Chapter 1

Risk Models and Ruin Theory

Three particular questions of interest in classical ruin theory are (a) *the time of ruin*, (b) *the deficit at ruin*, and (c) *the surplus immediately before ruin.* From a mathematical point of view, a crucial role is played by the amounts of surplus before and after ruin. In this chapter we mainly present the major results about ruin probabilities, the distribution of surplus before and after ruin for a compound Poisson model with a constant premium rate and a constant interest rate. The compound Poisson risk process with a constant interest force is an interesting stochastic model in risk theory and it provides a basic understanding about how investments will affect the ruin probability and related ruin functions. At first, we provide some results on the severity of ruin for a compound Poisson model with a constant interest rate. Then, we investigate the distribution of surplus process immediately before ruin in particular. Equations satisfied by the distributions of surplus immediately before ruin and their Laplace transform are given. Some special cases are also discussed and Lundberg type bounds are presented. Next, by using the techniques of [Kalashnikov and Konstantinides(2000)] and a formula obtained by [Yang and Zhang (2001a)], we give asymptotic formulas of the low and upper bounds for the distribution of the surplus immediately after ruin under subexponential claims. Finally, a class of risk processes in which claims occur as a renewal process is studied. A clear expression for Laplace transform of the finite time ruin probability time is well given when the claim amount distribution is a mixed exponential. As its consequence, a well-known result about ultimate ruin probability in the classical risk model is obtained. All results in Sections 1 and 2 of this chapter are from [Yang and Zhang (2001a)] and [Yang and Zhang (2001b)]. The main results in Sections 3 and 4 are from [Wang *et. al.* (2004)] and [Wang and Liu(2002)].

1.1 On the Distribution of Surplus Immediately after Ruin under Interest Force

In this section, we consider the problem of the severity of ruin for a compound Poisson model with a constant interest rate. By using the techniques of [Sundt and Teugels (1995)], equations satisfied by the distributions of surplus immediately after ruin have been obtained. Some special cases are also discussed. Some results on the severity of ruin given in this section are similar to those in [Sundt and Teugels (1995)] on ruin probability.

1.1.1 *The Risk Model*

We use the same model as in [Sundt and Teugels (1995)]. Let $U_\delta(t)$ denote the value of the reserve at time t. $U_\delta(t)$ is given by

$$dU_\delta(t) = p\,dt + U_\delta(t)\,\delta\,dt - dX(t),$$

where p is a constant which denotes the premium rate that the insurance company receives and δ is the interest force,

$$X(t) = \sum_{j=1}^{N(t)} Y_j\ ,$$

where $N(t)$ denotes the number of claims occurring in an insurance portfolio in the time interval $(0, t]$ and Y_i denotes the amount of the ith claim. We assume that $\{N(t), t \geqslant 0\}$ is a homogeneous Poisson process with intensity λ, and $N(t)$ and Y_j are two independent processes. We further assume that Y_j (j $=$ 1, 2, ...) are positive and mutually independent and $\{Y_j\}$ is an identically distributed random sequence with common distribution F. We also assume that $F(0) = 0$ and $\mu_k = \int_0^\infty x^k dF(x)$. When $k = 1$, we denote $\mu = \mu_1$.

It follows from [Sundt and Teugels (1995)] that

$$U_\delta(t) = u\,e^{\delta t} + p\bar{s}_{\overline{t}|}^{(\delta)} - \int_0^t e^{\delta(t-v)} dX(v)\ ,$$

where $u = U(0) \geqslant 0$ and

$$\bar{s}_{\overline{t}|}^{(\delta)} = \int_0^t e^{\delta v} dv = \begin{cases} t, & \text{if} \quad \delta = 0, \\ \frac{e^{\delta t} - 1}{\delta}, & \text{if} \quad \delta > 0. \end{cases}$$

For convenience, we will drop the index δ when the force of interest is zero.

Let $\psi_\delta(u)$ denote the ultimate ruin probability with initial reserve u. That is

$$\psi_\delta(u) = P\left\{ \bigcup_{t \geqslant 0} (U_\delta(t) < 0) \,\middle|\, U_\delta(0) = u \right\}.$$

We use $\bar{\psi}_\delta(u) = 1 - \psi_\delta(u)$ to denote the non-ruin probability (i.e. the probability that ruin never occurs).

It follows immediately from [Sundt and Teugels (1995)] that the integral equation is satisfied

$$\bar{\psi}_\delta(u) = \frac{p}{p+\delta u}\bar{\psi}_\delta(0) + \frac{1}{p+\delta u}\int_0^u \bar{\psi}_\delta(u-y)\{\delta + \lambda(1-F(y))\}dy. \tag{1.1}$$

Next, we are interested in the function $G_\delta(u, y)$, representing the probability of ruin beginning with initial reserve u and that the deficit at the time of ruin is less than $y > 0$.

$$G_\delta(u,y) = \Pr\left(T < \infty, -y < U_\delta(T) < 0 \,\middle|\, U_\delta(0) = u\right),$$

where T is the ruin time defined by

$$T = \inf\{t \geqslant 0 : U_\delta(t) < 0\}.$$

It is easy to see that $\psi_\delta(u) = \lim_{y \to +\infty} G_\delta(u, y)$.

For notational convenience, we define

$$\begin{aligned} \bar{G}_\delta(u,y) &= \psi_\delta(u) - G_\delta(u,y) \\ &= P\left(T < \infty, U_\delta(T) \leqslant -y \,\middle|\, U_\delta(0) = u\right), \\ \bar{\bar{G}}_\delta(u,y) &= 1 - \bar{G}_\delta(u,y). \end{aligned}$$

1.1.2 *Equations for $\bar{\bar{G}}_\delta(u, y)$*

1.1.2.1 *Integral Equations for $\bar{\bar{G}}_\delta(u,y)$, $\bar{G}_\delta(u,y)$ and $G_\delta(u,y)$*

In this subsection, we will try to obtain integral equations for $\bar{\bar{G}}_\delta(u,y)$, $\bar{G}_\delta(u,y)$ and $G_\delta(u,y)$. Using the renewal property of the surplus process, an integral equation, satisfied by the function which we are interested in, can be obtained. This is a commonly used technique in risk theory. Although we cannot, in general, solve the integral equation, some asymptotic results can often be obtained by using the integral equation. The main result of this subsection is given in the following theorem.

Theorem 1.1

$$\begin{aligned}\bar{\bar{G}}_\delta(u,y) &= \frac{p}{p+\delta u}\bar{\bar{G}}_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u \bar{\bar{G}}_\delta(u-z,y)[\delta+\lambda(1-F(z))]dz \\ &\quad + \frac{\lambda}{p+\delta u}\int_0^y (F(z)-F(u+z))dz, \end{aligned} \tag{1.2}$$

$$\begin{aligned}\bar{G}_\delta(u,y) &= \frac{p}{p+\delta u}\bar{G}_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u \bar{G}_\delta(u-z,y)[\delta+\lambda(1-F(z))]dz \\ &\quad - \frac{\lambda}{p+\delta u}\int_0^u (1-F(z))dz - \frac{\lambda}{p+\delta u}\int_0^y (F(z)-F(u+z))dz, \end{aligned} \tag{1.3}$$

$$\begin{aligned}G_\delta(u,y) &= \frac{p}{p+\delta u}G_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u G_\delta(u-z,y)[\delta+\lambda(1-F(z))]dz \\ &\quad + \frac{\lambda}{p+\delta u}\int_0^y (F(z)-F(u+z))dz. \end{aligned} \tag{1.4}$$

Proof. The proof goes along the lines of page 10 of [Sundt and Teugels (1995)]. We only present the main steps of the proof. Notice that, given the first claim time $T_1 = t$ and the first claim amount $Y_1 = z$, the reserve just after the first claim is $ue^{\delta t} + p \cdot \frac{e^{\delta t}-1}{\delta} - z$. Therefore

$$\begin{aligned}\bar{\bar{G}}_\delta(u,y) &= E[\bar{\bar{G}}_\delta(ue^{\delta T_1} + p\frac{e^{\delta T_1}-1}{\delta} - Y_1, y)] \\ &= \int_0^{+\infty} \lambda e^{-\lambda t} \int_{(0,ue^{\delta t}+p\frac{e^{\delta t}-1}{\delta}+y)} \bar{\bar{G}}_\delta\left(ue^{\delta t} + p\frac{e^{\delta t}-1}{\delta} - z, y\right) \\ &\quad dF(z)dt \\ &= \lambda(p+\delta u)^{\frac{\lambda}{\delta}} \int_u^{+\infty} (p+\delta s)^{-\frac{\lambda}{\delta}-1} \int_0^{s+y} \bar{\bar{G}}_\delta(s-z,y)dF(z)ds, \end{aligned}$$

where the last equality is obtained by using the substitution

$$s = ue^{\delta t} + p\frac{e^{\delta t}-1}{\delta}.$$

By taking partial derivative of the above expression with respect to u, and rearrange the terms, we have

$$(p+\delta u)\frac{\partial}{\partial u}\bar{\bar{G}}_\delta(u,y) = \lambda\bar{\bar{G}}_\delta(u,y) - \lambda\int_0^{u+y}\bar{\bar{G}}_\delta(u-z,y)dF(z). \tag{1.5}$$

By integrating both sides of (1.5) from 0 to u, we have that the left hand side of (1.5) equals

$$(p+\delta u)\,\bar{\bar{G}}_\delta\,(u,y) - p\,\bar{\bar{G}}_\delta\,(0,y) - \delta\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv$$

and the right hand side of (1.5) equals

$$\lambda\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv + \lambda\int_0^u\int_0^{v+y} \bar{\bar{G}}_\delta\,(v-z,y)d(1-F(z))dv. \quad (1.6)$$

After some calculation and rearrangement of the terms, the second term of the right hand side of (1.6) equals

$$\begin{aligned}&-\lambda\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv + \lambda\int_0^u (1-F(z))\,\bar{\bar{G}}_\delta\,(u-z,y)dz\\&+\lambda\int_0^y \bar{\bar{G}}_\delta\,(-z,y)(F(z)-F(u+z))dz.\end{aligned}$$

So we have

$$\begin{aligned}&(p+\delta u)\,\bar{\bar{G}}_\delta\,(u,y) - p\,\bar{\bar{G}}_\delta\,(0,y) - \delta\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv\\=&\ \lambda\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv - \lambda\int_0^u \bar{\bar{G}}_\delta\,(v,y)dv\\&+\lambda\int_0^u (1-F(z))\,\bar{\bar{G}}_\delta\,(u-z,y)dz\\&+\lambda\int_0^y \bar{\bar{G}}_\delta\,(-z,y)(F(z)-F(u+z))dz.\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\bar{G}}_\delta\,(u,y) =&\ \frac{p}{p+\delta u}\,\bar{\bar{G}}_\delta\,(0,y)\\&+\frac{1}{p+\delta u}\int_0^u \bar{\bar{G}}_\delta\,(u-z,y)\big(\delta+\lambda(1-F(z))\big)dz\\&+\frac{\lambda}{p+\delta u}\int_0^y \bar{\bar{G}}_\delta\,(-z,y)(F(z)-F(u+z))dz.\end{aligned}$$

By the definition of $\bar{\bar{G}}_\delta\,(u,y)$, we know that

$$\bar{\bar{G}}_\delta\,(-z,y) = 1 \quad \text{for all} \quad z\in(0,y).$$

Based on this, we obtain Equation (1.2). Equations (1.3) and (1.4) can be obtained from Equations (1.1) and (1.2). □

1.1.2.2 *The Case $\delta = 0$*

Denote the equilibrium distribution of F by F_1, which is defined by

$$F_1(x) = \frac{1}{\mu}\int_0^x (1 - F(v))dv.$$

Let $\nu_k = \int_0^\infty x^k dF_1(x)$ be the moments of the equilibrium distribution. From [Sundt and Teugels (1995)], we know that

$$\nu_k = \frac{1}{k+1}\cdot\frac{\mu_{k+1}}{\mu}.$$

Suppose the Laplace transforms of F_1 are given by

$$\phi(s) = \int_0^{+\infty} e^{-su}dF_1(u) = \phi_0(s),$$
$$\phi_y(s) = \int_y^{+\infty} e^{-su}dF_1(u).$$

The following result generalizes the asymptotic result for the probability of ruin, as given in [Grandell (1991)], to the case where the severity of ruin is taken into account.

Theorem 1.2 *Assume that the net profit condition $\lambda\mu < p$ is true and that there exists a positive constant R, called adjustment coefficient, such that*

$$\frac{\lambda}{p}\int_0^{+\infty} e^{Rz}(1 - F(z))dz = 1. \tag{1.7}$$

Then, for $u \longrightarrow +\infty$ we have that

$$G(u,y) \sim \frac{1 - e^{-Ry}\int_y^{+\infty}\frac{\lambda}{p}e^{Rz}(1-F(z))dz - \frac{\lambda\mu}{p}F_1(y)}{\frac{\lambda}{p}(-\phi'(-R) - \frac{p}{\lambda})}\, e^{-Ru}. \tag{1.8}$$

Proof. The proof here is similar to that in [Grandell (1991)] for the ruin probability case. When $\delta = 0$, (1.4) becomes

$$G(u,y) = G(0,y) + \frac{\lambda}{p}\int_0^u G(u-z,y)(1-F(z))dz$$
$$+\frac{\lambda}{p}\int_0^y (F(z) - F(u+z))dz. \tag{1.9}$$

Since $\int_0^{+\infty}\frac{\lambda}{p}(1 - F(z))dz = \frac{\lambda\mu}{p} = m < 1$, equation (1.9) is a defective renewal equation. From equation (1.7), we know that $\frac{\lambda}{p}e^{Rz}(1 - F(z))$ is the density of a proper probability distribution. Multiplication of (1.9) by e^{Ru} yields

$$e^{Ru}G(u,y) = e^{Ru}G(0,y) + \int_0^u G(u-z,y)e^{R(u-z)}\frac{\lambda e^{Rz}}{p}(1-F(z))dz$$
$$+\frac{\lambda}{p}e^{Ru}\int_0^y (F(z)-F(u+z))dz,$$

which is a proper renewal equation. From the key renewal theorem, it follows that

$$\lim_{u\to\infty} e^{Ru}G(u,y) = \frac{c_1(y)}{c_2(y)},$$

where

$$c_1(y) = \int_0^{+\infty}\Big(e^{Ru}G(0,y) + \frac{\lambda}{p}e^{Ru}\int_0^y \big(F(z)-F(u+z)\big)dz\Big)du, \tag{1.10}$$

$$\begin{aligned} c_2(y) &= \frac{\lambda}{p}\int_0^{+\infty} z\,e^{Rz}(1-F(z))dz \\ &= \frac{\lambda}{p}\cdot\frac{1}{R}(-\phi'(-R)-p/\lambda). \end{aligned} \tag{1.11}$$

From [Willmot and Lin (1998)], we know that

$$G(0,y) = \frac{\lambda\mu}{p}F_1(y) = \frac{\lambda}{p}\int_0^y (1-F(z))dz.$$

Plugging this expression of $G(0,y)$ into (1.10), we have

$$\begin{aligned} c_1(y) &= \int_0^{+\infty}\frac{\lambda}{p}e^{Ru}\Big(\int_0^y \big(1-F(u+z)\big)dz\Big)du \\ &= \int_y^{+\infty}\frac{\lambda}{p}\int_{z-y}^z e^{Ru}\big(1-F(z)\big)du\,dz + \int_0^y\frac{\lambda}{p}\int_0^z e^{Ru}(1-F(z))du\,dz \\ &= \frac{1}{R}[1-e^{-Ry}\int_y^{+\infty}\frac{\lambda}{p}\big(1-F(z)\big)\cdot e^{Rz}dz] - \frac{\lambda\mu}{pR}F_1(y). \end{aligned}$$

From this and (1.11), we obtain (1.8). □

Remark 1.1 *Theorem 1.2 shows that $G(u,y)$ converges to zero at an exponential rate when the initial surplus becomes large. This is the same* *as in the case for ruin probability. If we let* $y \to +\infty$, *we have, for* u *sufficiently large,*

$$\lim_{y\to\infty} G(u,y) = \psi(u) \sim \frac{1-\frac{\lambda\mu}{p}}{\frac{\lambda}{p}(-\phi'(-R)-\frac{p}{\lambda})}\, e^{-Ru}.$$

This gives the same result as in [Grandell (1991)].

Remark 1.2 *Following [Sundt and Teugels (1995)], if we introduce the auxiliary distribution*

$$B_\delta(u,y) = \frac{\bar{\bar{G}}_\delta\,(u,y) - \bar{\bar{G}}_\delta\,(0,y)}{1-\bar{\bar{G}}_\delta\,(0,y)},$$

the integral equation satisfied by B_δ *can be obtained from (1.2). Following the exact same steps as those in [Sundt and Teugels (1995)], we can obtain a solution for the Laplace transform of* B_δ.

Remark 1.3 *Similar to [Sundt and Teugels (1995)], we can also obtain the following expression for* $\bar{\bar{G}}_\delta\,(0,y)$:

$$\begin{aligned}\bar{\bar{G}}_\delta\,(0,y) = 1 - \lambda\mu\Big(p\int_0^{+\infty} e^{-\int_0^z (p-\lambda\mu\phi(\delta w))dw}dz\Big)^{-1} \\ \cdot\Big(\int_0^{+\infty} e^{\delta zy}\phi_y(\delta z)e^{-\int_0^z (p-\lambda\mu\phi(\delta w))dw}dz\Big). \qquad (1.12)\end{aligned}$$

1.1.3 *Upper and Lower Bounds for $\bar{G}_\delta(0,y)$*

Equation (1.12) provides an expression for $\bar{\bar{G}}_\delta\,(0,y)$. However, the expression is complex and difficult to compute. Similar to the case of ruin probability, in many situations, we do not need the exact value of $\bar{\bar{G}}_\delta\,(0,y)$. An upper or lower (or both) bound will be sufficient. In this subsection, we provide both upper and lower bounds for $\bar{G}_\delta(0,y)$. The main result of this subsection is the following theorem.

Theorem 1.3

$$\begin{aligned}&p^{-1}(p-\lambda\mu)\Big[\sqrt{\frac{\lambda\mu}{\nu_1\delta}}\Big(1-F_1(y)+\sqrt{\frac{\underline{\nu_1^2}\delta}{\lambda\mu\nu_1}}\beta\Big)R(\beta) - \frac{\underline{\underline{\nu_1}}}{\nu_1}e^{\frac{\beta^2}{2}}\Big] \\ &\leqslant \bar{G}_\delta(0,y) \leqslant \frac{(1-F_1(y))\lambda\mu}{(p-\lambda\mu)\Big[R(\alpha)\cdot p\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}\Big]}, \qquad (1.13)\end{aligned}$$

where $R(\alpha)$ is the well known Mills' ratio, that is

$$R(\alpha) = e^{\frac{\alpha^2}{2}} \int_{\alpha}^{+\infty} e^{-\frac{x^2}{2}}\, dx$$

and $\alpha = \frac{\lambda\mu - p}{\sqrt{\lambda\mu\delta\nu_1}}$, $\phi_y'(0) = \int_y^{+\infty} x\, dF_1(x) = \underline{\nu_1}$.

Proof. Put

$$\begin{aligned} A_1 &= 1 + \int_0^{+\infty} \lambda\mu\phi(\delta z) e^{\int_0^z (\lambda\mu\phi(\delta w) - p)dw} dz \\ &\leqslant 1 + \int_0^{+\infty} \lambda\mu e^{\int_0^z (\lambda\mu - p)dw} dz = \frac{p}{p - \lambda\mu}, \end{aligned}$$

where the inequality holds because $\phi(s)$ is a strictly decreasing convex function. On the other hand,

$$\begin{aligned} A_1 &= p \int_0^{+\infty} e^{-\int_0^z (p - \lambda\mu\phi(\delta w))dw} dz \\ &\geqslant p \int_0^{+\infty} e^{\int_0^z (\lambda\mu(1 - \nu_1\delta w) - p)dw} dz = p \int_0^{+\infty} e^{-\frac{(z - \frac{p - \lambda\mu}{\lambda\mu\delta\nu_1})^2}{2\frac{1}{\lambda\mu\delta\nu_1}}} \cdot e^{\frac{(p - \lambda\mu)^2}{2\lambda\mu\delta\nu_1}} dz. \end{aligned}$$

Using the substitution

$$x = \frac{z - \frac{p - \lambda\mu}{\nu_1\delta\lambda\mu}}{\sqrt{\frac{1}{\nu_1\delta\lambda\mu}}} = \frac{z}{\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}} + \alpha,$$

we have

$$\begin{aligned} A_1 &\geqslant p \int_{\alpha}^{+\infty} e^{-\frac{x^2}{2}} \cdot e^{\frac{\alpha}{2}} \cdot \sqrt{\frac{1}{\lambda\mu\delta\nu_1}} dx = p\sqrt{\frac{1}{\lambda\mu\delta\nu_1}} e^{\frac{\alpha^2}{2}} \int_{\alpha}^{+\infty} e^{-\frac{x^2}{2}} dx \\ &= p\sqrt{\frac{1}{\lambda\mu\delta\nu_1}} R(\alpha). \end{aligned}$$

Therefore,

$$p\sqrt{\frac{1}{\lambda\mu\delta\nu_1}} \cdot R(\alpha) \leqslant A_1 \leqslant \frac{p}{p - \lambda\mu}.$$

Put

$$\begin{aligned} A_2 &= \int_0^{+\infty} e^{\delta z y} \phi_y(z\delta) e^{\int_0^z (\lambda\mu\phi(\delta w) - p)dw} dz \\ &\leqslant \int_0^{+\infty} \left(\int_0^{+\infty} dF_1(y + s) \right) e^{(\lambda\mu - p)z} dz = \frac{1 - F_1(y)}{p - \lambda\mu}, \end{aligned}$$

where the inequality holds because $\phi_y(s)$ is a strictly decreasing convex function. On the other hand,

$$\begin{aligned}
A_2 &= \int_0^{+\infty} e^{\delta zy}\phi_y(z\delta)e^{\int_0^z(\lambda\mu\phi(\delta w)-p)dw}dz \\
&\geqslant \int_0^{+\infty} e^{\delta yz}(1-F_1(y)-\underline{\nu_1}z\delta)e^{-\nu_1\delta\lambda\mu\frac{z^2}{2}-pz+\lambda\mu z}dz \\
&= \int_0^{+\infty}(1-F_1(y)-\underline{\nu_1}z\delta)e^{-\frac{(z-\frac{p-\lambda\mu-\delta y}{\nu_1\delta\lambda\mu})^2}{\frac{1}{2}\frac{1}{\nu_1\delta\lambda\mu}}}\cdot e^{\frac{(p-\lambda\mu-\delta y)^2}{2\nu_1\delta\lambda\mu}}dz.
\end{aligned}$$

Using the substitution

$$x = \frac{z-\frac{p-\lambda\mu-\delta y}{2\nu_1\delta\lambda\mu}}{\sqrt{\frac{1}{\nu_1\delta\lambda\mu}}}, \quad \beta = -\frac{p-\lambda\mu-\delta y}{\sqrt{\nu_1\delta\lambda\mu}},$$

we have

$$\begin{aligned}
A_2 &\geqslant \int_\beta^{+\infty}\left[(1-F_1(y))-x\cdot\sqrt{\frac{\underline{\nu_1}^2\delta}{\lambda\mu\nu_1}}-\frac{(p-\lambda\mu-\delta y)\underline{\nu_1}}{\lambda\mu\nu_1}\right]e^{-\frac{x^2}{2}} \\
&\quad\cdot e^{\frac{\beta^2}{2}}\cdot\sqrt{\frac{1}{\nu_1\delta\lambda\mu}}dx \\
&\geqslant \sqrt{\frac{1}{\nu_1\delta\lambda\mu}}\left(1-F_1(y)+\beta\sqrt{\frac{\underline{\nu_1}^2\delta}{\lambda\mu\nu_1}}\right)R(\beta)-\frac{\underline{\nu_1}}{\lambda\mu\nu_1}e^{\frac{\beta^2}{2}}.
\end{aligned}$$

Note that

$$\int_\beta^{+\infty} x\,e^{-\frac{x^2}{2}}dx \leqslant \int_0^{+\infty} x\,e^{-\frac{x^2}{2}}dx = 1$$

holds for all real β. Therefore, we have

$$\sqrt{\frac{1}{\nu_1\delta\lambda\mu}}\left(1-F_1(y)+\beta\sqrt{\frac{\underline{\nu_1}^2\delta}{\lambda\mu\nu_1}}\right)R(\beta)-\frac{\underline{\nu_1}^2}{\lambda\mu\nu_1}e^{\frac{\beta^2}{2}} \leqslant A_2 \leqslant \frac{1-F_1(y)}{p-\lambda\mu}.$$

From these inequalities and (1.12), we can obtain an inequality for $\overset{=}{G}_\delta(0,y)$. Using $\bar{G}_\delta(0,y) = 1 - \overset{=}{G}_\delta(0,y)$, (1.13) can be obtained. ☐

Remark 1.4 *Same as in [Sundt and Teugels (1995)], we can expand* $\bar{G}_\delta(0,y)$ *as a McLaurin series with respect to variable* δ. *We have*

$$\begin{aligned}\bar{G}_\delta(0,y) &= \frac{\lambda\mu}{p}(1-F_1(y)) + \frac{\lambda\mu}{p}\cdot\frac{y(1-F_1(y)) - \int_y^{+\infty} s dF_1(s)}{(p-\lambda\mu)^2}\,\delta \\ &\quad + o(\delta). \end{aligned} \tag{1.14}$$

Remark 1.5 *Similar to [Sundt and Teugels (1995)], we can obtain a Lundberg type upper bound for* $1 - B_\delta(u,y)$, *where* $B_\delta(u,y)$ *is given in Remark 1.2 of Subsection 1.1.2.*

Remark 1.6 *As remarked in Subsection 1.1.2 (Remark 1.2), we can obtain an expression for the Laplace transform of* $B_\delta(u,y)$. *If we assume that the claim random variables are exponentially distributed, similar to [Sundt and Teugels (1995)], we can obtain an expression for* $B_\delta(u,y)$ *by inverting its Laplace transform.*

1.2 On the Distribution of Surplus Immediately before Ruin under Interest Force

In last section, the distribution of surplus immediately after ruin, denoted by $G_\delta(u,x)$, was studied. In this section, we consider another interesting function, $B_\delta(u,y)$, which denotes the probability that ruin occurs beginning with the initial reserve u and that the surplus immediately prior to ruin is less than y, i.e.

$$\begin{aligned}B_\delta(u,y) &= P\big(T < +\infty, 0 < U(T-) \leqslant y | U(0) = u\big) \\ &= \psi_\delta(u) - P\big(T < +\infty, U(T-) > y | U(0) = u\big),\end{aligned} \tag{1.15}$$

where T is the ruin time. Clearly,

$$\lim_{y\to+\infty} B_\delta(u,y) = \psi_\delta(u).$$

Equations satisfied by the distributions of surplus immediately before ruin and their Laplace transform have been obtained. Some special cases are also discussed and Lundberg type bounds are presented in this section.

1.2.1 *Equations for $B_\delta(u, y)$*

1.2.1.1 *Integral Equations for $B_\delta(u, y)$*

Theorem 1.4

$$
\begin{aligned}
B_\delta(u,y) &= \frac{p}{p+\delta u} B_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u B_\delta(u-z,y)\big[\delta + \lambda(1-F(z))\big]\,dz \\
&-\frac{\lambda}{p+\delta u}\Big[I_{(u\leqslant y)}\int_0^u (1-F(v))\,dv + (1-I_{(u\leqslant y)})\int_0^y (1-F(v))\,dv\Big], \quad (1.16)
\end{aligned}
$$

where

$$
I_{\{u\geqslant y\}} = \begin{cases} 1, & \textit{if } u \geqslant y, \\ 0, & \textit{otherwise} \end{cases}
$$

is an indicator function.

Proof. We condition on the first claim time T_1 and the amount of the first claim Y_1. Given that $T_1 = t$ and $Y_1 = z$, the reserve just before the occurrence of the claim is $ue^{\delta t} + p \cdot \frac{e^{\delta t}-1}{\delta}$. Then,

$$
\begin{aligned}
B_\delta(u,y) &= E\left[B_\delta\Big(ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta} - Y_1, y\Big)\right] I_{(ue^{\delta T_1}+p\cdot\frac{e^{\delta T_1}-1}{\delta}\leqslant y)} \\
&= E\left[B_\delta\Big(ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta} - Y_1, y\Big)\,\middle|\, Y_1 \leqslant ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta}\right] \\
&\quad \cdot P\left[Y_1 \leqslant ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta}\right] I_{(ue^{\delta T_1}+p\cdot\frac{e^{\delta T_1}-1}{\delta}\leqslant y)} \\
&\quad + E\left[B_\delta\Big(ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta} - Y_1, y\Big)\,\middle|\, Y_1 > ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta}\right] \\
&\quad \cdot P\left[Y_1 > ue^{\delta T_1} + p\cdot\frac{e^{\delta T_1}-1}{\delta}\right] I_{(ue^{\delta T_1}+p\cdot\frac{e^{\delta T_1}-1}{\delta}\leqslant y)} \\
&= \int_0^{+\infty} \lambda e^{-\lambda t}\int_0^{ue^{\delta t}+p\cdot\frac{e^{\delta t}-1}{\delta}} B_\delta\Big(ue^{\delta t} + p\cdot\frac{e^{\delta t}-1}{\delta} - z, y\Big)\,dF(z)\,dt \\
&\quad + I_{(u\leqslant y)}\int_0^{\frac{\ln(\frac{\delta y+p}{\delta u+p})}{\delta}} \lambda e^{-\lambda t}\int_{ue^{\delta t}+p\cdot\frac{e^{\delta t}-1}{\delta}}^{+\infty} dF(z)\,dt.
\end{aligned}
$$

By using the substitution $s = ue^{\delta t} + p \cdot \frac{e^{\delta t}-1}{\delta}$, we obtain

$$\begin{aligned} B_\delta(u,y) &= \lambda(p+\delta u)^{\frac{\lambda}{\delta}} \int_u^{+\infty} (p+\delta s)^{-\frac{\lambda}{\delta}-1} \int_0^s B_\delta(s-z,y)\, dF(z)\, ds \\ &\quad + \lambda(p+\delta u)^{\frac{\lambda}{\delta}} I_{(u\leqslant y)} \int_u^y (\delta s + p)^{-\frac{\lambda}{\delta}-1}(1-F(s))\, ds. \end{aligned}$$

Assume that F is continuous, differentiation of the above expression gives

$$\begin{aligned} \frac{\partial B_\delta(u,y)}{\partial u} &= \frac{\lambda}{p+\delta u} B_\delta(u,y) - \frac{\lambda}{p+\delta u}\int_0^u B_\delta(u-z,y)dF(z) \\ &\quad - \frac{\lambda}{p+\delta u} I_{(u\leqslant y)}(1-F(u)). \end{aligned} \tag{1.17}$$

For $u \leqslant y$, integrating both sides of (1.17) from 0 to u, we have

$$\begin{aligned} B_\delta(u,y) &= \frac{p}{p+\delta u} B_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u B_\delta(u-z,y)\big[\delta + \lambda(1-F(z))\big]\, dz \\ &\quad - \frac{\lambda}{p+\delta u}\int_0^u (1-F(v))\, dv. \end{aligned} \tag{1.18}$$

From (1.18), we obtain (1.16) in the case where $u \leqslant y$.

For $u \geqslant y$, integrating both sides of (1.17) from y to u, we have

$$\begin{aligned} &(p+\delta u)B_\delta(u,y) - (p+\delta y)B_\delta(y,y) - \delta \int_y^u B_\delta(v,y)\, dv \\ &= \lambda \int_y^u B_\delta(v,y)dv - \lambda \int_y^u \int_0^v B_\delta(v-z,y)\, dF(z)\, dv, \end{aligned}$$

the second term on the right hand side of the above equation equals

$$\begin{aligned} &\lambda \int_y^u \int_0^v B_\delta(v-z,y)\, dF(z)\, dv \\ &= \lambda \int_y^u B_\delta(v-z,y)\, F(z)\Big|_{z=0}^{v} dv + \lambda \int_y^u \int_0^v \frac{\partial B_\delta(v-z,y)}{\partial v} F(z)\, dz\, dv \\ &= \lambda \int_0^u B_\delta(u-z,y)\, F(z)\, dz + \lambda \int_0^y \big[-B_\delta(y-z,y)\big] F(z)\, dz, \end{aligned}$$

so we have

$$\begin{aligned}&(p+\delta u)B_\delta(u,y)-(p+\delta y)B_\delta(y,y)-\delta\int_y^u B_\delta(v,y)\,dv\\ =&\ \lambda\int_y^u B_\delta(v,y)\,dv-\lambda\int_0^u B_\delta(u-z,y)\,F(z)\,dz\\ &+\lambda\int_0^y B_\delta(y-z,y)\,F(z)\,dz\\ =&\ \lambda\int_0^u \big(B_\delta(u-z,y)(1-F(z))\big)\,dz-\lambda\int_0^y B_\delta(y-z,y)(1-F(z))\,dz.\end{aligned}$$

Hence

$$\begin{aligned}B_\delta(u,y)=&\ \frac{p+\delta y}{p+\delta u}B_\delta(y,y)+\frac{1}{p+\delta u}\int_0^u B_\delta(u-z,y)[\delta+\lambda(1-F(z))]\,dz\\ &-\frac{1}{p+\delta u}\int_0^y B_\delta(y-z,y)[\delta+\lambda(1-F(z))]\,dz.\end{aligned}\tag{1.19}$$

From the definition of $B_\delta(u,y)$, it follows that $B_\delta(u,y)$ is a continuous function of $u\geqslant 0$ for a given value of y. Let $u=y$ in (1.18), we then have

$$\begin{aligned}B_\delta(y,y)=&\ \frac{p}{p+\delta y}B_\delta(0,y)+\frac{1}{p+\delta y}\int_0^y B_\delta(y-z,y)\big(\delta+\lambda(1-F(z))\big)\,dz\\ &-\frac{\lambda}{p+\delta y}\int_0^y (1-F(v))\,dv.\end{aligned}$$

Plugging $B_\delta(y,y)$ into (1.19),

$$\begin{aligned}B_\delta(u,y)=&\ \frac{p}{p+\delta u}B_\delta(0,y)+\frac{1}{p+\delta u}\int_0^u B_\delta(u-z,y)\big(\delta+\lambda(1-F(z))\big)\,dz\\ &-\frac{\lambda}{p+\delta u}\int_0^y (1-F(v))\,dv.\end{aligned}\tag{1.20}$$

This is (1.16) for $u\geqslant y$. □

1.2.1.2 *The Case $\delta=0$*

When $\delta=0$, denote $B_0(u,y)$ by $F(u,y)$, we have the following theorem.

Theorem 1.5 *When the adjustment coefficient R exists(see Theorem 1.2), we have when $u\leqslant y$ (in this case, when $u\to\infty$, $y\to\infty$)*

$$\lim_{u\to\infty}e^{Ru}F(u,y)=\frac{(1-\frac{\lambda\mu}{p})}{\frac{\lambda}{p}(-\phi'(-R)-\frac{p}{\lambda})},\tag{1.21}$$

and when $u \geqslant y$

$$\lim_{u\to\infty} e^{Ru}F(u,y) = \frac{F^*(y) - \frac{\lambda\mu}{p}F_1(y)}{\frac{\lambda}{p}\big(-\phi'(-R) - \frac{p}{\lambda}\big)}, \tag{1.22}$$

where

$$F^*(x) = \int_o^x \frac{\lambda}{p} e^{Rz}(1 - F(z))\,dz.$$

Proof. In the case of $u \leqslant y$, $\lim_{u\to\infty} e^{Ru}F(u,y) = \lim_{u\to\infty} e^{Ru}\psi(u)$. From [Grandell (1991)], we have $\psi(u) \sim \frac{1-\frac{\lambda\mu}{p}}{\frac{\lambda}{p}\left(-\phi'(-R)-\frac{p}{\lambda}\right)} e^{-Ru}$. This proves that (1.21) is true. For $u \geqslant y$, it comes from [Dickson(1992)] that

$$F(u,y) = G(u-y,y) - \frac{1-G(0,y)}{1-\psi(0)}(\psi(u-y) - \psi(u)) \quad \text{for } u \geqslant y,$$

where $G(u,y)$ denotes the distribution of the surplus at ruin. So

$$\begin{aligned} &\lim_{u\to\infty} e^{Ru}F(u,y) \\ &= \lim_{u\to\infty}\left[G(u-y,y) - \frac{1-G(0,y)}{1-\psi(0)}(\psi(u-y) - \psi(u))\right]e^{Ru}. \end{aligned} \tag{1.23}$$

Using the asymptotic result for $G(u,y)$ obtained in last section, and the asymptotic result of $\psi(u)$ in [Grandell (1991)], we obtain (1.22).

1.2.1.3 *Solution of the Integral Equation*

Following [Sundt and Teugels (1995)], we introduce the following auxiliary function

$$A_\delta(u,y) = \frac{B_\delta(u,y) - B_\delta(0,y)}{1 - B_\delta(0,y)}.$$

Equation (1.16), in terms of $A_\delta(u,y)$ and $F_1(x)$, the equilibrium distribution of F, can be rewritten as

$$\begin{aligned} pA_\delta(u,y) + \delta\, u\, A_\delta(u,y) &= \delta \int_0^u A_\delta(v,y)\,dv + \lambda\mu\, A_\delta * F_1(u,y) \\ &\quad - \lambda \int_0^u (1-F(z))\,dz \\ &\quad + \frac{\lambda}{B_\delta(0,y)}\Big[I_{\{u\geqslant y\}} \int_0^y (1-F(v))\,dv \\ &\quad + (1-I_{\{u\geqslant y\}})\cdot \int_0^u (1-F(v))\,dv\Big]. \qquad (1.24) \end{aligned}$$

Let

$$\gamma_{\delta 1}(s,y) = \int_0^{+\infty} e^{-sv} d\,A_\delta(v,y),$$

where y is regarded as a parameter.

The first order differential equation for the function, $\gamma_{\delta 1}(s,y)$, can be obtained by taking the Laplace transform of (1.24) with respect to u

$$\begin{aligned} &-\delta\,\gamma'_{\delta 1}(s,y) + (p - \lambda\mu\phi(s))\gamma_{\delta 1}(s,y) \\ &= \frac{\lambda\mu}{B_\delta(0,y)}(\phi(s) - \phi_y(s)) - \lambda\mu\phi(s), \qquad (1.25) \end{aligned}$$

and the initial condition of the above equation is given by $\lim_{s\to\infty} \gamma_{\delta 1}(s,y) = 0$.

Similar to [Sundt and Teugels (1995)], the solution of Equation (1.25) is given by

$$\begin{aligned} \gamma_{\delta 1}(s,y) &= \delta^{-1} \int_s^{+\infty} e^{-\{\chi(v)-\chi(s)\}/\delta} \Big[\frac{\lambda\mu(\phi(v)-\phi_y(v))}{B_\delta(0,y)} - \lambda\mu\phi(v)\Big]\,dv \\ &= \lambda\mu \int_0^{+\infty} e^{-\int_0^w [p-\lambda\mu\phi(s+\delta z)]dz} \Big[\frac{\phi(s+\delta w)-\phi_y(s+\delta w)}{B_\delta(0,y)} \\ &\quad -\phi(s+\delta w)\Big]\,dw, \qquad (1.26) \end{aligned}$$

where

$$\chi(s) := \int_0^s (p - \lambda\mu\phi(v))\,dv.$$

By letting $s = 0$ we have

$$\begin{aligned} B_\delta(0,y) &= \Big(\frac{\delta}{\lambda\mu} + \int_0^{+\infty} e^{-\chi(v)/\delta}\phi(v)dv\Big)^{-1}\Big(\int_0^{+\infty} e^{-\chi(v)/\delta}(\phi(v)-\phi_y(v))dv\Big) \\ &= \Big[\frac{1}{\lambda\mu} + \int_0^{+\infty} e^{-\int_0^w [p-\lambda\mu\phi(\delta z)]dz}\phi(\delta w)dw\Big]^{-1} \cdot \\ &\quad \Big[\int_0^{+\infty} e^{-\int_0^w [p-\lambda\mu\phi(\delta z)]dz}(\phi(\delta w)-\phi_y(\delta w))dw\Big]. \qquad (1.27) \end{aligned}$$

From [Sundt and Teugels (1995)], we know

$$\psi_\delta(0) = 1 - \Big(p\int_0^{+\infty} e^{-\int_0^z (p-\lambda\mu\phi(\delta w))dw}dz\Big)^{-1}.$$

Also by last section, we have

$$\begin{aligned} G_\delta(0,y) = &\Big(p\int_0^{+\infty} e^{-\int_0^z (p-\lambda\mu\phi(\delta w))dw}dz\Big)^{-1} \\ &\Big(\int_0^{+\infty} e^{\int_0^z (p-\lambda\mu\phi(\delta w))dw}\cdot\big(\lambda\mu\phi(\delta z) - \lambda\mu e^{\delta z y}\phi_y(\delta z)\big)\,dz\Big), \end{aligned}$$

so, when $\delta = 0$, $F(0,y) = G(0,y)$, but for $\delta > 0$, $G_\delta(0,y) \neq B_\delta(0,y)$. Hence, the reflection principle is not valid when $\delta \neq 0$. In addition, the relationships among $G(u,y)$, $F(u,y)$ and $\psi(u)$ in [Dickson(1992)], [Willmot and Lin (1998)] are not valid when $\delta \neq 0$.

1.2.2 $B_\delta(u,y)$ *with Zero Initial Reserve*

The main result of this subsection is the following theorem.

Theorem 1.6

$$\begin{aligned} &\Big(\frac{1}{p-\lambda\mu} + \frac{1}{\lambda\mu}\Big)^{-1}\cdot \\ &\Big[\Big(F_1(y) + \delta\alpha\nu_{11}(y)\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}\Big)\cdot\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}R(\alpha) - e^{\frac{\alpha^2}{2}}\frac{\nu_{11}(y)}{\lambda\mu\nu_1}\Big] \\ &\leqslant B_\delta(0,y) \\ &\leqslant \Big[\Big(\sqrt{\frac{1}{\lambda\mu\delta\nu_1}} + \frac{\alpha}{\lambda\mu}\Big)R(\alpha) - \frac{e^{\frac{\alpha^2}{2}}}{\lambda\mu} + \frac{1}{\lambda\mu}\Big]^{-1}\cdot\frac{F_1(y)}{p-\lambda\mu}, \qquad (1.28) \end{aligned}$$

where $R(\alpha)$ *is the well known Mills' ratio given by* $R(\alpha) = e^{\frac{\alpha^2}{2}}\int_\alpha^{+\infty} e^{-\frac{x^2}{2}}dx$, *and* $\alpha = \frac{\lambda\mu - p}{\sqrt{\nu_1\delta\lambda\mu}}$ *and* $\nu_{11}(y) = \int_0^y z\,dF_1(z)$.

Proof. Let

$$\begin{aligned} A_1 :&= \int_0^{+\infty} e^{-\int_0^w (p-\lambda\mu\phi(\delta z))dz}\phi(\delta w)\,dw \\ &= \int_0^{+\infty} e^{\int_0^w (\lambda\mu\phi(\delta z)-p)dz}\phi(\delta w)\,dw. \end{aligned} \tag{1.29}$$

Using the property that $\phi(s)$ is a strictly decreasing convex function, we have

$$\begin{aligned} &\int_0^{+\infty} e^{\int_0^w ((1-\nu_1\delta z)\lambda\mu-p)dz}(1-\nu_1\delta w)\,dw \\ &\leqslant A_1 \\ &\leqslant \int_0^{+\infty} e^{\int_0^w (\lambda\mu-p)dz}dw = \frac{1}{p-\lambda\mu}. \end{aligned}$$

Using a similar method given by [Sundt and Teugels (1995)], we have the following inequality for A_1:

$$\left(\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}+\frac{\alpha}{\lambda\mu}\right)R(\alpha)-\frac{e^{\frac{\alpha^2}{2}}}{\lambda\mu}\leqslant A_1\leqslant\frac{1}{p-\lambda\mu}. \tag{1.30}$$

Put

$$A_2 := \int_0^{+\infty} e^{-\int_0^w (p-\lambda\mu\phi(\delta z))dz}(\phi(\delta w)-\phi_y(\delta w))\,dw. \tag{1.31}$$

Since $\phi(s)$, $\phi_y(s)$ and $\phi(s)-\phi_y(s)$ are strictly decreasing convex functions, we get

$$\begin{aligned} A_2 &\geqslant \int_0^{+\infty} e^{\int_0^w (\lambda\mu-\nu_1\delta z\lambda\mu-p)dz}\left(F_1(y)-\delta w\int_0^y u\,dF_1(u)\right)dw \\ &= \int_0^{+\infty} e^{-\frac{(w-\frac{p-\lambda\mu}{\lambda\delta\mu\nu_1})^2}{2\cdot\frac{1}{\lambda\mu\delta\nu_1}}+\frac{(p-\lambda\mu)^2}{2\lambda\mu\delta\nu_1}}(F_1(y)-\delta w\nu_{11})\,dw \end{aligned}$$

and

$$A_2 \leqslant \int_0^{+\infty} e^{\int_0^2 (\lambda\mu-p)dz}F_1(y)dw = \frac{F_1(y)}{p-\lambda\mu}.$$

Therefore, we have the following inequality for A_2:

$$\begin{aligned}&\left(F_1(y) + \delta\alpha\nu_{11}(y)\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}\right)\cdot\sqrt{\frac{1}{\lambda\mu\delta\nu_1}}R(\alpha) - e^{\frac{\alpha^2}{2}}\frac{\delta\nu_{11}(y)}{\lambda\mu\delta\nu_1}\\ &\leqslant A_2 \leqslant \frac{F_1(y)}{p-\lambda\mu}. \end{aligned} \tag{1.32}$$

Notice that (1.27) can be rearranged as

$$B_\delta(0,y) = \left(A_1 + \frac{1}{\lambda\mu}\right)^{-1} A_2.$$

So Theorem 1.6 is proved. ☐

Similar to [Sundt and Teugels (1995)], we can expand $B_\delta(0,y)$ as a McLaurin series with respect to the variable δ, and we have

$$B_\delta(0,y) = \frac{\lambda\mu}{p}F_1(y) + \frac{-\lambda\mu\nu_1}{p(p-\lambda\mu)}\cdot\delta + o(\delta). \tag{1.33}$$

1.2.3 *Exponential Claim Size*

In this subsection, we will try to evaluate expression (1.27) in the case of an exponential claim size. First, we have

$$\begin{aligned}B_\delta(0,y) &= \left[\frac{\delta}{\lambda\mu} + \int_0^{+\infty} e^{-(pv-\lambda\log(1+\mu v))/\delta}(1+\mu v)^{-1}dv\right]^{-1}\\ &\quad\cdot\left[\int_0^{+\infty} e^{-(pv-\lambda\log(1+\mu v))/\delta}(1+\mu v)^{-1}(1-e^{-(v+\frac{1}{\mu})y})\,dv\right]\\ &= \left[\frac{\delta}{\lambda\mu} + \int_0^{+\infty} e^{-\frac{p}{\delta}v}(1+\mu v)^{\frac{\lambda}{\delta}-1}dv\right]^{-1}\\ &\quad\cdot\left[\int_0^{+\infty} e^{-\frac{p}{\delta}v}(1+\mu v)^{\frac{\lambda}{\delta}-1}(1-e^{-(v+\frac{1}{\mu})y})\,dv\right], \end{aligned} \tag{1.34}$$

then

$$
\begin{aligned}
&\gamma_{\delta 1}(s,y)\\
&= \delta^{-1}\int_s^{+\infty} e^{-[p(v-s)-\lambda\log\frac{1+\mu v}{1+\mu s}]/\delta}\lambda\mu(1+\mu v)^{-1}\Big[\frac{1-e^{-(v+\frac{1}{\mu})y}}{B_\delta(0,y)}-1\Big]\,dv\\
&= \delta^{-1}\lambda\mu\Big[\int_s^{+\infty} e^{-\frac{p}{\delta}(v-s)}(1+\mu v)^{-1}\Big(\frac{1+\mu v}{1+\mu s}\Big)^{\frac{\lambda}{\delta}}\Big[\frac{1}{B_\delta(0,y)}-1\Big]\,dv\\
&\quad -\frac{1}{B_\delta(0,y)}\int_s^{+\infty} e^{-\frac{p}{s}(v-s)}(1+\mu v)^{-1}\Big(\frac{1+\mu v}{1+\mu s}\Big)^{\frac{\lambda}{\delta}} e^{-vy}\cdot e^{-\frac{y}{\mu}}dv\Big]\\
&= \int_0^{+\infty} e^{-sx}\cdot e^{-\frac{x}{\mu}}(1+\frac{\delta}{p}x)^{\frac{\lambda}{\delta}-1}\frac{\lambda}{p}\cdot\Big(\frac{1}{B_\delta(0,y)}-1\Big)dx\\
&\quad +\frac{\lambda}{p}\int_0^{+\infty} e^{-s(x+\frac{\delta y}{p}x+y)}\cdot e^{-(\frac{x+y}{\mu}+\frac{\delta y}{p\mu}x)}\Big(1+\frac{\delta}{p}x\Big)^{\frac{\lambda}{\delta}-1}dx\cdot\frac{1}{B_\delta(0,y)}. \qquad (1.35)
\end{aligned}
$$

Using the substitution

$$x+\frac{\delta y}{p}x+y=t,$$

we have

$$
\begin{aligned}
\gamma_{\delta 1}(s,y) &= \int_0^{+\infty} e^{-st}e^{-\frac{t}{\mu}}(1+\frac{\delta}{p}x)^{\frac{\lambda}{\delta}-1}\frac{\lambda}{p}\Big(\frac{1}{B_\delta(0,y)}-1\Big)\,dt\\
&\quad -\int_y^{+\infty} e^{-st}e^{-\frac{t}{\mu}}\Big(1+\frac{\delta(t-y)}{p+\delta y}\Big)^{\frac{\lambda}{\delta}-1}\cdot\frac{\lambda}{p+\delta y}\cdot\frac{1}{B_\delta(0,y)}\,dt. \qquad (1.36)
\end{aligned}
$$

Remember that $\gamma_{\delta 1}(s,y)$ was introduced as a Laplace transformation of $A_\delta(u,y)$ with respect to u. Hence, by inverting, we find that

$$
\begin{aligned}
A_\delta(u,y) &= \int_0^{u} e^{-\frac{t}{\mu}}(1+\frac{\delta}{p}x)^{\frac{\lambda}{\delta}-1}\frac{\lambda}{p}\Big(\frac{1}{B_\delta(0,y)}-1\Big)\,dt\\
&\quad -\int_y^{u} e^{-\frac{t}{\mu}}\Big(1+\frac{\delta(t-y)}{p+\delta y}\Big)^{\frac{\lambda}{\delta}-1}\cdot\frac{\lambda}{p+\delta y}\cdot\frac{1}{B_\delta(0,y)}\cdot I_{\{u\geqslant y\}}dt, \qquad (1.37)
\end{aligned}
$$

where $I_{\{u\geqslant y\}}$ is the indicator function.

1.2.4 *Lundberg Bound*

Following the arguments in [Sundt and Teugels (1995)], we are also able to obtain the Lundberg type inequality. Same as in [Sundt and Teugels (1995)], we assume that $\phi(s)$ has $-\sigma$ as an abscissa of convergence. So for any $u > 0$ and for $s \in I := (-\sigma, 0)$, we have

$$\gamma_{\delta 1}(s,y) \geqslant \int_u^{+\infty} e^{-sx} d A_\delta(x,y) \geqslant e^{-su}\{1 - A_\delta(u,y)\},$$

so we obtain the Chernoff type estimate

$$1 - A_\delta(u,y) \leqslant \exp\left\{ \inf_{s\in I} \{su + \log \gamma_{\delta 1}(s,y)\} \right\}. \tag{1.38}$$

Let $s_\delta(u,y)$ denote the solution of the following equation:

$$u\,\gamma_{\delta 1}(s,y) = -\gamma'_{\delta 1}(s,y) \ , \tag{1.39}$$

and we call $|s_\delta(u,y)|$ the adjustment function.

Similar to [Sundt and Teugels (1995)], put

$$\underline{u}_\delta = \frac{\left(\frac{\lambda\mu}{B_\delta(u,y)} F_1(y) - p\right)}{\delta} \ , \tag{1.40}$$

we have the following result.

Theorem 1.7 *When the initial surplus u satisfies $u \geqslant \underline{u}_\delta$, then $1 - A_\delta(u,y)$ satisfies a Lundberg type inequality:*

$$1 - A_\delta(u,y) \leqslant \frac{\frac{\lambda\mu}{B_\delta(0,y)}[\phi(s_\delta(u,y)) - \phi_y(s_\delta(u,y))] - \lambda\mu\phi(s_\delta(u,y))}{u\delta + p - \lambda\mu\phi(s_\delta(u,y))} \cdot e^{u s_\delta(u,y)}.$$

1.3 Asymptotic Estimates of the Low and Upper Bounds for the Distribution of the Surplus Immediately after Ruin under Subexponential Claims

The asymptotic estimates of ruin probability is one of the most important topics in risk theory. A lot of results have been obtained since the establishment of Cramér-Lundberg model. These results range from the exponential type estimates as given by Cramér, till the estimates when exponential moment of the claim size distribution does not exist. The purpose of this section is to show that the techniques of [Kalashnikov and Konstantinides(2000)] can be used to deal with the distribution of surplus immediately after ruin. In this section, we give asymptotic formulas of the low and upper bounds for the distribution of the surplus immediately after ruin under subexponential claims. The result is similar to that in [Kalashnikov and Konstantinides(2000)] on ruin probability.

1.3.1 *Preliminaries and Auxiliary Relations*

We also consider the same model as in [Sundt and Teugels (1995)]. We are interested in the function $G_\delta(u,y)$, which is a special form of the Gerber-Shiu penalty function. A detailed study of the Gerber-Shiu penalty function under the model can be found in [Cai and Dickson(2002)]. This is interesting in theory and application to discuss the function $G_\delta(u,y)$. From (1.4), we know that

$$\begin{aligned} G_\delta(u,y) = {} & \frac{p}{p+\delta u} G_\delta(0,y) + \frac{1}{p+\delta u}\int_0^u G_\delta(u-z,y)[\delta+\lambda(1-F(z))]dz \\ & -\frac{\lambda}{p+\delta u}\int_0^y [F(u+z)-F(z)]dz. \end{aligned} \tag{1.41}$$

This result is often referred to as the *integral equation* satisfied by $G_\delta(u,y)$. Following [Kalashnikov and Konstantinides(2000)], if we introduce the auxiliary distribution

$$E_\delta(u,y) = \frac{G_\delta(0,y) - G_\delta(u,y)}{G_\delta(0,y)},$$

then we have

$$G_\delta(u,y) = G_\delta(0,y) - G_\delta(0,y)E_\delta(u,y). \tag{1.42}$$

Replacing $G_\delta(u,y)$ in (1.41) by the expression of (1.42) and making some calculations, we get

$$\begin{aligned} (p+\delta u)E_\delta(u,y) = {} & \delta\int_0^u E_\delta(z,y)dz - \lambda\mu F_1(u) + \lambda\mu E_\delta(\cdot,y) * F_1(u) \\ & + \frac{\lambda\mu}{G_\delta(0,y)}(F_1(u) - F_1(u+y) + F_1(y)), \end{aligned} \tag{1.43}$$

where $*$ stands for the Stieltjes convolution. Put

$$k_\delta(u,y) = \int_u^\infty v\frac{\partial E_\delta(v,y)}{\partial v}dv.$$

Then, using integral by parts, we have

$$\int_0^u E_\delta(v,y)dv = uE_\delta(u,y) - (k_\delta(0,y) - k_\delta(u,y)).$$

Plugging this expression of $\int_0^u E_\delta(v,y)dv$ into (1.43), we have that

$$\begin{aligned}E_\delta(u,y) &= -\frac{\delta}{p}(k_\delta(0,y) - k_\delta(u,y)) + \frac{k_\delta}{p}F_1(u)\\ &+\frac{k_\delta+\rho}{p}(F_1(y) - F_1(u+y)) + \frac{\rho}{p}E_\delta(\cdot,y) * F_1(u),\end{aligned}$$

where

$$\rho = \lambda\mu, \quad k_\delta = \frac{\rho(1-G_\delta(0,y))}{G_\delta(0,y)}.$$

With the abbreviations

$$\begin{aligned}m &= \frac{\rho}{p},\\ C_\delta(u,y) &= -\frac{\delta}{p}(k_\delta(0,y) - k_\delta(u,y)) + \frac{k_\delta}{p}F_1(u)\\ &+\frac{k_\delta+\rho}{p}(F_1(y) - F_1(u+y))\end{aligned}$$

then we obtain

$$E_\delta(u,y) = C_\delta(u,y) + mE_\delta(\cdot,y) * F_1(u). \tag{1.44}$$

Let us assume that the positive safety loading condition $\rho < p$ takes place, which yields $m < 1$. So, the solution of (1.44) is

$$E_\delta(u,y) = C_\delta(\cdot,y) * H(u), H(u) = \sum_{n=0}^{\infty} m^n F_1^{n*}(u).$$

Thus we can rewrite (1.44) in the form

$$\begin{aligned}E_\delta(u,y) = &-\frac{\delta}{p}(\sum_{n=0}^{\infty} m^n F_1^{n*}) * (k_\delta(0,y) - k_\delta(\cdot,y))(u) + \frac{k_\delta}{p}\sum_{n=0}^{\infty} m^n F_1^{(n+1)*}(u)\\ &+\frac{(k_\delta+\rho)F_1(y)}{p}\sum_{n=0}^{\infty} m^n F_1^{n*}(u)\\ &-\frac{k_\delta+\rho}{p}F_1(\cdot+y) * (\sum_{n=0}^{\infty} m^n F_1^{n*})(u).\end{aligned} \tag{1.45}$$

Letting $\delta = 0$ in (1.41) yields

$$\begin{aligned} G_0(u,y) &= G_0(0,y) + \frac{\lambda}{p}\int_0^u G_0(u-z,y)(1-F(z))dz \\ &\quad -\frac{\lambda}{p}\int_0^y (F(u+z)-F(z))dz \\ &= D_0(u,y) + mG_0(\cdot,y) * F_1(u), \end{aligned} \tag{1.46}$$

here $D_0(u,y)$ is defined by

$$D_0(u,y) = G_0(0,y) - \frac{\lambda}{p}\int_0^y (F(u+z)-F(z))dz.$$

Noting that

$$\begin{aligned} D_0(u,y) &= G_0(0,y) - \frac{\lambda}{p}\int_0^y (F(u+z)-F(z))dz \\ &= G_0(0,y) - \frac{\lambda\mu}{p}\int_0^y \frac{F(u+z)-F(z)}{\mu}dz \\ &= G_0(0,y) - m\int_0^y \left[\frac{1-F(z)}{\mu} - \frac{1-F(u+z)}{\mu}\right] dz \\ &= G_0(0,y) - mF_1(y) + m\int_0^y \frac{1-F(u+z)}{\mu}dz \\ &= G_0(0,y) - mF_1(y) + m\int_u^{u+y} \frac{1-F(z)}{\mu}dz \\ &= G_0(0,y) - mF_1(y) + m[F_1(u+y) - F_1(u)], \end{aligned}$$

the solution of (1.46) is then

$$\begin{aligned} G_0(u,y) &= D_0(\cdot,y) * H(u) \\ &= D_0(\cdot,y) * (\sum_{n=0}^{\infty} m^n F_1^{n*})(u) \\ &= (G_0(0,y) - mF_1(y))\sum_{n=0}^{\infty} m^n F_1^{n*}(u) - \sum_{n=0}^{\infty} m^{n+1} F_1^{(n+1)*}(u) \\ &\quad + \sum_{n=0}^{\infty} m^{n+1} F_1(\cdot + y) * F_1^{n*}(u). \end{aligned}$$

From [Willmot and Lin(2001)], $G_0(0,y) = mF_1(y)$. Thus

$$G_0(u,y) = \sum_{n=0}^{\infty} m^{n+1} F_1(\cdot + y) * F_1^{n*}(u) - \sum_{n=0}^{\infty} m^{n+1} F_1^{(n+1)*}(u). \tag{1.47}$$

It immediately follows from the above expression of $G_0(u,y)$ that

$$\sum_{n=0}^{\infty} m^n F_1(\cdot + y) * F_1^{n*}(u) = \frac{1}{m} G_0(u,y) + \sum_{n=0}^{\infty} m^n F_1^{(n+1)*}(u). \tag{1.48}$$

On the other hand, by the Pollaczek-Khinchine formula (or, which is the same, by the Beckman convolution formula, see [Rolski *et. al.* (1999)] , we have

$$\sum_{n=0}^{\infty} m^n F_1^{n*}(u) = \frac{1-\psi_0(u)}{1-m}. \tag{1.49}$$

In terms of (1.45), (1.48) and (1.49) we arrive at

$$\begin{aligned} E_\delta(u,y) = & -\frac{\delta}{p}(\frac{1-\psi_0(\cdot)}{1-m} * (k_\delta(0,y) - k_\delta(\cdot,y))(u) + 1 - \frac{1-\psi_0(u)}{1-m} \\ & + \frac{k_\delta + \rho}{p} F_1(y)(\frac{1-\psi_0(u)}{1-m}) - \frac{k_\delta + \rho}{p} \frac{G_0(u,y)}{m}. \end{aligned}$$

Further,

$$\begin{aligned} 1 - E_\delta(u,y) = & \frac{\delta}{p-\rho} \phi_0 * (k_\delta(0,y) - k_\delta(\cdot,y))(u) + \frac{p\phi_0(u)}{p-\rho} \\ & - \frac{k_\delta + \rho}{p-\rho} F_1(y)\phi_0(u) + \frac{k_\delta + \rho}{\rho} G_0(u,y), \end{aligned} \tag{1.50}$$

where $\phi_0(u) = 1 - \psi_0(u)$ is the survival probability (without interest rate).

Now we are in a position to take Laplace-Stieltjes transform (see, [Rolski *et. al.* (1999)] on both sides of (1.50). The Laplace-Stieltjes transform will be denoted by equipping the corresponding original function with hat and using s as its argument. It is easy to see from (1.50) that

$$\begin{aligned} 1 - \hat{E}_\delta(s,y) = & \frac{\delta}{p-\rho}(k_\delta(0,y)\hat{\phi}_0(s) - \hat{k}_\delta(s,y)\hat{\phi}_0(s)) + \frac{p}{p-\rho}\hat{\phi}_0(s) \\ & - \frac{k_\delta + \rho}{p-\rho} F_1(y)\hat{\phi}_0(s) + \frac{k_\delta + \rho}{\rho} \hat{G}_0(s,y), \end{aligned} \tag{1.51}$$

where $\hat{k}_\delta(s,y) = \int_0^\infty e^{-su} dk_\delta(u,y), \hat{G}_0(s,y) = \int_0^\infty e^{-su} dG_0(u,y)$, and by the Pollaczek-Khinchine formula (see [Rolski *et. al.* (1999)], or [Kalashnikov and Konstantinides(2000)]),

$$\hat{\phi}_0(s) = \int_0^\infty e^{-su} d\phi_0(u) = \frac{p-\rho}{p-\rho\hat{F}_1(s)}. \tag{1.52}$$

Because of (1.47), in order to get the Laplace-Stieltjes transform of $G_0(u,y)$, it suffices to compute that of $F_1(u+y)$. In fact,

$$\begin{aligned}\int_0^\infty e^{-su}dF_1(u+y) &= e^{sy}\int_0^\infty e^{-s(u+y)}dF_1(u+y)\\ &= e^{sy}\int_y^\infty e^{-su}dF_1(u)\\ &= e^{sy}\int_0^\infty e^{-su}dF_1(u) - e^{sy}\int_0^y e^{-su}dF_1(u)\\ &= e^{sy}\hat{F}_1(s) - e^{sy}\int_0^y e^{-su}dF_1(u). \qquad (1.53)\end{aligned}$$

Thus we can obtain from (1.47) the Laplace-Stieltjes transform of $G_0(u,y)$ that

$$\begin{aligned}\hat{G}_0(s,y) &= \sum_{n=0}^\infty m^{n+1}\hat{F}_1^n(s)(e^{sy}\hat{F}_1(s) - e^{sy}\int_0^y e^{-su}dF_1(u))\\ &\quad -\sum_{n=0}^\infty m^{n+1}\hat{F}_1^{n+1}(s)\\ &= e^{sy}\sum_{n=0}^\infty m^{n+1}\hat{F}_1^{n+1}(s) - e^{sy}\int_0^y e^{-su}dF_1(u)\sum_{n=0}^\infty m^{n+1}\hat{F}_1^n(s)\\ &\quad -\sum_{n=0}^\infty m^{n+1}\hat{F}_1^{n+1}(s)\\ &= (e^{sy}-1)\frac{m\hat{F}_1(s)}{1-m\hat{F}_1(s)} - e^{sy}\int_0^y e^{-su}dF_1(u)\frac{m}{1-m\hat{F}_1(s)}\\ &= (e^{sy}-1)\frac{\rho\hat{F}_1(s)}{p-\rho\hat{F}_1(s)} - e^{sy}\int_0^y e^{-su}dF_1(u)\frac{\rho}{p-\rho\hat{F}_1(s)}. \qquad (1.54)\end{aligned}$$

By virtue of (1.51), (1.52) and (1.54) we have that

$$\begin{aligned}(p-\rho\hat{F}_1(s))(1-\hat{E}_\delta(s,y)) &= \delta(k_\delta(0,y) - \hat{k}_\delta(s,y)) + p\\ &\quad -(k_\delta+\rho)F_1(y) + (k_\delta+\rho)(e^{sy}-1)\hat{F}_1(s)\\ &\quad -(k_\delta+\rho)e^{sy}\int_0^y e^{-su}dF_1(u). \qquad (1.55)\end{aligned}$$

Now, we need find the value of $k_\delta(0,y)$. Letting $u\to\infty$ in the expression of (1.44) leads to $1 = C_\delta(\infty,y)+m$, which means that $C_\delta(\infty,y) = 1-m$.

Since

$$C_\delta(u,y)=-\frac{\delta}{p}(k_\delta(0,y)-k_\delta(u,y))+\frac{k_\delta}{p}F_1(u)+\frac{k_\delta+\rho}{p}(F_1(y)-F_1(u+y)),$$

and letting $u\to\infty$ on both sides of above equation yields

$$1-m=-\frac{\delta}{p}k_\delta(0,y)+\frac{k_\delta}{p}+\frac{k_\delta+\rho}{p}(F_1(y)-1).$$

Hence

$$k_\delta(0,y)=\frac{(k_\delta+\rho)F_1(y)-p}{\delta}. \tag{1.56}$$

By making use of (1.55) and (1.56) and making some calculation we get

$$\begin{aligned}\hat{k}_\delta(s,y)=&\frac{k_\delta+\rho}{\delta}(1-\hat{F}_1(s))-\frac{k_\delta+\rho}{\delta}[1-(e^{sy}\hat{F}_1(s)-e^{sy}\int_0^y e^{-su}dF_1(u))]\\&-\frac{p}{\delta}(1-\hat{E}_\delta(s,y))+\frac{\rho}{\delta}(1-\hat{E}_\delta(s,y))\hat{F}_1(s).\end{aligned} \tag{1.57}$$

Upon inverting Laplace-Stieltjes transform, and applying (1.53), the above relation yields

$$\begin{aligned}k_\delta(u,y)=&\frac{k_\delta+\rho}{\delta}(1-F_1(u))-\frac{k_\delta+\rho}{\delta}(1-F_1(u+y))\\&-\frac{p}{\delta}(1-E_\delta(u,y))+\frac{\rho}{\delta}F_1*(1-E_\delta(\cdot,y))(u).\end{aligned} \tag{1.58}$$

1.3.2 *Asymptotic Estimates of the Low and Upper Bounds*

Proposition 1.1 *If $\rho<p$, then*

$$k_*(u,y)\leqslant k_\delta(u,y)\leqslant k^*(u,y), \tag{1.59}$$

where

$$\begin{aligned}k_*(u,y)=&(1+\frac{p}{\delta u})^{-1}\frac{k_\delta+\rho}{\delta}(1-F_1(u))\\&-(1+\frac{p}{\delta u})^{-1}\frac{k_\delta+\rho}{\delta}(1-F_1(u+y)),\end{aligned} \tag{1.60}$$

$$k^*(u,y)=\frac{p-\rho}{\rho}\frac{(k_\delta+\rho)G_0(u,y)}{\delta\phi_0(u)}. \tag{1.61}$$

Proof. It is obvious that

$$k_\delta(u,y)=\int_u^\infty z\frac{\partial(E_\delta(z,y))}{\partial z}dz\geqslant u(1-E_\delta(u,y)).$$

So

$$1 - E_\delta(u,y) \leqslant \frac{k_\delta(u,y)}{u}.$$

According to the last inequality and (1.58) we have

$$k_\delta(u,y) \geqslant \frac{k_\delta+\rho}{\delta}(1-F_1(u)) - \frac{p}{\delta}\frac{k_\delta(u,y)}{u} - \frac{k_\delta+\rho}{\delta}(1-F_1(u+y)).$$

Hence

$$\begin{aligned} k_\delta(u,y) &\geqslant (1+\frac{p}{\delta u})^{-1}\frac{k_\delta+\rho}{\delta}(1-F_1(u)) \\ &\quad -(1+\frac{p}{\delta u})^{-1}\frac{k_\delta+\rho}{\delta}(1-F_1(u+y)) \\ &= k_*(u,y). \end{aligned}$$

In addition, it is easy to see that

$$\phi_0 * k_\delta(\cdot,y)(u) \geqslant \phi_0(u)k_\delta(u,y).$$

In view of (1.50), (1.56) and last inequality we get

$$\begin{aligned} 1-E_\delta(u,y) &\leqslant \frac{\delta}{p-\rho}\phi_0(u)k_\delta(0,y) - \frac{\delta}{p-\rho}\phi_0(u)k_\delta(u,y) \\ &\quad + \frac{p\phi_0(u)}{p-\rho} - \frac{k_\delta+\rho}{p-\rho}F_1(y)\phi_0(u) + \frac{k_\delta+\rho}{\rho}G_0(u,y) \\ &= \frac{k_\delta+\rho}{\rho}G_0(u,y) - \frac{\delta}{p-\rho}\phi_0(u)k_\delta(u,y). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_0(u)\frac{\delta}{p-\rho}k_\delta(u,y) &\leqslant \frac{k_\delta+\rho}{\rho}G_0(u,y) - (1-E_\delta(u,y)) \\ &\leqslant \frac{k_\delta+\rho}{\rho}G_0(u,y). \end{aligned}$$

Thus

$$k_\delta(u,y) \leqslant \frac{k_\delta+\rho}{\delta}\frac{(p-\rho)G_0(u,y)}{\rho\phi_0(u)} = k^*(u,y).$$

This completes the proof of Proposition 1.1.

Proposition 1.2 *If $\rho < p$, for subexponential F_1,*

$$\begin{aligned} k_*(u,y) \sim\ & \frac{k_\delta + \rho}{\delta}(1 - F_1(u)) \\ & -\frac{k_\delta + \rho}{\delta}(1 - F_1(u+y)), \quad u \to \infty, \end{aligned} \tag{1.62}$$

$$\begin{aligned} k^*(u,y) \sim\ & \frac{k_\delta + \rho}{\delta}(1 - F_1(u)) - \frac{k_\delta + \rho}{\delta}(1 - F_1(u+y)) \\ & +\frac{k_\delta + \rho}{\delta}\frac{\rho F_1(y)}{p - \rho}(1 - F_1(u)), \quad u \to \infty. \end{aligned} \tag{1.63}$$

Proof. Clearly, by the definition of $k_*(u,y)$,

$$\begin{aligned} k_*(u,y) &= (1 + \frac{p}{\delta u})^{-1}\frac{k_\delta + \rho}{\delta}(1 - F_1(u)) \\ &\quad -(1 + \frac{p}{\delta u})^{-1}\frac{k_\delta + \rho}{\delta}(1 - F_1(u+y)) \\ &\sim \frac{k_\delta + \rho}{\delta}(1 - F_1(u)) - \frac{k_\delta + \rho}{\delta}(1 - F_1(u+y)),\ u \to \infty. \end{aligned}$$

Because

$$\begin{aligned} G_0(u,y) &= \sum_{n=0}^{\infty} m^{n+1} F_1^{n*} * F_1(\cdot + y)(u) - \sum_{n=0}^{\infty} m^{n+1} F_1^{(n+1)*}(u) \\ &= \sum_{n=0}^{\infty} m^{n+1}(1 - \overline{F_1^{n*}}) * F_1(\cdot + y)(u) - \sum_{n=0}^{\infty} m^{n+1}(1 - \overline{F_1^{(n+1)*}}(u)) \\ &= \sum_{n=0}^{\infty} m^{n+1} F_1(u+y) - \sum_{n=0}^{\infty} m^{n+1} + \sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{(n+1)*}}(u) \\ &\quad -\sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{n*}} * F_1(\cdot + y)(u) \\ &= \frac{-\rho}{p - \rho}(1 - F_1(u+y)) + \sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{(n+1)*}}(u) \\ &\quad -\sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{n*}} * F_1(\cdot + y)(u). \end{aligned}$$

Noting that

$$
\begin{aligned}
&\overline{F_1^{n*}} * F_1(\cdot + y)(u) \\
&= \int_0^u \overline{F_1^{n*}}(u-z)dF_1(z+y) = \int_y^{y+u} \overline{F_1^{n*}}(u+y-z)dF_1(z) \\
&= \int_0^{y+u} \overline{F_1^{n*}}(u+y-z)dF_1(z) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z) \\
&= \int_0^{y+u} (1 - F_1^{n*}(u+y-z))dF_1(z) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z) \\
&= F_1(y+u) - \int_0^{y+u} F_1^{n*}(u+y-z)dF_1(z) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z) \\
&= F_1(y+u) - F_1^{(n+1)*}(y+u) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z),
\end{aligned}
$$

and making use of Lemma 2.5.1, Lemma 2.5.3 and Theorem 2.5.3 in [Rolski *et al.* (1999)] and the dominated convergence theorem we obtain

$$
\begin{aligned}
&\lim_{u\to\infty} \frac{\sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{(n+1)*}}(u) - \sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{n*}} * F_1(\cdot + y)(u)}{\frac{\rho}{p-\rho}\overline{F_1}(u)} \\
&= \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1} \lim_{u\to\infty} \frac{\overline{F_1^{(n+1)*}}(u)}{\overline{F_1}(u)} - \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1} \cdot \\
&\quad \lim_{u\to\infty}\left[\frac{F_1(y+u) - F_1^{(n+1)*}(y+u) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z)}{\overline{F_1}(u)}\right] \\
&= \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1}(n+1) - \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1} \cdot \\
&\quad \lim_{u\to\infty}\left[\frac{\overline{F_1^{(n+1)*}}(y+u) - \overline{F_1}(y+u) - \int_0^y \overline{F_1^{n*}}(u+y-z)dF_1(z)}{\overline{F_1}(u)}\right] \\
&= \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1}(n+1) - \frac{p-\rho}{\rho}\sum_{n=0}^{\infty} m^{n+1}(n - nF_1(y)) \\
&= \frac{p - \rho\overline{F_1}(y)}{p-\rho}.
\end{aligned}
$$

The above limit is due to the subexponentiality of F_1. Hence,

$$\begin{aligned}&\sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{(n+1)*}}(u) - \sum_{n=0}^{\infty} m^{n+1}\overline{F_1^{n*}} * F_1(\cdot + y)(u)\\ &\sim \frac{\rho\overline{F_1}(u)(p-\rho\overline{F_1}(y))}{(p-\rho)^2}, \quad u \to \infty.\end{aligned}$$

So we have that

$$G_0(u,y) \sim \frac{-\rho}{p-\rho}(1-F_1(u+y)) + \frac{\rho\overline{F_1}(u)(p-\rho\overline{F_1}(y))}{(p-\rho)^2}, \quad u \to \infty.$$

Thus

$$\begin{aligned}k^*(u,y) &\sim \frac{k_\delta+\rho}{\delta}\cdot\frac{p-\rho}{\rho}\cdot\frac{\rho\overline{F_1}(u)(p-\rho\overline{F_1}(y))}{(p-\rho)^2}\\ &\quad -\frac{k_\delta+\rho}{\delta}\cdot\frac{p-\rho}{\rho}\cdot\frac{\rho}{p-\rho}(1-F_1(u+y))\\ &= \frac{k_\delta+\rho}{\delta}\left[\frac{\overline{F_1}(u)(p-\rho+\rho F_1(y))}{p-\rho} - (1-F_1(u+y))\right]\\ &= \frac{k_\delta+\rho}{\delta}(1-F_1(u)) - \frac{k_\delta+\rho}{\delta}(1-F_1(u+y))\\ &\quad +\frac{k_\delta+\rho}{\delta}\cdot\frac{\rho F_1(y)}{p-\rho}(1-F_1(u)).\end{aligned}$$

Therefore, (1.62) and (1.63) are all true, and this completes the proof of Proposition 1.2.

From (1.42) and the definition of $k_\delta(u,y)$ it follows that

$$\begin{aligned}\frac{G_\delta(u,y)}{G_\delta(0,y)} &= 1-E_\delta(u,y) = E_\delta(\infty,y)-E_\delta(u,y) = \int_u^\infty \frac{\partial E_\delta(v,y)}{\partial v}dv\\ &= -\int_u^\infty \frac{1}{v}\frac{\partial k_\delta(v,y)}{\partial v}dv = \frac{k_\delta(u,y)}{u} - \int_u^\infty \frac{k_\delta(v,y)}{v^2}dv. \qquad (1.64)\end{aligned}$$

Evidently, $k_*(z,y) \leqslant k_\delta(z,y) \leqslant k^*(z,y)$, it is easy to verify that

$$\frac{k_*(u,y)}{u} - \int_u^\infty \frac{k^*(v,y)}{v^2}dv \leqslant \frac{G_\delta(u,y)}{G_\delta(0,y)} \leqslant \frac{k^*(u,y)}{u} - \int_u^\infty \frac{k_*(v,y)}{v^2}dv. \quad (1.65)$$

Put

$$\begin{aligned}G^*(u,y) &= G_\delta(0,y)\left[\tfrac{k^*(u,y)}{u} - \int_u^\infty \tfrac{k_*(v,y)}{v^2}dv\right],\\ G_*(u,y) &= G_\delta(0,y)\left[\tfrac{k_*(u,y)}{u} - \int_u^\infty \tfrac{k^*(v,y)}{v^2}dv\right].\end{aligned}$$

Then (1.65) implies

$$G_*(u,y) \leqslant G_\delta(u,y) \leqslant G^*(u,y). \tag{1.66}$$

Next, we are going to estimate $G_*(u,y)$ and $G^*(u,y)$, the lower and upper bounds of $G_\delta(u,y)$. Before we give our main result of this section, we need one lemma, which is used in the proof of following Theorem 1.8.

Lemma 1.1 *Suppose that* $f_1(u) \sim g_1(u)$ *and* $f_2(u) \sim g_2(u), u \to \infty$. *If* $\lim\limits_{u\to\infty} \frac{f_2(u)}{f_1(u)} \neq 1$, *then* $f_1(u) - f_2(u) \sim g_1(u) - g_2(u), u \to \infty$.

Proof. By the assumptions of Lemma we have that

$$\begin{aligned}\lim_{u\to\infty} \frac{f_1(u)-f_2(u)}{g_1(u)-g_2(u)} &= \lim_{u\to\infty} \frac{1-\frac{f_2(u)}{f_1(u)}}{\frac{g_1(u)}{f_1(u)}-\frac{g_2(u)}{f_1(u)}} \\ &= \frac{1-\lim\limits_{u\to\infty}\frac{f_2(u)}{f_1(u)}}{\lim\limits_{u\to\infty}\frac{g_1(u)}{f_1(u)}-\lim\limits_{u\to\infty}\frac{g_2(u)}{f_2(u)}\cdot\frac{f_2(u)}{f_1(u)}} \\ &= \frac{1-\lim\limits_{u\to\infty}\frac{f_2(u)}{f_1(u)}}{1-\lim\limits_{u\to\infty}\frac{f_2(u)}{f_1(u)}} = 1.\end{aligned}$$

Hence, we obtain immediately the required assertion.

Theorem 1.8 *If* $\rho < p$, F_1 *is subexponential. Then for* $u \to \infty$,

$$G_*(u,y) \sim \frac{\lambda}{\delta}\int_u^{u+y}\frac{1-F(z)}{z}dz - \frac{\rho^2 F_1(y)}{(p-\rho)\delta}\int_u^\infty \frac{1-F(z)}{z^2}dz, \tag{1.67}$$

$$G^*(u,y) \sim \frac{\lambda}{\delta}\int_u^{u+y}\frac{1-F(z)}{z}dz + \frac{\rho^2 F_1(y)}{(p-\rho)\delta}\cdot\frac{1-F(u)}{u}. \tag{1.68}$$

Proof. Firstly, it is easy to see from (1.60), (1.61) and (1.47) that

$$\lim_{u\to\infty}\frac{\int_u^\infty \frac{k^*(v,y)}{v^2}dv}{\frac{k_*(u,y)}{u}} \neq 1.$$

So, according to (1.62) and (1.63) and Lemma 1.1, a straightforward computation gives

$$
\begin{aligned}
\frac{G_*(u,y)}{G_\delta(0,y)} &= \frac{k_*(u,y)}{u} - \int_u^\infty \frac{k^*(v,y)}{v^2}dv \\
&\sim \frac{k_\delta+\rho}{\delta}\left[\frac{1-F_1(u)}{u} - \frac{1-F_1(u+y)}{u}\right] \\
&\quad - \frac{k_\delta+\rho}{\delta}\int_u^\infty \left(\frac{1-F_1(z)}{z^2} - \frac{1-F_1(z+y)}{z^2}\right. \\
&\quad \left. + \frac{\rho F_1(y)}{p-\rho}\cdot\frac{1-F_1(z)}{z^2}\right) dz \\
&\sim \frac{k_\delta+\rho}{\delta}\left[\frac{F_1(u+y)}{u+y} - \frac{F_1(u)}{u}\right] \\
&\quad - \frac{k_\delta+\rho}{\delta}\int_u^\infty \left(\frac{F_1(z+y)}{(z+y)^2} - \frac{F_1(z)}{z^2}\right) dz \\
&\quad - \frac{k_\delta+\rho}{\delta}\cdot\frac{\rho F_1(y)}{p-\rho}\int_u^\infty \frac{1-F_1(z)}{z^2}dz \\
&= \frac{k_\delta+\rho}{\delta}\left[\frac{F_1(u+y)}{u+y} - \frac{F_1(u)}{u}\right] + \frac{k_\delta+\rho}{\delta}\int_u^\infty \frac{F_1(z)}{z^2}dz \\
&\quad - \frac{k_\delta+\rho}{\delta}\int_u^\infty \frac{F_1(z+y)}{(z+y)^2}dz - \frac{k_\delta+\rho}{\delta}\cdot\frac{\rho F_1(y)}{p-\rho}\int_u^\infty \frac{1-F_1(z)}{z^2}dz \\
&= \frac{k_\delta+\rho}{\delta}\int_u^{u+y} \frac{1}{z}dF_1(z) - \frac{k_\delta+\rho}{\delta}\cdot\frac{\rho F_1(y)}{p-\rho}\int_u^\infty \frac{1-F_1(z)}{z^2}dz \\
&= \frac{\lambda}{\delta}\int_u^{u+y} \frac{1-F(z)}{z}dz\cdot\frac{1}{G_\delta(0,y)} \\
&\quad - \frac{\rho^2 F_1(y)}{\delta(p-\rho)}\int_u^\infty \frac{1-F_1(z)}{z^2}dz\cdot\frac{1}{G_\delta(0,y)},
\end{aligned}
$$

and this yields (1.67). (1.68) is proven similarly by using (1.62) and (1.63) and making some calculations.

Remark 1.7 *Although Theorem 1.8 gives asymptotic formulas of the low and upper bounds for the distribution of the surplus immediately after ruin under subexponential claims, unfortunately, the asymptotic formulas for* $G_*(u,y)$ *and* $G^*(u,y)$ *given in Theorem 1.8 are not equal. Thus, we still do not know what the asymptotic formula for* $G_\delta(u,y)$ *is.*

1.4 On the Ruin Probability under a Class of Risk Processes

In this section a class of collective risk model with non-Poissonian claims' arrival processes is considered. A clear expression for Laplace transform of the finite time ruin probability is well given when the claim amount distribution is the mixture of two exponentials. As its consequence, one result by [Malinovskii(1998)] about the expression for Laplace transform of the ruin probability within finite time is obtained when the claim amount distribution is a exponential. Finally, a well-known result in [Gerber(1979)] about ultimate ruin probability in the classical risk model is proved again.

1.4.1 *The Risk Model*

In this section we consider a Sparre Andersen risk process

$$R(t) = u + ct - \sum_{i=1}^{N(t)} Y_i,$$

defined in terms of the following values: $u = R(0) \geqslant 0$ is the initial risk reserve, $c > 0$ is the premium received continuously per unit time, $\{T_i, i \geqslant 1\}$ are the (iid) interclaim time, $N(t)$ denotes number of claims having occurred up to time t, i.e., $N(t) = \max\{n : T_1 + T_2 + \cdots + T_n \leqslant t\}$, and $\{Y_i, i \geqslant 1\}$ are the (iid) amounts of claims. Throughout this section, we suppose that $\{T_i, i \geqslant 1\}$ and $\{Y_j, j \geqslant 1\}$ are independent, and the relative security loading $\Lambda = \frac{c \cdot ET_1}{EY_1} - 1 > 0$, which means that the premium received per unit time exceed the expected claim payments per unit time. Denote by $\psi(t, u), \psi(u)$ and $\phi(t, u)$ the probability of ruin within finite time, the probability of ultimate ruin and the probability of survival to time t, respectively. Clearly, $\psi(t, u) = 1 - \phi(t, u)$.

1.4.2 *The Laplace Transform of the Ruin Probability with Finite Time*

Our main result in this subsection is the following theorem.

Theorem 1.9 *Let the claim sizes* $\{Y_i, i \geqslant 1\}$ *and interoccurrence times* $\{T_i, i \geqslant 1\}$ *be mutually independent and i.i.d. Let* Y_1 *be a mixed exponential and its p.d.f. be* $p\lambda_1 e^{-\lambda_1 y} + q\lambda_2 e^{-\lambda_2 y}, y > 0, 0 < \lambda_1 < \lambda_2, p + q = 1, 0 \leqslant p, q \leqslant 1$. *Assume that* $\gamma(\alpha) = \int_0^\infty e^{-\alpha u} P_T(du)$ *is the Laplace transform of*

T_1, where $P_T(u)$ is the distribution function of T_1. Then for $\alpha > 0$,

$$\begin{aligned}&\alpha\int_0^\infty e^{-\alpha t}\psi(t,u)dt\\&=\frac{(\lambda_1-\lambda_2)[y_1(\alpha)\beta_2(\alpha)e^{-\beta_1(\alpha)u}-y_2(\alpha)\beta_1(\alpha)e^{-\beta_2(\alpha)u}]}{(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))},\end{aligned} \tag{1.69}$$

where $y_1(\alpha)=(1-\frac{\beta_1(\alpha)}{\lambda_1})(1-\frac{\beta_1(\alpha)}{\lambda_2})$, $y_2(\alpha)=(1-\frac{\beta_2(\alpha)}{\lambda_1})(1-\frac{\beta_2(\alpha)}{\lambda_2})$, $\beta_1(\alpha),\beta_2(\alpha)$ are, respectively, the unique solutions of the equation

$$(\lambda_1-\beta)(\lambda_2-\beta)-[\lambda_1\lambda_2-(p\lambda_1+q\lambda_2)\beta]\gamma(\alpha+c\beta)=0 \tag{1.70}$$

in $(0,\lambda_1]$ and $[\beta_0,\lambda_2]$, and $\beta_0=\frac{\lambda_1\lambda_2}{p\lambda_1+q\lambda_2}$.

Remark 1.8 *Equation (1.70) has the unique root, respectively, in $(0,\lambda_1]$ and $[\beta_0,\lambda_2]$. In fact, Let*

$$f(\beta)=(\lambda_1-\beta)(\lambda_2-\beta)-[\lambda_1\lambda_2-(p\lambda_1+q\lambda_2)\beta]\gamma(\alpha+c\beta),$$

then $f(0)>0, f(\lambda_1)\leqslant 0, f(\beta_0)\leqslant 0, f(\lambda_2)\geqslant 0$. By the existence theorem of root we know that Equation (1.70) has the roots in $(0,\lambda_1]$ and $[\beta_0,\lambda_2]$. On the other hand, it is very easy to prove the uniqueness of roots by the positiveness of relative security loading and the convexity of $\gamma(\alpha+c\beta)$ with respect to β.

Proof. Let $\tau=\inf\{t>0: R(t)<0\}$ be the *first time of ruin* with the understanding that $\tau=\infty$ if $R(t)\geqslant 0$ for all t. Let $U_n=\sum_{i=1}^n T_i, U_0=0, \nu=\inf\{n\geqslant 1: R(U_n)<0\}$, and ν is the index of that claim which causes the first ruin. If for each $n, R(U_n)\geqslant 0$, then $\nu=\infty$. Clearly, $\tau=U_\nu$. For $t,x\geqslant 0$, denote $Q^n(t,x)=P\{U_n\leqslant t, R(U_n)\leqslant x, n<\nu\}, H(t,x)=\sum_{n\geqslant 0}Q^n(t,x)$, and Laplace transform of $H(t,x)$ denoted by $\hat{H}(\alpha,\beta)$. Then for $\alpha,\beta>0$,

$$\begin{aligned}&\hat{H}(\alpha,\beta)\\&=\int_0^\infty\int_0^\infty e^{-\alpha t-\beta x}H(dt,dx)=\sum_{n\geqslant 0}E[e^{-\alpha U_n-\beta R(U_n)}I_{\{\nu>n\}}].\end{aligned} \tag{1.71}$$

Obviously, (1.71) is *analytical*. Since

$$\begin{aligned}
\phi(t,u) &= P(\tau > t) \\
&= \sum_{n\geqslant 0} P(\tau > t, U_n \leqslant t < U_{n+1}) \\
&= \sum_{n\geqslant 0} P(\nu > n, U_n \leqslant t, T_{n+1} > t - U_n) \\
&= \sum_{n\geqslant 0} \int_0^t \int_0^\infty Q^n(d\mu, dz)(1 - P_T(t-\mu)) \\
&= \int_0^t H(d\mu, \infty)(1 - P_T(t-\mu)), \qquad (1.72)
\end{aligned}$$

then we get

$$d\phi(t,u) = -\int_0^t P_T'(t-\mu)H(d\mu,\infty)dt + H(dt,\infty). \qquad (1.73)$$

Consequently,

$$\begin{aligned}
\alpha\int_0^\infty e^{-\alpha t}\phi(t,u)dt &= -\int_0^\infty \phi(t,u)de^{-\alpha t} \\
&= \int_0^\infty e^{-\alpha t}d\phi(t,u) \\
&= \int_0^\infty \int_0^t e^{-\alpha t}(-P_T'(t-\mu))H(d\mu,\infty)dt \\
&\quad + \int_0^\infty e^{-\alpha t}H(dt,\infty) \\
&= \int_0^\infty \int_0^t e^{-\alpha\mu}e^{-\alpha(t-\mu)}(-P_T'(t-\mu))H(d\mu,\infty)dt \\
&\quad +\hat{H}(\alpha,0) \\
&= \int_0^\infty e^{-\alpha\mu}(\int_\mu^\infty e^{-\alpha(t-\mu)}(-P_T'(t-\mu))dt)H(d\mu,\infty) \\
&\quad +\hat{H}(\alpha,0) \\
&= \int_0^\infty e^{-\alpha\mu}\int_0^\infty e^{-\alpha y}(-P_T'(y))dyH(d\mu,\infty) + \hat{H}(\alpha,0) \\
&= -\int_0^\infty \gamma(\alpha)e^{-\alpha\mu}H(d\mu,\infty) + \hat{H}(\alpha,0) \\
&= \hat{H}(\alpha,0)(1-\gamma(\alpha)). \qquad (1.74)
\end{aligned}$$

Now we turn our attention to computation of $\hat{H}(\alpha, 0)$. To this end, denote $\Omega_n = \sigma\{Y_1, T_1, Y_2, T_2, \cdots, Y_n, T_n\}$. Noting that

$$\begin{aligned}
&E[e^{-\beta R(U_n)} I_{\{R(U_n)\geqslant 0\}} | \Omega_{n-1}, T_n] \\
&= \int_0^{R(U_{n-1})+cT_n} \exp\{-\beta(R(U_{n-1}) + cT_n - y)\}(p\lambda_1 e^{-\lambda_1 y} + q\lambda_2 e^{-\lambda_2 y})dy \\
&= \frac{p\lambda_1}{\lambda_1 - \beta}[e^{-\beta(R(U_{n-1})+cT_n)} - e^{-\lambda_1(R(U_{n-1})+cT_n)}] \\
&\quad + \frac{q\lambda_2}{\lambda_2 - \beta}[e^{-\beta(R(U_{n-1})+cT_n)} - e^{-\lambda_2(R(U_{n-1})+cT_n)}],
\end{aligned}$$

we get

$$\begin{aligned}
&E[e^{-\alpha U_n - \beta R(U_n)} I_{\{R(U_n)\geqslant 0\}} | \Omega_{n-1}] \\
&= E[E(e^{-\alpha U_n - \beta R(U_n)} I_{\{R(U_n)\geqslant 0\}} | \Omega_{n-1}, T_n) | \Omega_{n-1}] \\
&= E[e^{-\alpha U_n} (\frac{p\lambda_1}{\lambda_1 - \beta}[e^{-\beta(R(U_{n-1})+cT_n)} - e^{-\lambda_1(R(U_{n-1})+cT_n)}] \\
&\quad + \frac{q\lambda_2}{\lambda_2 - \beta}[e^{-\beta(R(U_{n-1})+cT_n)} - e^{-\lambda_2(R(U_{n-1})+cT_n)}]) | \Omega_{n-1}] \\
&= \frac{p\lambda_1}{\lambda_1 - \beta} e^{-\alpha U_{n-1}} [e^{-\beta R(U_{n-1})} \gamma(\alpha + c\beta) - e^{-\lambda_1 R(U_{n-1})} \gamma(\alpha + c\lambda_1)] \\
&\quad + \frac{q\lambda_2}{\lambda_2 - \beta} e^{-\alpha U_{n-1}} [e^{-\beta R(U_{n-1})} \gamma(\alpha + c\beta) - e^{-\lambda_2 R(U_{n-1})} \gamma(\alpha + c\lambda_2)].
\end{aligned} \tag{1.75}$$

Therefore, by using (1.75) we get

$$\begin{aligned}
\hat{H}(\alpha, \beta) - e^{-\beta u} &= \sum_{n\geqslant 1} E[e^{-\alpha U_n - \beta R(U_n)} I_{\{\nu > n\}}] \\
&= \sum_{n\geqslant 1} E[E(e^{-\alpha U_n - \beta R(U_n)} I_{\{\nu > n\}} | \Omega_{n-1})] \\
&= \sum_{n\geqslant 1} E[I_{\{\nu > n-1\}} E(e^{-\alpha U_n - \beta R(U_n)} I_{\{R(U_n)\geqslant 0\}} | \Omega_{n-1})] \\
&= \frac{p\lambda_1}{\lambda_1 - \beta}[\gamma(\alpha + c\beta)\hat{H}(\alpha, \beta) - \gamma(\alpha + c\lambda_1)\hat{H}(\alpha, \lambda_1)] \\
&\quad + \frac{q\lambda_2}{\lambda_2 - \beta}[\gamma(\alpha + c\beta)\hat{H}(\alpha, \beta) - \gamma(\alpha + c\lambda_2)\hat{H}(\alpha, \lambda_2)].
\end{aligned} \tag{1.76}$$

It is easy from (1.76) to get that

$$\hat{H}(\alpha,\beta)=\frac{e^{-\beta u}-\frac{p\lambda_1}{\lambda_1-\beta}\gamma(\alpha+c\lambda_1)\hat{H}(\alpha,\lambda_1)-\frac{q\lambda_2}{\lambda_2-\beta}\gamma(\alpha+c\lambda_2)\hat{H}(\alpha,\lambda_2)}{1-(\frac{p\lambda_1}{\lambda_1-\beta}+\frac{q\lambda_2}{\lambda_2-\beta})\gamma(\alpha+c\beta)}$$

$$=\frac{A-B-C}{(\lambda_1-\beta)(\lambda_2-\beta)-[\lambda_1\lambda_2-(p\lambda_1+q\lambda_2)\beta]\gamma(\alpha+c\beta)},\tag{1.77}$$

where

$$A=(\lambda_1-\beta)(\lambda_2-\beta)e^{-\beta u},$$

$$B=p\lambda_1(\lambda_2-\beta)\gamma(\alpha+c\lambda_1)\hat{H}(\alpha,\lambda_1),$$

$$C=q\lambda_2(\lambda_1-\beta)\gamma(\alpha+c\lambda_2)\hat{H}(\alpha,\lambda_2).$$

Since $\beta_1(\alpha)$ and $\beta_2(\alpha)$ are two roots of Equation (1.70), and $\hat{H}(\alpha,\beta)$ is *analytical*, thus $\beta_1(\alpha),\beta_2(\alpha)$ satisfy the following equations:

$$\begin{aligned}&(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_1(\alpha))e^{-\beta_1(\alpha)u}-p\lambda_1(\lambda_2-\beta_1(\alpha))\gamma(\alpha+c\lambda_1)\hat{H}(\alpha,\lambda_1)\\&=q\lambda_2(\lambda_1-\beta_1(\alpha))\gamma(\alpha+c\lambda_2)\hat{H}(\alpha,\lambda_2),\end{aligned}$$

$$\begin{aligned}&(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_2(\alpha))e^{-\beta_2(\alpha)u}-p\lambda_1(\lambda_2-\beta_2(\alpha))\gamma(\alpha+c\lambda_1)\hat{H}(\alpha,\lambda_1)\\&=q\lambda_2(\lambda_1-\beta_2(\alpha))\gamma(\alpha+c\lambda_2)\hat{H}(\alpha,\lambda_2).\end{aligned}$$

Solving the above equations we obtain

$$\begin{aligned}&\gamma(\alpha+c\lambda_1)\hat{H}(\alpha,\lambda_1)\\&=\frac{(\lambda_1-\beta_1(\alpha))(\lambda_1-\beta_2(\alpha))[(\lambda_2-\beta_1(\alpha))e^{-\beta_1(\alpha)u}-(\lambda_2-\beta_2(\alpha))e^{-\beta_2(\alpha)u}]}{p\lambda_1[(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))]},\end{aligned}$$

$$\begin{aligned}&\gamma(\alpha+c\lambda_2)\hat{H}(\alpha,\lambda_2)\\&=\frac{(\lambda_2-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))[(\lambda_1-\beta_2(\alpha))e^{-\beta_2(\alpha)u}-(\lambda_1-\beta_1(\alpha))e^{-\beta_1(\alpha)u}]}{q\lambda_2[(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))]}.\end{aligned}$$

Thus

$$\begin{aligned}
&\hat{H}(\alpha,\beta)\\
=&\ \big[(\lambda_1-\beta)(\lambda_2-\beta)-[\lambda_1\lambda_2-(p\lambda_1+q\lambda_2)\beta]\gamma(\alpha+c\beta)\big]^{-1}\\
&\times\big\{(\lambda_1-\beta)(\lambda_2-\beta)e^{-\beta u}\\
&-[(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))]^{-1}\\
&\times\big[(\lambda_2-\beta)(\lambda_1-\beta_1(\alpha))(\lambda_1-\beta_2(\alpha))[(\lambda_2-\beta_1(\alpha))e^{-\beta_1(\alpha)u}\\
&-(\lambda_2-\beta_2(\alpha))e^{-\beta_2(\alpha)u}]\\
&+(\lambda_1-\beta)(\lambda_2-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))[(\lambda_1-\beta_2(\alpha))e^{-\beta_2(\alpha)u}\\
&-(\lambda_1-\beta_1(\alpha))e^{-\beta_1(\alpha)u}]]\big\}.
\end{aligned}$$

Let $\beta=0$, then

$$\begin{aligned}
&\hat{H}(\alpha,0)\\
=&\ \frac{1}{1-\gamma(\alpha)}\cdot\left\{1-\frac{(\lambda_1-\lambda_2)D}{(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))}\right\},
\end{aligned}\tag{1.78}$$

where

$$\begin{aligned}
D=&\ [(1-\frac{\beta_1(\alpha)}{\lambda_1})(1-\frac{\beta_1(\alpha)}{\lambda_2})\beta_2(\alpha)e^{-\beta_1(\alpha)u}\\
&-(1-\frac{\beta_2(\alpha)}{\lambda_1})(1-\frac{\beta_2(\alpha)}{\lambda_2})\beta_1(\alpha)e^{-\beta_2(\alpha)u}].
\end{aligned}$$

By (1.74) and (1.78) we have

$$\begin{aligned}
&\alpha\int_0^\infty e^{-\alpha t}\phi(t,u)dt=\hat{H}(\alpha,0)(1-\gamma(\alpha))\\
&=1-(\lambda_1-\lambda_2)\frac{y_1(\alpha)\beta_2(\alpha)e^{-\beta_1(\alpha)u}-y_2(\alpha)\beta_1(\alpha)e^{-\beta_2(\alpha)u}}{(\lambda_1-\beta_2(\alpha))(\lambda_2-\beta_1(\alpha))-(\lambda_1-\beta_1(\alpha))(\lambda_2-\beta_2(\alpha))},
\end{aligned}\tag{1.79}$$

where $y_1(\alpha)=(1-\frac{\beta_1(\alpha)}{\lambda_1})(1-\frac{\beta_1(\alpha)}{\lambda_2}), y_2(\alpha)=(1-\frac{\beta_2(\alpha)}{\lambda_1})(1-\frac{\beta_2(\alpha)}{\lambda_2})$. Since $\psi(t,u)=1-\phi(t,u)$, (1.69) follows immediately from (1.79).

Remark 1.9 *Theorem 1.9 shows an exact numerical technique which requires merely numerical inversion of the Laplace transform of the ruin probability within finite time. Numerical methods of such an inversion could be found in [Abate and Whitt(1992)], and we don't discuss it here. On the other hand, it is well known that the probabilities of ruin* $\psi(t,u)$ *and* $\psi(u)$

can be identified with the virtual and limiting waiting time distributions, respectively, in a single server queue fed by a renewal process and having the service time distribution B. So, it is possible to prove Theorem 1.9 by virtue of related results from the theory of queues (see, for example, [Prabhu(1965)] and [Prabhu(1980)]).

1.4.3 *Two Corollaries*

As one application of Theorem 1.9, we can get the following two corollaries. The first corollary is about the expression for Laplace transform of the ruin probability within finite time when the claim amount distribution is a exponential, which was obtained by [Malinovskii(1998)]. The second corollary is concerned with a result about ultimate ruin probability in the classical risk model which was proved by [Gerber(1979)].

Corollary 1.1 *Let the sizes of claims $\{Y_i, i \geqslant 1\}$ and the inter-claims $\{T_i, i \geqslant 1\}$ be iid and mutually independent. Assume that $Y_1 \sim$ Exponential$(\lambda), \lambda > 0$, and $\gamma(\alpha) = \int_0^\infty e^{-\alpha u} P_T(du)$ is the Laplace transform of T_1. Then*

$$\alpha \int_0^\infty e^{-\alpha t} \phi(t, u) dt = 1 - y(\alpha) \exp\{-u\lambda(1 - y(\alpha))\}, \alpha > 0, \tag{1.80}$$

where $y(\alpha)$ is the unique root in $(0, 1)$ of Equation

$$y = \gamma(\alpha + c\lambda(1 - y)), \alpha > 0. \tag{1.81}$$

Proof. Taking $p \doteq 1, \lambda_1 = \lambda$ in (1.70) of Theorem 1.9 yields

$$(\lambda_2 - \beta)(\lambda - \beta - \lambda\gamma(\alpha + c\beta)) = 0, \alpha > 0. \tag{1.82}$$

Therefore we get that $\beta_2(\alpha) = \lambda_2$, and $\beta_1(\alpha)$ is the unique root of Equation

$$\lambda - \beta - \lambda\gamma(\alpha + c\beta) = 0, \alpha > 0. \tag{1.83}$$

Moreover, $y_1(\alpha) = (1 - \frac{\beta_1(\alpha)}{\lambda})(1 - \frac{\beta_1(\alpha)}{\lambda_2}), y_2(\alpha) = 0$, and the right side of (1.69) is $(1 - \frac{\beta_1(\alpha)}{\lambda})e^{-\beta_1(\alpha)u}$. Put $y(\alpha) = 1 - \frac{\beta_1(\alpha)}{\lambda}$. Then

$$(1 - \frac{\beta_1(\alpha)}{\lambda})e^{-\beta_1(\alpha)u} = y(\alpha) \exp\{-u\lambda(1 - y(\alpha))\}.$$

Again by (1.83) we see that $y(\alpha)$ is the unique root of the Equation (1.81) in $(0, 1)$. Finally, (1.80) follows immediately from (1.69) by $\psi(t, u) = 1 - \phi(t, u)$.

Corollary 1.2 *For the classical risk model, in the case of exponential claim amounts, the ultimate ruin probability is an exponential function of the initial surplus measured in mean claim amounts. In other words, if the interclaims* $\{T_i, i \geqslant 1\} \sim$ *Exponential*$(\mu), \mu > 0$, *the sizes of claims* $\{Y_i, i \geqslant 1\} \sim$ *Exponential*$(\lambda), \lambda > 0$. *Then* $\psi(u) = \frac{1}{1+\Lambda}\exp\{-\frac{\Lambda}{1+\Lambda}\frac{u}{E(Y_1)}\}$, *where* $\Lambda = \frac{c}{\mu EY_1} - 1 = \frac{c\lambda}{\mu} - 1$.

Before we give the proof of this Corollary, we need the following Lemma, whose proof is very easy by induction, and so is omitted.

Lemma 1.2 *When* $0 < x \leqslant \frac{1}{4}$, *for any nonnegative integer* n, *the following equality always holds:*

$$\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!(n+k+1)!} x^k = \frac{1}{(n+1)} \left(\frac{1+\sqrt{1-4x}}{2}\right)^{-(n+1)}. \tag{1.84}$$

Remark 1.10 *When* $0 < p < 1, 0 < q < 1, p+q = 1$, *for any nonnegative interger* n,

$$\sum_{k=0}^{\infty} \frac{(n+1)(n+2k)!}{k!(n+k+1)!} p^k q^{n+k+1} = \begin{cases} 1, & 0 < p \leqslant \frac{1}{2}, \\ (\frac{q}{p})^{n+1}, & \frac{1}{2} < p < 1. \end{cases} \tag{1.85}$$

Proof. Evidently, $0 < pq \leqslant \frac{1}{4}$. Therefore by Lemma 1.2 we see that

$$\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!(n+k+1)!} (pq)^k = \frac{1}{n+1} \left(\frac{1+\sqrt{1-4pq}}{2}\right)^{-(n+1)}.$$

Consequently,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(n+1)(n+2k)!}{k!(n+k+1)!} p^k q^{n+k+1} &= q^{n+1} \left(\frac{1+|1-2p|}{2}\right)^{-(n+1)} \\ &= \begin{cases} 1, & 0 < p \leqslant \frac{1}{2}, \\ (\frac{q}{p})^{n+1}, & \frac{1}{2} < p < 1. \end{cases} \end{aligned}$$

The following (1.86) comes immediately from the above (1.85).

Remark 1.11 *Let* $p = \frac{1}{2+\Lambda}, q = \frac{1+\Lambda}{2+\Lambda}$, *where* Λ *is the relative safety loading,* $\Lambda > 0$. *Then for any nonnegative integer* n *we have*

$$\sum_{k=0}^{\infty} \frac{(n+1)(n+2k)!}{k!(n+k+1)!} \left(\frac{1}{2+\Lambda}\right)^k \left(\frac{1+\Lambda}{2+\Lambda}\right)^{n+k+1} = 1. \tag{1.86}$$

The Proof of Corollary 1.2 Since $\{T_i, i \geqslant 1\} \sim E(\mu), \mu > 0, \gamma(\alpha) = \frac{\mu}{\alpha+\mu}$, by Corollary 1.1 we have $y(\alpha) = \gamma(\alpha + c\lambda(1 - y(\alpha))) = \frac{\mu}{\alpha + c\lambda(1-y(\alpha))+\mu}$, i.e.,

$$c\lambda y^2(\alpha) - (\alpha + \mu + c\lambda)y(\alpha) + \mu = 0.$$

Again by Corollary 1.1 we get

$$y(\alpha) = 4\mu\left(\sqrt{\alpha + (\sqrt{\mu} + \sqrt{c\lambda})^2} + \sqrt{\alpha + (\sqrt{\mu} - \sqrt{c\lambda})^2}\right)^{-2}. \tag{1.87}$$

Put $x = (\sqrt{\mu} + \sqrt{c\lambda})^2, y = (\sqrt{\mu} - \sqrt{c\lambda})^2$, then

$$y(\alpha) = \sqrt{\frac{\mu}{c\lambda}}(x - y)(\sqrt{\alpha + x} + \sqrt{\alpha + y})^{-2}. \tag{1.88}$$

Thus $y(\alpha)$ is the Laplace transform of $\sqrt{\frac{\mu}{c\lambda}}e^{-(\mu+c\lambda)t}t^{-1}\mathrm{I}_1(2\sqrt{\mu c\lambda}t)$, where $\mathrm{I}_1(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+2)}(\frac{z}{2})^{2k+1}$ is the first kind of Bessel function (see [Feller(1971)]). On the other hand, $\alpha\int_0^\infty e^{-\alpha t}\phi(t,u)dt = 1 - y(\alpha)\exp\{-u\lambda(1 - y(\alpha)\}, \alpha > 0$, and $\alpha\int_0^\infty e^{-\alpha t}\phi(t,u)dt = 1 - E(e^{-\alpha\tau})$. Hence

$$\begin{aligned}
E(e^{-\alpha\tau}) &= e^{-u\lambda}\sum_{n=0}^{\infty}\frac{(u\lambda)^n}{n!}y^{n+1}(\alpha) \\
&= e^{-u\lambda}\sum_{n=0}^{\infty}\frac{(u\lambda)^n}{n!}(\sqrt{\frac{\mu}{c\lambda}})^{n+1}((x - y)(\sqrt{\alpha + x} + \sqrt{\alpha + y})^{-2})^{n+1}.
\end{aligned} \tag{1.89}$$

The (1.89) is the Laplace transform of $e^{-u\lambda}\sum_{n=0}^{\infty}\frac{(u\lambda\sqrt{\frac{\mu}{c\lambda}})^n}{n!t}\sqrt{\frac{\mu}{c\lambda}}(n + 1)e^{-(\mu+c\lambda)t}\mathrm{I}_{n+1}(2\sqrt{\mu c\lambda}t)$, where $\mathrm{I}_n(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(n+k+1)}(\frac{z}{2})^{n+2k}$ (see Feller(1971)). Hence

$$\begin{aligned}
&\psi(t,u) = P(\tau \leqslant t) \\
&= \int_0^t e^{-u\lambda}\sum_{n=0}^{\infty}\frac{(u\lambda\sqrt{\mu/c\lambda})^n}{n!s}\sqrt{\frac{\mu}{c\lambda}}(n + 1)e^{-(\mu+c\lambda)s}\mathrm{I}_{n+1}(2\sqrt{\mu c\lambda}s)ds \\
&= \frac{\mu}{c\lambda}e^{-u\lambda}\sum_{n=0}^{\infty}\frac{(\frac{u\mu}{c})^n}{n!}(\frac{\mu}{c\lambda})^{-\frac{n+1}{2}}\int_0^t e^{-(\mu+c\lambda)s}\frac{n+1}{s}\mathrm{I}_{n+1}(2\sqrt{\mu c\lambda}s)ds.
\end{aligned}$$

Thus, letting $t \to \infty$ and using Remark 1.11 gives

$$\begin{aligned}
&\psi(u) = P(\tau < \infty) \\
&= \frac{\mu}{c\lambda} e^{-u\lambda} \sum_{n=0}^{\infty} \frac{(\frac{u\mu}{c})^n}{n!} (\frac{\mu}{c\lambda})^{-\frac{n+1}{2}} \int_0^{\infty} e^{-(\mu+c\lambda)s} \frac{n+1}{s} I_{n+1}(2\sqrt{\mu c\lambda s}) ds \\
&= \frac{\mu}{c\lambda} e^{-u\lambda} \sum_{n=0}^{\infty} \frac{(\frac{u\mu}{c})^n}{n!} (\frac{\mu}{c\lambda})^{-\frac{n+1}{2}} \cdot \\
&\quad \int_0^{\infty} e^{-(\mu+c\lambda)s} \frac{n+1}{s} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n+k+2)} (\sqrt{\mu c\lambda}s)^{n+2k+1} ds \\
&= \frac{\mu}{c\lambda} e^{-u\lambda} \sum_{n=0}^{\infty} \frac{(\frac{u\mu}{c})^n}{n!} (\frac{\mu}{c\lambda})^{-\frac{n+1}{2}} (n+1) \cdot \\
&\quad \sum_{k=0}^{\infty} \frac{\Gamma(n+2k+1)}{k!\Gamma(n+k+2)} \frac{(\sqrt{\mu c\lambda})^{n+2k+1}}{(\mu+c\lambda)^{n+2k+1}} \\
&= \frac{\mu}{c\lambda} e^{-u\lambda} \sum_{n=0}^{\infty} \frac{n+1}{n!} (\frac{u\mu}{c})^n \cdot \\
&\quad \sum_{k=0}^{\infty} \frac{\Gamma(n+2k+1)}{k!\Gamma(n+k+2)} (\frac{c\lambda}{\mu})^{n+k+1} (1+\frac{c\lambda}{\mu})^{-(n+2k+1)} \\
&= \frac{1}{1+\Lambda} e^{-u\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{u\lambda}{1+\Lambda})^n (n+1) \cdot \\
&\quad \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!(n+k+1)!} (\frac{1+\Lambda}{2+\Lambda})^{n+k+1} (\frac{1}{2+\Lambda})^k \\
&= \frac{1}{1+\Lambda} e^{-u\lambda} e^{\frac{\lambda u}{1+\Lambda}} \\
&= \frac{1}{1+\Lambda} \exp\{-\frac{\Lambda}{1+\Lambda} \lambda u\} \\
&= \frac{1}{1+\Lambda} \exp\{-\frac{\Lambda}{1+\Lambda} \frac{u}{E(Y_1)}\}.
\end{aligned}$$

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