Chapter 1

Deterministic and random oscillators

This chapter reviews all types of differential equations which will be analyzed later on in more detail. We present here the second order (underdamped) differential equations. The first order (overdamped) equations, which we will consider first can be obtained from it by leaving only the first derivative in these equations.

1.1 Simple harmonic oscillator

Such an oscillator is a simple system where the net force is directly proportional to the displacement x of the mass m from the equilibrium position $x = 0$ and pointing in the opposite direction (the Hook's law). According to Newton's law of motion,

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0. \tag{1.1}$$

The solution of this equation has a form

$$x = C\cos\left(\omega_0 t + \phi\right), \tag{1.2}$$

where the angular frequency $\omega_0 = \sqrt{k/m}$ depends only upon the parameters of the system, while the amplitude A and the phase ϕ are constants determined by the initial displacement $x\left(t = 0\right)$ and velocity $\frac{dx}{dt}\left(t = 0\right)$.

1.2 Damped harmonic oscillator

The preceding analysis can be supplemented by a dissipative force which points in the opposite direction and is usually assumed to be proportional to the velocity,

$$m\frac{d^2x}{dt^2} = -kx - 2\gamma\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + kx = 0. \tag{1.3}$$

The solution of equation (1.3) for $k > \gamma$ has the form

$$x = C\exp\left(-\frac{\gamma t}{m}\right)\cos\left(\omega_1 t + \phi\right), \tag{1.4}$$

with $\omega_1 = \sqrt{\omega_0^2 - \left(\frac{\gamma}{m}\right)^2}$.

1.3 Driven harmonic oscillator

Another generalization of the equations for a simple harmonic oscillator (1.1) and a damped harmonic oscillator (1.3) is one driven by some time-dependent external force, the simplest form of which is a sinusoidal function of time, i.e.,

$$m\frac{d^2x}{dt^2} + kx = A\cos\left(\Omega t + \beta\right). \tag{1.5}$$

This has a driven solution of the form

$$x = \frac{A}{m\left(\omega_0^2 - \Omega^2\right)}\cos\left(\Omega t + \beta\right). \tag{1.6}$$

When the external frequency Ω approaches the intrinsic frequency ω_0, the steady state amplitude approaches infinity (dynamic resonance). Classic demonstrations of this dynamic resonance are two architectural flaws in the US. The first is the Takoma bridge which was destroyed by the wind's force with the resonance frequency, and the second is the Paramount Communication building in New York which was transformed into a luxury apartment building. Due to the winds the top floors were twisted and windows pried loose from their casements.

1.4 Driven, damped harmonic oscillator

On combining the damping and driving forces, one arrives at the following equation

$$m\frac{d^2x}{dt^2} + kx + 2\gamma\frac{dx}{dt} = A\cos\left(\Omega t + \beta\right). \tag{1.7}$$

The general solution of equation (1.7) is a sum of a transient and a steady state solutions

$$\begin{aligned} x = {} & C\exp\left(-\frac{\gamma t}{m}\right)\cos\left(\omega_1 t + \phi\right) \\ & + \frac{A}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + \frac{4\gamma^2\Omega^2}{m^2}}}\sin\left(\Omega t + \phi + \beta\right) \end{aligned} \tag{1.8}$$

with $\beta = \tan^{-1}\frac{\omega_0^2 - \Omega^2}{2\gamma\Omega}$.

1.5 Nonlinear oscillator in a double-well potential

Up to now, we have considered the linear oscillator with the restoring force f proportional to the displacement x, $f \approx -kx$. In a more general case this dependence can be nonlinear, $f \approx ax - bx^n$. Hereafter, we restrict our attention to the most interesting cases $n = 2$ and $n = 3$.

The linear force in equations (1.1), (1.3), (1.7) corresponds to the potential $U = kx^2/2$. For the more complicated case of a double-well potential energy $U = -ax^2/2 + bx^4/4$, with two minima located at $x = \pm\sqrt{a/b}$ and a maximum at $x = 0$, the equation of motion has the form

$$m\frac{d^2x}{dt^2} - ax + bx^3 = 0. \tag{1.9}$$

1.6 Nonlinear oscillator in a single-well potential

Another much used form of the potential energy is one of the form $U = -ax^2/2 + bx^3/3$, with an unstable minimum at $x = a/b$, which results in the following equation of motion

$$m\frac{d^2x}{dt^2} - ax + bx^2 = 0. \tag{1.10}$$

Both equations (1.9) and (1.10) can be easily generalized to include a damping term and an external force. However, one cannot find the exact solutions of these equations even for a simplified form of these equations.

1.7 Harmonic oscillator with external noise

In the foregoing we considered a pure mechanical system (zero temperature). However, for all finite temperatures the dynamic equations (1.1), (1.4) have to be supplemented by thermal noise, i.e.,

$$m\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + kx = \xi\left(t\right), \tag{1.11}$$

where $\xi\left(t\right)$ is the random variable with zero mean $\langle\xi\left(t\right)\rangle = 0$ and the variance $\left\langle\xi^2\left(t\right)\right\rangle$ which for the thermal noise must satisfy the fluctuation-dissipation theorem [5], $\left\langle\xi^2\left(t\right)\right\rangle = 4\gamma\kappa T$, where κ is the Boltzmann constant. The noise acting on a system can also be external (and not thermal noise) with no special requirement for the value of $\left\langle\xi^2\left(t\right)\right\rangle$. Still another way to justify the validity of equation (1.11) is as follows: in considering only one (slow) mode $x\left(t\right)$ of a complex system, one may take into account the influence of other (fast) modes by introducing a random force into the dynamic equations.

1.8 Brownian motion

Equation (1.11) with $m = 0$ and x meaning the particle's velocity v describes Brownian motion, where the force acting on the Brownian particle consists of the systematic force $-kv$ and the random force $\xi(t)$

$$\gamma \frac{dv}{dt} + kv = \xi(t)\,. \tag{1.12}$$

Note, that caution is required in going from the oscillator equation to that of the Brownian particle, since the simple substitution of $m \to 0$ in equation (1.11) will decrease the order of the differential equation, and one has to use the singular perturbation theory [6].

1.9 Harmonic oscillator with random frequency

The random force $\xi(t)$ enters equation (1.11) additively. When the noise has an external origin rather than an internal one, for instance if it arises from the fluctuations of the potential energy $U = \frac{kx^2[1+\xi(t)]}{2}$, the equation of motion of an oscillator with an external periodic force $A\sin(\Omega t)$, will take the following form:

$$m\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + k\,[1+\xi(t)]\,x = A\sin(\Omega t)\,. \tag{1.13}$$

The internal dynamics of a force-free harmonic oscillator $(A = 0)$ with random frequency is a subject that has been extensively investigated in different fields, including physics (on-off intermittence [7], dye lasers [8], wave propagation in random media [9], turbulent flows [10]), biology (population dynamics [11]), economics (stock market prices [12]) and so on.

Note that for $A = \gamma = 0$ and for x and t being replaced by the electric field and coordinate, respectively, equation (1.13) transforms into the Maxwell equation with a random dielectric constant which has also been intensively studied [13]. Recently we considered equation (1.13) with an external force [14], [15].

1.10 Harmonic oscillator with random damping

Another possibility for the generalization of the dynamic equation (1.7) is the incorporation of random damping

$$m\frac{d^2x}{dt^2}+2\gamma\left[1+\xi\left(t\right)\right]\frac{dx}{dt}+kx=A\sin\left(\Omega t\right) \tag{1.14}$$

The first time this equation (with $A=0$) was used [16] for the problem of water waves influenced by a turbulent wind field. However, this equation (with the coordinate x and time t replaced by the order parameter and the coordinate, respectively) is transformed into the Ginzburg-Landau equation with a convective term which describes phase transitions in moving systems [17]. There is an increasing number of problems where the particles advected by the mean flow pass through the region under study. These include problems of phase transition under shear [18], open flows of liquids [19], Rayleigh-Benard and Taylor-Couette problems in fluid dynamics [20], dendritic growth [21], chemical waves [22], and the motion of vortices [23].

Notice the important difference between the linear equation (1.11) with an additive noise and equations (1.13), (1.14) with multiplicative noise. The latter show, in fact, "quasi-nonlinear behavior" including the stochastic resonance phenomenon [14], [24].