1-c)^2f^2K[(1-c)f]}{(f+ch)^2} \\ &= \frac{c(f+h)^2K(f+h) + (1-c)f^2K(f)}{(f+ch)^2} \\ &\leqq \frac{c(f+h)^2K(f+h) - (1-c)f^2k'}{(f+ch)^2}. \end{aligned}$$

If we take a sufficiently small $c$, then $K(g)$ is bounded above by a negative constant on the compact set we are considering.

We have thus constructed a complete metric $ds^2 = 2g\,dz\,d\bar z$ on $\mathbf{C}-\{0,1\}$ whose Gaussian curvature $K(g)$ is bounded above by a negative constant, say $k$. Multiplying the metric by a suitable constant, we may assume that $k=-4$.

## 4 Schottky's Theorem

One of the best known consequences of the generalized Schwarz lemma is the following theorem of Schottky [1].

**Theorem 4.1** *Given a complex number $a$ with $a \neq 0, 1$ and a real number $r$ with $0 \leqq r < 1$, there exists a positive number $S = S(a, r)$ with the following property: If $f$ is a holomorphic map from the open unit disk $D = \{z \in \mathbf{C}; |z| < 1\}$ into $\mathbf{C} - \{0, 1\}$ such that $f(0) = a$, then $|f(z)| \leqq S(a, r)$ for $|z| \leqq r$.*

**Proof** Let $ds_D^2$ denote the Poincaré metric on $D$ with curvature $-4$. We set $M = \mathbf{C} - \{0, 1\}$ and let $ds_M^2$ be a complete metric with curvature $\leqq -4$; such a metric was constructed in the preceding section. If we set $r' = \frac{1}{2}\log(1+r)/(1-r)$, then the set $\{z \in \mathbf{C}; |z| < r\}$ coincides with the set of points in $D$ whose distances from 0 with respect to $ds_D^2$ do not exceed $r'$. Let $N(a; r')$ be the set of points in $M$ which are at distance $r'$ or less from $a$ with respect to the metric $ds_M^2$. Since the metric $ds_M^2$ is complete, $N(a; r')$ is compact and hence is contained in the set $\{z \in \mathbf{C}; |z| \leqq S\}$ for a suitable positive number $S$. Let $f : D \to M$ be a holomorphic map such that $f(0) = a$. Since $f$ is distance-decreasing, i.e., $f^*(ds_M^2) \leqq ds_D^2$ by Theorem 2.1, it maps the set $\{z \in D; |z| \leqq r\}$ into $N(a, r')$. □

Since the construction of the metric $ds_M^2$ in the preceding section is fairly explicit, we can give an estimate on $S(a, r)$. Let $ds_M^2 = 2g\, dz\, d\bar{z}$ be the complete metric with curvature $\leqq -4$ constructed on $\mathbf{C} - \{0, 1\}$ in the preceding section. From our construction, we see that $g > A^2 h$ on $\mathbf{C} - \{0, 1\}$ for a suitable positive number $A$, where $h$ was defined in Sec. 3. From the definition of $h$, we see that, in the domain $\{z \in \mathbf{C}; |z| \geqq 3\}$, $h$ coincides with

$$\frac{1}{|z|^2(\log|z|^2)^2}.$$

If we use the polar coordinate system $(r, \theta)$, then we have

$$\begin{aligned} ds_M^2 &= 2g\, dz\, d\bar{z} > 2A^2 h\, dz\, d\bar{z} \\ &\geqq \frac{A^2\, dr\, dr}{2r^2(\log r)^2} \quad \text{on } \{z \in \mathbf{C}; |z| \geqq 3\}. \end{aligned}$$

Let $a$ and $b$ be two points of $\mathbf{C}$ such that $3 \leqq |a| \leqq |b|$. Then the distance from $a$ to $b$ with respect to $ds_M^2$ is greater than

$$\int_{|a|}^{|b|} \frac{A\, dr}{\sqrt{2} r \log r} = \frac{A}{\sqrt{2}} \log \frac{\log|b|}{\log|a|}.$$

If $|a| < 3$, then it suffices to replace $\log |a|$ by $\log 3$. On the other hand, the distance from $0 \in D$ to $z \in D$ with respect to $ds_D^2$ is equal to

$$\frac{1}{2} \log \frac{1+|z|}{1-|z|}.$$

Let $f : D \to M = \mathbf{C} - \{0, 1\}$ be a holomorphic map. Since it is distance-decreasing, we have

$$\frac{A}{\sqrt{2}} \log \frac{\log |f(z)|}{\log |f(0)|} < \frac{1}{2} \log \frac{1+|z|}{1-|z|},$$

provided that $|f(0)| \geqq 3$ and $|f(z)| \geqq 3$. If $|f(0)| < 3$ and $|f(z)| \geqq 3$, it suffices to replace $\log |f(0)|$ by $\log 3$. In general, we obtain

$$\log |f(z)| < \mathrm{Max}\left\{\log 3, B\left(\frac{1+|z|}{1-|z|}\right)^C\right\},$$

where $B = \mathrm{Max}\{\log 3, \log |f(0)|\}$ and $C = \sqrt{2}/A$.

If we wish to estimate the constant $C$, we have to write down the function $s(z, \bar{z})$ in Sec. 3 explicitly.

Similar estimates on $\log f(z)$ have been obtained by Ostrowski [1], Pfluger [1], and Ahlfors [1]. See also Heins [1].

## 5 Compact Riemann Surfaces of Genus $\geqq 2$

In this section we prove the following statement:

**Theorem 5.1** *Every compact Riemann surface $M$ of genus $g \geqq 2$ admits a Kaehler metric whose Gaussian curvature is bounded above by a negative constant.*

**Proof** If we make use of the classical result that every compact Riemann surface $M$ of genus $g \geqq 2$ is a quotient of the upper half-plane $H$ by a discontinuous group $\Gamma$ of linear fractional transformations acting freely on $H$, we see easily that $M$ admits a Kaehler metric of constant negative curvature. We shall give here a more elementary proof following Grauert and Reckziegel [1].

Since the genus of $M$ is $g$, there are $g$ linearly independent holomorphic 1-forms $\omega_1, \ldots, \omega_g$ on $M$. We set

$$ds^2 = \omega_1 \bar{\omega}_1 + \omega_2 \bar{\omega}_2.$$

In terms of a local coordinate system $z$ of $M$, we may write

$$ds^2 = (f_1 \bar{f}_1 + f_2 \bar{f}_2)\, dz\, d\bar{z}, \quad \text{where } \omega_i = f_i\, dz.$$

If $p_1, \ldots, p_k$ is the set of common zeros of $\omega_1$ and $\omega_2$, then $ds^2$ is a Kaehler metric $M - \{p_1, \ldots, p_k\}$. By a simple calculation we see that the Gaussian curvature $K$ is given by

$$K = -2|f_1' f_2 - f_1 f_2'|^2/(f_1\bar{f}_1 + f_2\bar{f}_2)^3.$$

Since $\omega_1$ and $\omega_2$ are linearly independent, it follows that $f_2 \not\equiv 0$ and $f_1/f_2 \not\equiv$ constant, and consequently $K \not\equiv 0$. Hence there are at most finitely many points $p_{k+1}, \ldots, p_m$ in $M - \{p_1, \ldots, p_k\}$ where the curvature $K$ vanishes. For each $p_i$, choose a neighborhood $V_i$ in such a way that $V_i \cap V_j = \emptyset$ for $i \neq j$. In each $V_i$ we shall modify the metric so that the Gaussian curvature becomes negative everywhere on $M$. We fix $i, i = 1, \ldots, m$. Choosing a local coordinate system $z$ in $V_i$ such that $z(p_i) = 0$, let $r$ be a positive number and

$$B = \{z; |z| < r\} \Subset V_i, \quad B' = \{z; |z| < r/2\}.$$

Let $a(z, \bar{z})$ be a $C^\infty$ function on $V_i$ such that

$$0 \leqq a(z, \bar{z}) \leqq 1, \quad a|B' \equiv 1, \quad a|(V_i - B) \equiv 0.$$

Let $c$ be a constant, $0 < c < 1$, and set

$$g = f + ch, \quad \text{where } f = f_1\bar{f}_1 + f_2\bar{f}_2 \quad \text{and} \quad h = a \cdot (1 + z\bar{z}).$$

Then the metric $g\, dz\, d\bar{z}$ coincides with $ds^2$ on $V_i - B$. Since the curvature of $ds^2$ is bounded above by a negative constant on the compact set $\bar{B} - B'$, the metric $g\, dz\, d\bar{z}$ has a strictly negative curvature on $\bar{B} - B'$ if $c$ is sufficiently small. From the estimate on $K(g)$ given at the end of Sec. 3, we see that the curvature of $g\, dz\, d\bar{z}$ is strictly negative in $B'$ if $c$ is sufficiently small. □

**Remark** It is known (see, for instance, Springer [1, p. 270]) that if the genus $g \geqq 1$, there is no point of $M$ where all holomorphic 1-forms vanish. It is therefore possible to choose $\omega_1$ and $\omega_2$ in the foregoing proof in such a way that $\omega_1\bar{\omega}_1 + \omega_2\bar{\omega}_2$ is positive definite everywhere on $M$. But this does not simplify the proof.

## 6 Holomorphic Mappings from an Annulus into an Annulus

Let $A$ be the annulus in $\mathbf{C}$ defined by

$$A = \{z \in \mathbf{C};\ 0 < r < |z| < R\}.$$

The number $M = \log(R/r)$ is called the *modulus* of $A$. Two annuli with the same modulus are clearly biholomorphic to each other under a suitable homothetic transformation. We may therefore assume that $rR = 1$.

Let $A_1$ and $A_2$ be two annuli with moduli $M_1$ and $M_2$, respectively:

$$A_k = \{z \in \mathbf{C};\ 0 < r_k < |z| < 1/r_k\}, \quad k = 1, 2.$$

Let $f : A_1 \to A_2$ be a holomorphic mapping and $f_* : \pi_1(A_1) \to \pi_1(A_2)$ the induced homomorphism on the fundamental groups of $A_1$ and $A_2$. Let $\alpha_k$ be the generator of $\pi_1(A_k) = Z, k = 1, 2$. Then the *degree of* $f$, denoted by $\deg f$, is defined by

$$f_*(\alpha_1) = (\deg f)\alpha_2.$$

For each integer $m$ such that $|m| \leqq M_2/M_1$, we define a holomorphic mapping $f_m : A_1 \to A_2$ with $\deg f = m$ as follows:

$$f_m(z) = z^m \quad \text{for } z \in A_1.$$

From topology we know that any mapping $f : A_1 \to A_2$ (holomorphic or not) of degree $m$ with $|m| \leqq M_2/M_1$ is homotopic to $f_m$. In fact, any two mappings $A_1$ into $A_2$ with the same degree are homotopic to each other. We know also that for any integer $m$ there is a mapping (not necessarily holomorphic) of $A_1$ into $A_2$ with degree $m$. But we have

**Theorem 6.1** *Let $A_1$ and $A_2$ be annuli with moduli $M_1$ and $M_2$ as above. If $f : A_1 \to A_2$ is a holomorphic mapping, then $|\deg f| \leqq M_2/M_1$. If $M_2/M_1$ is an integer, then a holomorphic mapping $f : A_1 \to A_2$ with degree $m = \pm M_2/M_1$ coincides with the mapping $f_m$ up to a rotation.*

**Proof** We consider the band $B = \{z \in \mathbf{C}; -b < \operatorname{Im} z < b\}$ of width $2b$. We know that it is conformally equivalent to the open unit disk or equivalently to the upper half-plane $H = \{w \in \mathbf{C};\ \operatorname{Im} w > 0\}$ in $\mathbf{C}$. Indeed, the mapping $z \in B \to ie^{\pi z/2b} \in H$ is a biholomorphic mapping. The invariant metric $ds_H^2$ of curvature $-1$ on $H$ is given by

$$ds_H^2 = \frac{dw\, d\bar{w}}{v^2} \quad \text{where } w = u + iv.$$

If we induce this metric to the band $B$ by the holomorphic mapping given above, we obtain the following invariant metric $ds_B^2$ of curvature $-1$:

$$ds_B^2 = \frac{\pi^2\, dz\, d\bar{z}}{4b^2 \cos^2(\pi y/b)} \quad \text{where } z = x + iy.$$

We consider now a holomorphic mapping $p$ from $B$ onto the annulus $A = \{w \in \mathbf{C}; r < |w| < 1/r\}, r = e^{-2\pi b}$, defined by

$$p(z) = e^{2\pi i z} \quad z \in B.$$

Then $p : B \to A$ is a covering projection. We denote by $ds_A^2$ the metric on $A$ induced by $ds_B^2$. Then the rectangle $\{z \in B; 0 \leqq x \leqq 1\}$ is a fundamental domain for this covering space $(B, A, p)$. The projection $p$ maps the upper edge, the lower edge, and the two vertical edges of this rectangle onto the inner boundary, the outer boundary of the annulus $A$, and the segment $\{w = u + iv \in A; u > 0, v = 0\}$, respectively. It is also clear that $p$ maps $\{z = x + iy \in B; 0 \leqq x \leqq 1, y = 0\}$ onto the unit circle $\{w \in A; |w| = 1\}$, which is the generator of the fundamental group $\pi_1(A)$. Consider a curve $w(t), 0 \leqq t \leqq 1$, in $A$ which represents the generator of $\pi_1(A)$. We may assume that $w(0) = w(1) > 0$. To compute its length with respect to $ds_A^2$, we consider a curve $z(t)$ in the rectangle $\{z \in B; 0 \leqq x \leqq 1\}$ such that $p[z(t)] = w(t), \mathrm{Re}[z(0)] = 0$ and $\mathrm{Re}[z(1)] = 1$, and compute the length of $z(t)$ with respect to $ds_B^2$. From the expression of $ds_B^2$ given above, we see that the curve $w(t)$ has the least length when it is the circle $|w(t)| = 1$, i.e., when $z(t)$ is real for all $t$. This least length is given by

$$\int_0^1 \frac{\pi\, dx}{2b} = \frac{\pi}{2b}.$$

We have shown that the circle $w(t) = e^{2\pi it}, 0 \leqq t \leqq 1$, represents the generator $\alpha$ of $\pi_1(A)$ and has arc-length $\pi/2b$ with respect to $ds_A^2$ and that any closed curve in $A$ representing $\alpha$ has arc-length $\geqq \pi/2b$, where the equality holds only when the closed curve coincides with the unit circle up to a parametrization. Similarly, the curve $w(t) = e^{2m\pi it}, 0 \leqq t \leqq 1$, representing $m\alpha \in \pi_1(A)$ has arc-length $|m\pi/2b|$ and any closed curve representing $m\alpha$ has arc-length $> |m\pi/2b|$ unless it coincides with $w(t) = e^{2m\pi it}$ up to a parametrization.

Given two annuli $A_1$ and $A_2$, we consider the bands $B_1$ and $B_2$ of widths $b_1$ and $b_2$ and covering projections $p_1 : B_1 \to A_1$ and $p_2 : B_2 \to A_2$ as above. Let $f : A_1 \to A_2$ be a holomorphic mapping. Since $B_1$ is holomorphically equivalent to the unit open disk, we may apply Schwarz–Pick–Ahlfors lemma (Theorem 2.1) to the map $f \circ p_1 : B_1 \to A_2$ and see that $f \circ p_1$ is distance-decreasing, i.e., $(f \circ p_1)^* ds_{A_2}^2 \leqq ds_{B_1}^2$. Since $p_1$ is a local isometry, we see that $f$ itself is also distance-decreasing, i.e., $f^* ds_{A_2}^2 \leqq ds_{A_1}^2$. We consider the unit circle $w_1(t) = e^{2\pi it}, 0 \leqq t \leqq 1$, in $A_1$ and its image curve $w_2(t) = f[w_1(t)]$, $0 \leqq t \leqq 1$, in $A_2$. Since $w_1(t)$ has arc-length $\pi/2b_1$ and $f$ is distance-decreasing, $w_2(t)$ has arc-length $\leqq \pi/2b_1$. If $\deg f = m$ so that $w_2(t)$ represents $m\alpha_2 \in \pi_1(A_2)$, where $\alpha_2$ is the generator of $\pi_1(A_2)$, then $w_2(t)$ has arc-length $\geqq |m\pi/2b_2|$. Hence, $|m\pi/2b_2| \leqq \pi/2b_1$, that is, $|m| \leqq b_2/b_1$. Since $r_k = e^{-2\pi b_k}$ and $M_k = \log(1/r_k^2) = -2\log r_k$ for $k = 1, 2$, we have $b_2/b_1 = M_2/M_1$. Hence, $|m| \leqq M_2/M_1$, which proves the first assertion of the theorem.

Assume that $M_2/M_1$ is an integer and $|m| = M_2/M_1$. Then $w_2(t)$ has arc-length $|m\pi/2b_2| = \pi/2b_1$. Hence, the closed curve $w_2(t)$ coincides with $e^{2m\pi it}, 0 \leqq t \leqq 1$, up to a parametrization. Since $f$ is distance-decreasing and $w_2(t)$ has arc-length $\pi/2b_1$, $f$ maps $w_1(t)$ onto $w_2(t)$ isometrically. This implies that $w_2(t)$ coincides actually with $e^{2m\pi it}$ up to a rotation. It follows that $f$ and $f_m$ coincide on the unit circle $w_1(t)$ of $A_1$ up to a rotation and hence they coincide of $A_1$ up to a rotation. □

**Corollary 6.2** *Let $f$ be a holomorphic mapping from a annulus $A = \{z \in \mathbf{C}; r < |z| < 1/r\}$ into itself. Then either $f$ is homotopic to a constant map or is of the form $f(z) = e^{\pm 2\pi i(z+a)}$, where $a$ is a real number.*

Theorem 6.1 is due to Huber [1]. The proof presented here is perhaps a little more differential geometric. In connection with the result of this section, see also Schiffer [1], Jenkins [1], and Landau and Osserman [1, 2].