Chapter 1

Markov and Semi-Markov Processes

1.1 Preliminaries

The aim of this chapter is to give a brief account of basic notions on Markov and semi-Markov processes, together with martingale characterization, needed throughout this book. Further introduction to semimartingales and stochastic linear operators is also given.

Let (E, r) be a complete, separable metric space, (that is, a *Polish space*), and let $\mathcal{E}$ be its Borel σ-algebra of subsets of E [52]. Throughout this book, we will call the measurable space $(E, \mathcal{E})$ a *standard state space.* The space $\mathbb{R}^d$, with the Euclidean metric, is a Polish space. We will denote by $\mathcal{B}_d$ its Borel σ-algebra, $d > 1$, with $\mathcal{B} := \mathcal{B}_1$.

In this book, the space where the trajectories of processes are considered is $\mathbf{D}[0, \infty)$, the space of right-continuous functions having left side limits with Skorokhod metric [16,45,70,132,153]. We call them cadlag trajectories and cadlag processes. Let also $\mathbf{C}[0, \infty)$ be the subspace of $\mathbf{D}[0, \infty)$, of continuous functions with the sup-norm, $\|x\| = \sup_{t \geq 0} |x(t)|$, $x \in \mathbf{C}[0, \infty)$. These two spaces are Polish spaces (see Appendix A).

Let $\mathbf{B}$ be the Banach space, that is a complete linear normed space, of all bounded real-valued measurable functions on E, with the sup-norm $\|\varphi\| = \sup_{x \in E} |\varphi(x)|$, $\varphi \in \mathbf{B}$.

Let us consider also a fixed *stochastic basis* $\Im = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t,\ t \geq 0), \mathbb{P})$, where $(\mathcal{F}_t,\ t \geq 0)$ (for discrete time note that $\mathbf{F} = (\mathcal{F}_n,\ n \geq 0)$) is a filtration of sub-σ-algebras of $\mathcal{F}$, that is $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$, for all $s < t$ and $t \geq 0$. The filtration $\mathbf{F} = (\mathcal{F}_t,\ t \geq 0)$ is said to be *complete*, if $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets. Set $\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s, t \geq 0$. If for any $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{t+}$, then the filtration $\mathcal{F}_t, t \geq 0$, is said to be *right-continuous.* If a filtration is complete and right-continuous, we say that it satisfies the *usual conditions.*

A mapping $T : \Omega \to [0, +\infty]$, such that $\{T \leq t\} \in \mathcal{F}_t$, is called a *stopping time.* If T is a stopping time, we denote by $\mathcal{F}_T$ the collection of all sets $A \in \mathcal{F}$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$.

An $(E, \mathcal{E})$-valued stochastic process $x(t), t \in I$, $(I = \mathbb{R}_+$ or $I = \mathbb{N})$, defined on the stochastic basis $\mathfrak{F}$, is *adapted,* if for any $t \in I$, $x(t)$ is $\mathcal{F}_t$-measurable. The set of values E is said to be the *state* (or *phase*) *space* of the process $x(t), t \geq 0$.

Given a probability measure μ on $(E, \mathcal{E})$, we define the probability measure $\mathbb{P}_\mu$, by

$$\mathbb{P}_\mu(B) = \mu\mathbb{P}(B) = \int_E \mu(dx)\mathbb{P}_x(B), \quad x \in E, B \in \mathcal{F},$$

where $\mathbb{P}_x(B) := \mathbb{P}(B \mid x(0) = x)$. We denote by $\mathbb{E}_\mu$ and $\mathbb{E}_x$ the expectations corresponding respectively to $\mathbb{P}_\mu$ and $\mathbb{P}_x$.

We will also consider the following spaces endowed by the corresponding sup-norms:

- **B** is the Banach space of real-valued measurable bounded functions $\varphi(u, x)$, $u \in \mathbb{R}^d$, $x \in E$;
- $\mathbf{B}^1 := C^1(\mathbb{R}^d \times E) \cap \mathbf{B}$ is the Banach space of continuously differentiable functions on $u \in \mathbb{R}^d$, uniformly on $x \in E$, with bounded first derivative;
- $\mathbf{B}^2 := C^2(\mathbb{R}^d \times E) \cap \mathbf{B}$ is the Banach space of twice continuously differentiable functions on $u \in \mathbb{R}^d$, uniformly on $x \in E$, with bounded first two derivatives.

1.2 Markov Processes

1.2.1 *Markov Chains*

Definition 1.1 A positive-valued function $P(x, B)$, $x \in E, B \in \mathcal{E}$, is called a *Markov transition function* or a *Markov kernel* or *transition (probability) kernel,* if

1) for any fixed $x \in E$, $P(x, \cdot)$ is a probability measure on $(E, \mathcal{E})$, and

2) for any fixed $B \in \mathcal{E}$, $P(\cdot, B)$ is a Borel measurable function, that is, an $(\mathcal{E}, \mathcal{B})$-measurable function.

If $P(x, E) \leq 1$, for a $x \in E$, then P is said to be a *sub-Markov kernel.* If for fixed $x \in E$, the $P(x, \cdot)$ is a *signed measure,* then it is said to be a *signed kernel.* In that case, we will suppose that the signed kernel P is of bounded variation, that is,

$$|P|(x,E) < +\infty. \tag{1.1}$$

If E is a finite or countable set, we take $\mathcal{E} = \mathcal{P}(E)$ (the set of all subsets of E), the Markov kernel is determined by the matrix $(P(i,j);\ i,j \in E)$, with $P(i,B) = \sum_{j\in B} P(i,j)$, $B \in \mathcal{E}$.

Definition 1.2 A *time-homogeneous Markov chain* associated to a Markov kernel $P(x,B)$ is an adapted sequence of random variables $x_n, n \geq 0$, defined on some stochastic basis $\mathfrak{F}$, satisfying, for every $n \in \mathbb{N}$, $x \in E$, and $B \in \mathcal{E}$, the following relation

$$\mathbb{P}(x_{n+1} \in B \mid \mathcal{F}_n) = \mathbb{P}(x_{n+1} \in B \mid x_n) =: P(x_n, B), \quad \text{(a.s.)}, \tag{1.2}$$

which is called the *Markov property.*

In most cases we consider the Markov property with respect to $\mathcal{F}_n := \sigma(x_k,\ k \leq n)$, $n \geq 0$, the *natural filtration* generated by the chain x_n, $n \geq 0$. The Markov property (1.2) is satisfied for any finite $\mathcal{F}_n$-stopping time and it is called *strong Markov property* and the chain a *strong Markov chain.*

The product of two Markov kernels P and Q defined on $(E, \mathcal{E})$, is also a Markov kernel, defined by

$$PQ(x,B) = \int_E P(x,dy)Q(y,B). \tag{1.3}$$

Let us denote by $P^n(x,B) = \mathbb{P}(x_n \in B \mid x_0 = x) = \mathbb{P}(x_{n+m} \in B \mid x_m = x)$ the n-step transition probability which is defined inductively by (1.3). By the Markov property we get, for $n, m \in \mathbb{N}$,

$$P^{n+m}(x,B) = \int_E P^n(x,dy)P^m(y,B) = \int_E P^m(x,dy)P^n(y,B),$$

which is the *Chapman-Kolmogorov equation.*

A subset $B \in \mathcal{E}$, is called accessible from a state $x \in E$, if

$$\mathbb{P}_x(x_n \in B, \text{ for some } n \geq 1) > 0,$$

or equivalently $P^n(x,B) > 0$.

Definition 1.3

1) A Markov chain $x_n, n \geq 0$, is called *Harris recurrent* if there exists a σ-finite measure ψ on $(E, \mathcal{E})$, with $\psi(E) > 0$, such that

$$\mathbb{P}_x(\cup_{n\geq 1}\{x_n \in A\}) = 1, \quad x \in E, \tag{1.4}$$

for any $A \in \mathcal{E}$ with $\psi(A) > 0$.

2) If the probability (1.4) is positive, the Markov chain is called *ψ-irreducible.*

3) The Markov chain is said to be *uniformly irreducible* if, for any $A \in \mathcal{E}$,

$$\sup_x \mathbb{P}_x(\tau_A > N) \longrightarrow 0, \quad N \to \infty, \tag{1.5}$$

where $\tau_A := \inf\{n \geq 0 : x_n \in A\}$, is the hitting time of set $A \in \mathcal{E}$.

Definition 1.4 A Markov chain $x_n, n \geq 0$, is said to be *d-periodic* ($d > 1$), if there exists a cycle, that is a sequence $(C_1, ..., C_d)$ of sets, $C_i \in \mathcal{E}$, $1 \leq i \leq d$, with $P(x, C_{j+1}) = 1$, $x \in C_j$, $1 \leq j \leq d-1$, and $P(x, C_1) = 1$, $x \in C_d$, such that:

- the set $E \setminus \cup_{i=1}^d C_i$ is ψ-null;

- if $(C'_1, ..., C'_{d'})$ is another cycle, then d' divides d and C'_i differs from a union of d/d' members of $(C_1, ..., C_d)$ only by a *ψ-null set* which is of type $\cup_{i\geq 1} V_i$, where, for any $i \geq 1$, $\mathbb{P}_x(\limsup\{x_n \in V_i\}) = 0$.

If $d = 1$ then the Markov chain is said to be *aperiodic.*

Definition 1.5 A probability measure ρ on $(E, \mathcal{E})$, is said to be a *stationary distribution* or *invariant probability* for the Markov chain x_n, $n \geq 0$, (or for the Markov kernel $P(x, B)$) if, for any $B \in \mathcal{E}$,

$$\rho(B) = \int_E \rho(dx) P(x, B).$$

Definition 1.6

1) If a Markov chain is ψ-irreducible and has an invariant probability, it is called *positive*, otherwise it is called *null.*

2) If a Markov chain is Harris recurrent and positive it is called *Harris positive.*

3) If a Markov chain is aperiodic and Harris positive it is called (Harris) *ergodic.*

Proposition 1.1 *Let $x_n, n \geq 0$, be an ergodic Markov chain, then:*

1) for any probability measure α on $(E, \mathcal{E})$, we have

$$\|\alpha P^n - \rho\| \to 0, \quad n \to \infty;$$

2) for any $\varphi \in \mathbf{B}$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(x_k) = \int_E \rho(dx)\varphi(x), \quad \mathbb{P}_\mu\text{-}a.s.,$$

for any probability measure μ *on* $(E, \mathcal{E})$.

Let us denote by P the operator of transition probabilities on $\mathbf{B}$ defined by

$$P\varphi(x) = \mathbb{E}[\varphi(x_{n+1}) \mid x_n = x] = \int_E P(x, dy)\varphi(y),$$

and denote by P^n the n-step transition operator corresponding to $P^n(x, B)$.

The Markov property (1.2) can be represented in the following form

$$\mathbb{E}[\varphi(x_{n+1}) \mid \mathcal{F}_n] = P\varphi(x_n). \tag{1.6}$$

Definition 1.7 Let us denote by Π the *stationary projector* in $\mathbf{B}$ defined by the stationary distribution $\rho(B), B \in \mathcal{E}$ of the Markov chain x_n, as follows

$$\Pi\varphi(x) := \int_E \rho(dy)\varphi(y)\mathbf{1}(x) = \widehat{\varphi}\mathbf{1}(x), \quad \widehat{\varphi} := \int_E \rho(dx)\varphi(x),$$

where $\mathbf{1}(x) = 1$ for all $x \in E$. Of course, we have $\Pi^2 = \Pi$.

Definition 1.8 The Markov chain x_n is called *uniformly ergodic* if

$$\sup_{\|\varphi\|\le 1} \|(P^n - \Pi)\varphi\| \longrightarrow 0, \quad n \to \infty, \tag{1.7}$$

Note that uniform ergodicity implies Harris recurrence [117,80,137,139].

Moreover, the convergence in (1.7) is of exponential rate (see, e.g. [137]). So the series

$$R_0 := \sum_{n=0}^{\infty} [P^n - \Pi],$$

is convergent and defines the *potential operator* of the Markov chain $x_n, n \geq 0$, satisfying the property (see Section 1.6)

$$R_0[I - P] = [I - P]R_0 = I - \Pi.$$

1.2.2 *Continuous-Time Markov Processes*

Let us consider a family of Markov kernels $(P_t = P_t(x,B),\ t \in \mathbb{R}_+)$ on $(E,\mathcal{E})$. Let an adapted $(E,\mathcal{E})$-valued stochastic process $x(t),\ t \geq 0$, be defined on some stochastic basis $\mathfrak{F}$.

Definition 1.9 A stochastic process $x(t),\ t \geq 0$, is said to be a *time-homogeneous Markov process*, if, for any fixed $s,t \in \mathbb{R}_+$ and $B \in \mathcal{E}$,

$$\mathbb{P}(x(t+s) \in B \,|\, \mathcal{F}_s) = \mathbb{P}(x(t+s) \in B \,|\, x(s)) = P_t(x(s),B), \quad \text{(a.s.)}. \tag{1.8}$$

When the Markov property (1.8) holds for any finite $\mathbf{F}$-stopping time τ, instead of a deterministic time s, we say that the Markov process $x(t), t \geq 0$, satisfies the *strong Markov property*, and that the process $x(t)$ is a *strong Markov process.*

Definition 1.10 On the Banach space $\mathbf{B}$, the operator P_t of transition probability, is defined by

$$P_t\varphi(x) = \mathbb{E}_x[\varphi(x(t))] = \int_E \varphi(y)P_t(x,dy), \quad \varphi \in \mathbf{B}. \tag{1.9}$$

This is a contractive operator (that is, $\|P_t\varphi\| \leq \|\varphi\|$).

The *Chapman-Kolmogorov equation* is equivalent to the following semigroup property of P_t,

$$P_tP_s = P_{t+s}, \text{ for all } t,s \in \mathbb{R}_+. \tag{1.10}$$

The Markov process $x(t),\ t \geq 0$, has a *stationary* (or invariant) *distribution*, π say, if, for any $B \in \mathcal{E}$,

$$\pi(B) = \int_E \pi(dx)P_t(x,B), \quad \pi(E) = 1, \quad t \geq 0.$$

Definition 1.11 The Markov process $x(t),\ t \geq 0$, is said to be *ergodic*, if for every $\varphi \in \mathbf{B}$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(x(s))ds = \int_E \pi(dx)\varphi(x), \quad \mathbb{P}_\mu\text{-a.s.},$$

for any probability measure μ on $(E,\mathcal{E})$.

The stationary projector Π, of an ergodic Markov process with stationary distribution π, is defined as follows (see Definition 1.7)

$$\Pi\varphi(x) := \int_E \pi(dy)\varphi(y)\mathbf{1}(x) = \widehat{\varphi}\mathbf{1}(x), \quad \widehat{\varphi} := \int_E \pi(dx)\varphi(x),$$

where $\mathbf{1}(x) = 1$ for all $x \in E$. Of course, we have $\Pi^2 = \Pi$.

Let us consider a Markov process $x(t),\ t \geq 0$, on the stochastic basis $\mathfrak{F}$, with trajectories in $\mathbf{D}[0, \infty)$, and semigroup $(P_t,\ t \geq 0)$.

There exists a linear operator Q acting on $\mathbf{B}$, defined by

$$\lim_{t \downarrow 0} \frac{1}{t}(P_t\varphi - \varphi) = Q\varphi, \tag{1.11}$$

where the limit exists in norm.

Let $\mathcal{D}(Q)$ be the subset of $\mathbf{B}$ for which the above limit exists, this is the *domain* of the operator Q. The operator Q is called a (strong) *generator* or (strong) *infinitesimal operator.*

Definition 1.12 A Markov semigroup $P_t, t \geq 0$, is said to be *uniformly continuous* on $\mathbf{B}$, if

$$\lim_{t \to 0} \|P_t - I\| = 0,$$

where I is the identity operator on $\mathbf{B}$.

A time-homogeneous Markov process is said to be (purely) *discontinuous* or of *jump type*, if its semigroup is uniformly continuous. In that case, the process stays in any state for a positive (strict) time, and after leaving a state it moves directly to another one. We will call it a *jump Markov process* [34,56,153,165].

Let $x(t),\ t \geq 0$, be a time-homogeneous jump Markov process. Let $\tau_n, n \geq 0$ be the jump times for which we have $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \cdots$. A Markov process is said to be *regular* (*non explosive*), if $\tau_n \to \infty$, as $n \to \infty$ (see, e.g. [56]).

Definition 1.13 The stochastic process $x_n, n \geq 0$ defined by

$$x_n = x(\tau_n), \quad n \geq 0,$$

is called the *embedded Markov chain* of the Markov process $x(t),\ t \geq 0$.

Let $P(x, B)$ be the transition probability of $x_n, n \geq 0$. The generator Q of the jump Markov process $x(t), t \geq 0$, is of the form (see, e.g. [56,34]),

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \tag{1.12}$$

where the kernel $P(x, dy)$ is the transition kernel of the embedded Markov chain, and $q(x), x \in E$, is the intensity of jumps function.

Proposition 1.2 *(see, e.g. [165,52]) Let $(P_t,\ t \geq 0)$ be a uniformly continuous semigroup on $\mathbf{B}$, and Q its generator with domain $\mathcal{D}(Q) \subset \mathbf{B}$. Then:*

1) the limit in (1.11) exists, and the operator Q is bounded with $\overline{\mathcal{D}}(Q) = \mathbf{B}$;

2) $dP_t/dt = QP_t = P_tQ$;

3) $P_t = \exp(tQ) = I + \sum_{k\geq 1}(tQ)^k/k!$.

If $x(t)$ has a *stationary distribution*, π, then x_n also has a stationary distribution, ρ, and we have

$$\pi(dx)q(x) = q\rho(dx), \quad q := \int_E \pi(dx)q(x).$$

Let us consider the *counting process*

$$\nu(t) = \max\{n : \tau_n \leq t\}, \tag{1.13}$$

with $\max \emptyset = 0$. That gives the number of jumps of the Markov process in $(0, t]$.

▷ **Example 1.1.** The generator $\mathbb{L}$ of a Poisson process, with intensity $\lambda > 0$, is

$$\mathbb{L}\varphi(u) = \lambda[\varphi(u+1) - \varphi(u)], \quad u \in \mathbb{N}.$$

▷ **Example 1.2.** Let $\tau_n, n \geq 0$, be a renewal process on $\mathbb{R}_+$, $\tau_0 = 0$, with distribution function F, and hazard rate of inter-arrival times $\theta_n := \tau_n - \tau_{n-1}$,

$$\lambda(t) := -\frac{\overline{F}'(t)}{\overline{F}(t)}.$$

Let $\nu(t), t \geq 0$ be the corresponding counting process, that is $\nu(t) := \sup\{n : \tau_n \leq t\}$.

The generator of the Markov process $x(t) := t - \tau(t), t \geq 0$, $\tau(t) := \tau_{\nu(t)}$, is given by

$$\mathbb{L}\varphi(x) = \varphi'(x) + \lambda(x)[\varphi(0) - \varphi(x)], \quad u \in \mathbb{N},\ t \in \mathbb{R}_+,$$

where $\overline{F}(t) := 1 - F(t)$. The domain of this generator is $\mathcal{D}(\mathbb{L}) = C^1(\mathbb{R})$.

▷ **Example 1.3.** Let $x(t), t \geq 0$, be a nonhomogeneous jump Markov process, the generator of the coupled Markov process $t, x(t), t \geq 0$, is defined as follows

$$\mathbb{L}\varphi(t,x) = \frac{\partial}{\partial t}\varphi(t,x) + Q\varphi(\cdot,x),$$

with $\mathcal{D}(\mathbb{L}) = C^{1,0}(\mathbb{R}_+ \times E)$.

▷ **Example 1.4.** Let $x(t), t \geq 0$, be a pure jump Markov process (that is, without drift and diffusion part) with state space E and generator Q, and let $\nu(t), t \geq 0$, be the corresponding counting process of jumps and $x_n, n \geq 0$ the embedded Markov chain. Let a be a real-valued measurable function on the state space E, and consider the *increment process*

$$\alpha(t) = \sum_{k=1}^{\nu(t)} a(x_k), \quad t \geq 0.$$

Then the generator of the coupled Markov process $\alpha(t), x(t), t \geq 0$ is

$$\mathbb{L} = Q + Q_0[\mathbb{\Gamma}(x) - I], \tag{1.14}$$

where:

$$\mathbb{\Gamma}(x)\varphi(u) := \varphi(u + a(x))$$

$$Q_0\varphi(x) := q(x)\int_E P(x,dy)\varphi(y),$$

and I is the identity operator.

▷ **Example 1.5.** For the jump Markov process $x(t), t \geq 0$, as in the previous example, let us consider the process

$$\xi(t) := \int_0^t a(x(s))ds.$$

Then the generator of the coupled process $\xi(t), x(t), t \geq 0$ is

$$\mathbb{L} = Q + \mathbb{A}(x),$$

where $\mathbb{A}(x)\varphi(u) := a(x)\varphi'(u)$.

1.2.3 *Diffusion Processes*

A diffusion process is a strong Markov process in continuous time with almost surely continuous paths. This process is used in order to describe mathematically the physical phenomenon of diffusion, that is the movement of particles in a chaotic environment. In this book we will use diffusions as limit processes of stochastic systems switched by Markov and semi-Markov processes.

Definition 1.14 A Markov nonhomogeneous in time process $x(t), t \geq 0$, defined on a stochastic basis $\Im$, with values in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, with transition function $P_{s,t}(x, B) := \mathbb{P}(x(t) \in B \mid x(s) = x)$, for $0 \leq s < t < \infty$, $x \in \mathbb{R}^d, B \in \mathcal{B}_d$, is said to be a *diffusion process*, with generator $\mathbb{L}_t$, if (a) it has continuous paths, and (b) for any $x \in \mathbb{R}^d$, and any $\varphi \in C^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} P_{t,t+h}(x, dy)[\varphi(y) - \varphi(x)] = h\mathbb{L}_t\varphi(x) + o(h), \quad h \to 0.$$

Let a be a real-valued measurable function, defined on $\mathbb{R}^d \times \mathbb{R}_+$, and B a function defined on $\mathbb{R}^d \times \mathbb{R}_+$ with values in the space of symmetric positive operators from $\mathbb{R}^d$ to $\mathbb{R}^d$.

For any $t \geq 0$, the generator $\mathbb{L}_t$, of the diffusion $x(t), t \geq 0$, acts on functions φ in $C^2(\mathbb{R}^d)$ as follows

$$\mathbb{L}_t\varphi(x) = a(x,t)\varphi'(x) + \frac{1}{2}B(x,t)\varphi''(x). \tag{1.15}$$

The above means that the *drift coefficient* $a(x,t)$, and the *diffusion operator* (or coefficient) $B(x,t)$, apply as follows:

$$a(x,t)\varphi'(x) = \sum_{i=1}^{d} a_i(x,t)\frac{\partial}{\partial x_i}\varphi(x),$$

and

$$B(x,t)\varphi''(x) = \sum_{i,j=1}^{d} b_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x).$$

For a *time-homogeneous diffusion* the generator does not depend on t, that means functions a and B are free of t.

If $a(x,t) \equiv 0$, and $B(x,t) = I$, the $x(t), t \geq 0$, is a *Wiener process* or a *Brownian motion.*

A diffusion process with a constant shift vector and a diffusion operator B has the following representation

$$x(t) = x(0) + ta + \sigma w(t),$$

where $w(t), t \geq 0$, is a Wiener process, with $\mathbb{E}w(t) = 0$, $\mathbb{E}(xw(t))^2 = t\,\|x\|^2$, $x \in \mathbb{R}^d$, and $\sigma := B^{1/2}$ $(B = \sigma\sigma^*)$ is the positive symmetric square root of the matrix B.

For the existence and weak uniqueness of a diffusion with generator (1.15), we suppose that (see, e.g. [56,140,151,165])

(a) $a(x,t)$ and $B(x,t)$ are continuous, and

(b) $\|a(x,t)\|+\|B(x,t)\| \leq C(1+\|x\|)$, where C is some positive constant.

Weak uniqueness here means uniqueness of the transition function.

1.2.4 *Processes with Independent Increments*

Let $\Im$ be a stochastic basis, and $x(t), t \geq 0$, a stochastic process adapted to $\Im$ with values in $\mathbb{R}^d$.

Definition 1.15 The process $x(t), t \geq 0$, is said to be with *independent increments* (PII), if for any $s, t \in \mathbb{R}_+$, with $s < t$, $x(t) - x(s)$ is independent of $\mathcal{F}_s$.

This property is equivalent to the increment independence property of the process $x(t)$, when the filtration $(\mathcal{F}_t)$ is the natural filtration of the process. That is, for any $n \geq 1$, and any $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the random variables $x(t_0), x(t_1) - x(t_0), ..., x(t_n) - x(t_{n-1})$ are independent.

If, moreover, $x(t)$ has *stationary increments*, that is, the law of $x(t) - x(s)$ depends only on $t - s$, then the process $x(t)$ is said to be a *process with stationary independent increments* (PSII).

The main properties of PSII are that their distributions are infinitely divisible and have the Markov property. Let

$$\phi_t(\lambda) := \mathbb{E}\exp(i\lambda(x(t+s) - x(s)),$$

be the *characteristic function* of increments. Then it satisfies the semigroup property

$$\phi_{t+s}(\lambda) = \phi_t(\lambda)\phi_s(\lambda).$$

The characteristic function has the following natural representation

$$\phi_t(\lambda) = \exp[t\psi(\lambda)].$$

The *cumulant* $\psi(\lambda)$, has the well known *Lévy-Khintchine formula* [13,164]

$$\psi(\lambda) = i\lambda a - \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} [e^{i\lambda z} - 1 - i\lambda z \mathbf{1}_{\{|z|\leq 1\}}]H(dz), \qquad (1.16)$$

where H is the *spectral measure* and satisfies the following conditions

$$\int_{|z|\leq 1} z^2 H(dz) < \infty, \qquad \int_{|z|>1} H(dz) < \infty.$$

In the case where the PSII process $x(t)$ is of finite variation (that is, it has trajectories of finite variation), the cumulant has the following form

$$\psi(\lambda) = i\lambda a + \int_{\mathbb{R}} [e^{i\lambda z} - 1]H(dz),$$

and the spectral measure satisfies the condition

$$\int_{|z|\leq 1} |z|\, H(dz) < \infty.$$

The most important PSII processes are Brownian motion, Poisson process, and Lévy process. [13]

▷ **Example 1.6.** The *Compound Poisson Process* is defined by

$$x(t) = \sum_{k=1}^{\nu(t)} \xi_k,$$

where $\nu(t), t \geq 0$, is a homogeneous Poisson process with intensity $\lambda > 0$, and $\xi_k, k \geq 1$, is an i.i.d. sequence of real random variables, independent of $\nu(t), t \geq 0$, with common distribution function F. Then we have $F(A) = H(A)/\lambda = \mathbb{P}(\xi_k \in A)$. And the cumulant of the compound Poisson process has the form

$$\psi(\lambda) = \lambda \int_{\mathbb{R}} [e^{i\lambda z} - 1]F(dz). \qquad (1.17)$$

As stated above, the PSII satisfy the Markov property, that is, the transition probabilities are generated by the Markov semigroup

$$\Gamma_t\varphi(u) := \mathbb{E}\varphi(u + x(t)). \qquad (1.18)$$

Lemma 1.1 *([163]) The generator* $\mathbb{\Gamma}$ *of semigroup (1.18) has the following representation*

$$\mathbb{\Gamma}\varphi(u) = \int_{\mathbb{R}} e^{i\lambda u}\psi(\lambda)\tilde{\varphi}(\lambda)d\lambda, \tag{1.19}$$

for $\varphi(u) = \int_{\mathbb{R}} e^{i\lambda u}\tilde{\varphi}(\lambda)d\lambda$*, where* $\tilde{\varphi}(\lambda)$ *and* $\lambda^2\tilde{\varphi}(\lambda)$ *are integrable functions.*

Proof. Let us consider the semigroup (1.18)

$$\begin{aligned}\Gamma_t\varphi(u) &= \mathbb{E}\varphi(u + x(t)) = \mathbb{E}\int_{\mathbb{R}} e^{i\lambda(u+x(t))}\tilde{\varphi}(\lambda)d\lambda \\ &= \int_{\mathbb{R}} e^{i\lambda u + t\psi(\lambda)}\tilde{\varphi}(\lambda)d\lambda.\end{aligned}$$

Note that, according to the Lévy-Khintchine formula, the cumulant has the asymptotic form $\psi(\lambda) = O(\lambda^2)$, as $|\lambda| \to \infty$. Hence, the latter integral is convergent, uniformly on t. So, we get the derivative

$$\frac{d}{dt}\Gamma_t\varphi(u) = \int_{\mathbb{R}} e^{i\lambda u + t\psi(\lambda)}\psi(\lambda)\tilde{\varphi}(\lambda)d\lambda.$$

By the evolutionary equation for the semigroup, we have

$$\frac{d}{dt}\Gamma_t\varphi(u) = \Gamma_t\mathbb{\Gamma}\varphi(u).$$

Comparing the two latter formulas, we get (1.19). □

The meaning of representation (1.19) is that the cumulant $\psi(\lambda)$ of a PSII is the symbol of the generator $\mathbb{\Gamma}$. In the particular case of a drift process $x(t) = at$, the corresponding generator is $\mathbb{\Gamma}\varphi(u) = a\varphi'(u)$ since for the corresponding semigroup

$$\frac{d}{dt}\Gamma_t\varphi(u) = \int_{\mathbb{R}} e^{i\lambda u} i\lambda a\tilde{\varphi}(\lambda)d\lambda.$$

It is well-known that the standard Wiener process with the cumulant $\psi(\lambda) = -\sigma^2\lambda^2/2$ has the generator $\mathbb{\Gamma}\varphi(u) = \sigma^2\varphi''(u)/2$.

Corollary 1.1 *The generator of the semigroup (1.18) with the cumulant (1.16) has the following representation*

$$\mathbb{\Gamma}\varphi(u) = a\varphi'(u) - \frac{\sigma^2}{2}\varphi''(u) + \int_{\mathbb{R}}[\varphi(u+v) - \varphi(u) - v\varphi'(u)\mathbf{1}_{(|v|\le 1)}]H(dv).$$

In the particular case of a compound Poisson process with the cumulant (1.17) the generator has the form

$$\mathbb{\Gamma}\varphi(u) = \lambda \int_{\mathbb{R}} [\varphi(u+v) - \varphi(u)] F(dv).$$

1.2.5 *Processes with Locally Independent Increments*

We consider now the Markov processes with *locally independent increments* (PLII). It is worth noticing that such processes include strictly the independent increment processes. Here we will restrict our interest to PLII without diffusion part. These processes are called also "Piecewise–deterministic Markov processes" [34], or "jump Markov process with drift", or "weak differentiable Markov processes" and have the same local structure as the PII (see [56]).

Roughly speaking, these processes are jump Markov processes with drift and without diffusion part. These processes are of increasing interest in the literature because of their importance in applications, for which they constitute an alternative to diffusion processes. For their detailed presentation and applications see [34].

These processes are defined by the generator $\mathbb{\Gamma}$ as follows

$$\mathbb{\Gamma}\varphi(u) = a(u)\varphi'(u) + \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u)] \Gamma(u, dv), \tag{1.20}$$

with the intensity kernel $\Gamma(u, dv)$ satisfying the boundedness property: $\Gamma(u, \mathbb{R}^d) \in \mathbb{R}_+$.

We can also write the above generator in the following form, by extracting the drift, due to the jump part, and add it into the initial drift a to obtain the drift coefficient g, that is, $g(u) := a(u) + \int_{\mathbb{R}^d} v\Gamma(u, dv)$,

$$\mathbb{\Gamma}\varphi(u) = g(u)\varphi'(u) + \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u) - v\varphi'(u)] \Gamma(u, dv). \tag{1.21}$$

Let us be given the Euclidean space $\mathbb{R}^d$ with the Borel σ-algebra $\mathcal{B}_d$ and the compact measurable space $(E, \mathcal{E})$. We consider the family of time-homogeneous Markov processes $\eta(t; x)$, $t \geq 0$, $x \in E$, with trajectories in $\mathbf{D}[0, \infty)$, with locally independent increments. These processes take values in the Euclidean space $\mathbb{R}^d$ $(d \geq 1)$, and depend on the state $x \in E$, and their generators are given by

$$\mathbb{\Gamma}(x)\varphi(u)=a(u;x)\varphi'(u)+\int_{\mathbb{R}^d}[\varphi(u+v)-\varphi(u)\ -v\varphi'(u)]\Gamma(u,dv;x).\quad(1.22)$$

These processes will be used in Chapter 2 as switched processes. A complete characterization of the above generators is given in [34]. Note also that $\eta(t;x)$ contains no diffusion part (see, e.g. [34,51,56,107]).

Of course, it is understood that when $d > 1$, we have

$$v\varphi'(u) = \sum_{k=1}^{d} v_k \frac{\partial\varphi}{\partial u_k}(u).$$

It is worth noting that slightly changed conditions allow one to include a locally compact space of values for the switched process.

The drift velocity $a(u;x)$ and the measure of the random jumps $\Gamma(u,dv;x)$ depend on the state $x \in E$. The time-homogeneous cadlag jump Markov process $x(t)$, $t \geq 0$, taking values in the state space $(E,\mathcal{E})$, is given by its generators (1.12)

$$Q\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y)-\varphi(x)],$$

where q is the intensity of jumps, which is a nonnegative element of the Banach space $\mathbf{B}(E)$ of real bounded functions defined on the state space E, with the sup–norm, that is $\|\varphi\| := \sup_{x\in E}|\varphi(x)|$.

We will consider additive functionals of the process $\eta(t;x)$, of the following form

$$\xi(t) = \xi_0 + \int_0^t \eta(ds;x(s)) = \xi_0 + \sum_{k=1}^{\nu(t)-1} \eta(\theta_k;x_k) + \eta(t-\tau(t);x(t)),$$

with generator $\mathbb{L} = Q + \mathbb{\Gamma}(x)$.

1.2.6 *Martingale Characterization of Markov Processes*

Let $\Im = (\Omega,\mathcal{F},\mathbf{F} = (\mathcal{F}_t,\ t\geq 0),\mathbb{P})$ be a stochastic basis. Let $z(t), t \geq 0$, be a process, defined on $\Im$ and adapted to $\mathbf{F}$.

Definition 1.16 (see, e.g. [70,132,93]) A real-valued process $z(t), t \geq 0$, adapted to the filtration $\mathbf{F}$, is called an $\mathbf{F}$-*martingale* (*submartingale*, *su-*

permartingale), if $\mathbb{E}\,|z(t)| < \infty$, for all $t \geq 0$, and, for $s < t$,

$$\mathbb{E}[z(t) \mid \mathcal{F}_s] = z(s), \quad (\mathbb{E}[z(t) \mid \mathcal{F}_s] \geq z(s),\ \mathbb{E}[z(t) \mid \mathcal{F}_s] \leq z(s)), \quad \text{a.s.} \tag{1.23}$$

A martingale $\mu_t, t \geq 0$, is called *square integrable*, if

$$\sup_{t \geq 0} \mathbb{E}\mu_t^2 < \infty. \tag{1.24}$$

Then the process $\mu_t^2, t \geq 0$, is an $\mathcal{F}_t$-submartingale.

The *Doob-Meyer decomposition* [132] of a square integrable martingale $\mu_t, t \geq 0$, is as follows

$$\mu_t^2 = \langle \mu \rangle_t + z(t),$$

where $z(t), t \geq 0$, is a martingale, and the increasing process $\langle \mu \rangle_t, t \geq 0$, is called the *square characteristic* of the martingale μ_t.

Definition 1.17 (see, e.g. [132]) A process $z(t), t \geq 0$, adapted to $\mathbf{F}$ is called an $\mathbf{F}$-*local martingale* if there exists an increasing to infinity sequence of *stopping times*, $\tau_n \to +\infty$, such that the stopped process $z^n(t) = z(t \wedge \tau_n), t \geq 0$, is a martingale for each $n \geq 1$.

It is clear that any martingale is a local martingale, by setting $\tau_n = n$, $n \geq 1$. The converse is false.

Let $x(t), t \geq 0$, be a Markov process with a standard state space $(E, \mathcal{E})$, defined on a stochastic basis $\Im$. Let $P_t(x, B), x \in E, B \in \mathcal{E}, t \geq 0$, be its transition function, and $P_t, t \geq 0$, its strongly continuous semigroup defined on the Banach space $\mathbf{B}$ of real-valued measurable functions defined on E, with the sup-norm, (see Definition 1.10). Let Q be the generator of the semigroup $P_t, t \geq 0$, with the dense domain of definition $\mathcal{D}(Q) \subset \mathbf{B}$.

For any function $\varphi \in \mathcal{D}(Q)$ and any $t > 0$, we have the *Dynkin formula* [45,34]

$$P_t\varphi(x) = \varphi(x) + \int_0^t QP_s\varphi(x)ds. \tag{1.25}$$

From this formula, using conditional expectation, we get

$$\mathbb{E}_x\left[\varphi(x(t)) - \varphi(x) - \int_0^t Q\varphi(x(s))ds\right] = 0.$$

Thus, the process

$$\mu(t) := \varphi(x(t)) - \varphi(x) - \int_0^t Q\varphi(x(s))ds \tag{1.26}$$

is an $\mathcal{F}_t^x = \sigma(x(s), s \leq t)$-martingale.

The following theorem gives the martingale characterization of Markov processes.

Theorem 1.1 *([45]) Let $(E, \mathcal{E})$ be a standard state space and let $x(t), t \geq 0$, be a stochastic process on it, adapted to the filtration $\mathbf{F} = (\mathcal{F}_t,\ t \geq 0)$. Let Q be the generator of a strongly continuous semigroup $P_t, t \geq 0$, on the Banach space $\mathbf{B}$, with dense domain $\mathcal{D}(Q) \subset \mathbf{B}$. If for any $\varphi \in \mathcal{D}(Q)$, the process $\mu(t), t \geq 0$, defined by (1.26) is an $\mathcal{F}_t$-martingale, then $x(t), t \geq 0$, is a Markov process generated by the infinitesimal generator Q.*

The process $x(t), t \geq 0$, is said to solve the *martingale problem for the generator Q.*

The martingale (1.26) is a square integrable one whose the square integrable characteristic is the process:

Theorem 1.2 *([132]) The square characteristic of the martingale $\mu(t), t \geq 0$, (see 1.26), denoted by $\langle\mu\rangle_t, t \geq 0$, is the process*

$$\langle\mu\rangle_t = \int_0^t [Q\varphi^2(x(s)) - 2\varphi(x(s))Q\varphi(x(s))]ds, \quad t \geq 0.$$

PROOF. Let us denote $\alpha_t := \int_0^t Q\varphi(x(s))ds$. Then, from the representation of the martingale $\mu(t)$, we have

$$\varphi^2 = (\mu + \alpha)^2 = \mu^2 + 2\mu\alpha + \alpha^2 = \mu^2 + L,$$

where $L := 2\mu\alpha + \alpha^2$. Differentiating L, we get

$$dL = 2d\mu\alpha + 2\mu Q\varphi ds + 2\alpha Q\varphi ds.$$

Now the martingale representation $\mu = \varphi - \alpha$, gives

$$dL = 2d\mu\alpha + 2\varphi Q\varphi ds,$$

and, by integration,

$$L = 2\int_0^t d\mu(s)\int_0^s Q\varphi dv + 2\int_0^t \varphi Q\varphi ds.$$

The first term is a martingale, since it is the integral with respect to a martingale $\mu(s)$. It is obvious that

$$\mu_1(t) := \varphi^2(x(t)) - \int_0^t Q\varphi^2(x(s))ds,$$

is a martingale. Hence

$$\begin{aligned}\mu^2 &= \varphi^2 - L \\ &= \varphi^2 - \int_0^t Q\varphi^2 ds + \int_0^t Q\varphi^2 ds - 2\int_0^t \varphi Q\varphi ds - \mu_1 \\ &= \int_0^t [Q\varphi^2 - 2\varphi Q\varphi]ds + \mu_2,\end{aligned}$$

where μ_2 is a martingale. So, the latter relation gives the square characteristic of the martingale $\mu(t)$. □

Let $x_n, n \geq 0$, be a Markov chain on a measurable state space $(E, \mathcal{E})$ induced by a stochastic kernel $P(x, B)$, $x \in E, B \in \mathcal{E}$. Let P be the corresponding transition operator defined on the Banach space $\mathbf{B}$.

Let us construct now the following martingale as a sum of martingale differences

$$\mu_n = \sum_{k=0}^{n-1}[\varphi(x_{k+1}) - \mathbb{E}(\varphi(x_{k+1}) \mid \mathcal{F}_k)]. \tag{1.27}$$

By using the Markov property and the rearrangement of terms in (1.27) the martingale takes the form

$$\mu_n = \varphi(x_n) - \varphi(x_0) - \sum_{k=1}^{n-1}[P - I]\varphi(x_k). \tag{1.28}$$

This representation of the martingale is associated with a Markov chain characterization.

Lemma 1.2 *Let $x_n, n \geq 0$, be a sequence of random variables taking values in a measurable space $(E, \mathcal{E})$ and adapted to the filtration $\mathcal{F}_n, n \geq 0$. Let P be a bounded linear positive operator on the Banach space $\mathbf{B}$ induced by a transition probability kernel $P(x, B)$ on $(E, \mathcal{E})$. If for every $\varphi \in \mathbf{B}$, the right hand side of (1.28) is a martingale $\mu_n, \mathcal{F}_n, n \geq 0$, then the sequence $x_n, n \geq 0$, is a Markov chain with transition probability kernel $P(x, B)$ induced by the operator P.*

PROOF. Using (1.28) we have

$$\begin{aligned}\mathbb{E}[\mu_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}[\varphi(x_n) \mid \mathcal{F}_{n-1}] - \varphi(x_0) - \sum_{k=1}^{n-1}[P - I]\varphi(x_k) \\ &= \mathbb{E}[\varphi(x_n) \mid \mathcal{F}_{n-1}] - P\varphi(x_{k-1}) + \varphi(x_{n-1}) - \varphi(x_0) \\ &= \mu_{n-1} + \mathbb{E}[\varphi(x_n) \mid \mathcal{F}_{n-1}] - P\varphi(x_{k-1}).\end{aligned}$$

So, the martingale property $\mathbb{E}[\mu_n \mid \mathcal{F}_{n-1}] = \mu_{n-1}$, is equivalent to the Markov property

$$\mathbb{E}[\varphi(x_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[\varphi(x_{n+1}) \mid x_n] = P\varphi(x_n). \tag{1.29}$$

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By the definition of square characteristic of martingale it is easy to check that

$$\langle \mu \rangle_n = \sum_{k=0}^{n-1}[P\varphi^2(x_k) - (P\varphi(x_k))^2]. \tag{1.30}$$

1.3 Semi-Markov Processes

The semi-Markov process is a generalization of the Markov and renewal processes. We will present shortly definitions and basic properties of semi-Markov process useful in the sequel of the book (see, e.g. [127,116]).

1.3.1 *Markov Renewal Processes*

Definition 1.18 A positive-valued function $Q(x, B, t)$, $x \in E$, $B \in \mathcal{E}$, $t \in \mathbb{R}_+$, is called a *semi-Markov kernel* on $(E, \mathcal{E})$ if

(i) $Q(x, B, \cdot)$, for $x \in E$, $B \in \mathcal{E}$, is a non-decreasing, right continuous real function, such that $Q(x, B, 0) = 0$;
(ii) $Q(\cdot, \cdot, t)$, for any $t \in \mathbb{R}_+$, is a sub-Markov kernel on $(E, \mathcal{E})$;
(iii) $P(\cdot, \cdot) = Q(\cdot, \cdot, \infty)$ is a Markov kernel on $(E, \mathcal{E})$.

For any fixed $x \in E$, the function $F_x(t) := Q(x, E, t)$ is a distribution function on $\mathbb{R}_+$. By Radon-Nikodym theorem, as $Q \ll P$ there exists a positive-valued function $F(x, y, \cdot)$, such that

$$Q(x, B, t) = \int_B F(x, y, t) P(x, dy), \quad B \in \mathcal{E}. \tag{1.31}$$

We consider a special class of semi-Markov processes where $F(x, y, t)$ does not depend on the second argument y, we have $F(x, y, t) =: F_x(t)$. Nevertheless, any semi-Markov process can be transformed in one of the above kind, (see, e.g. [127]), by representing the semi-Markov kernel Q as follows

$$Q(x, B, t) = P(x, B)F_x(t). \tag{1.32}$$

Let us consider a $(E \times \mathbb{R}_+, \mathcal{E} \otimes \mathcal{B}_+)$-valued stochastic process $(x_n, \tau_n;\ n \geq 0)$, with $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \tau_{n+1} \leq \cdots$.

Definition 1.19 A *Markov renewal process* is a two component Markov chain, $x_n, \tau_n,\ n \geq 0$, *homogeneous with respect to the second component* with transition probability defined by a semi-Markov kernel Q as follows,

$$\begin{aligned}\mathbb{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n \leq t \mid \mathcal{F}_n) &= \mathbb{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n \leq t \mid x_n) \\ &= Q(x_n, B, t) \quad (a.s.),\end{aligned} \tag{1.33}$$

for any $n \geq 0,\ t \geq 0$, and $B \in \mathcal{B}_+$.

Let us define the counting process of jumps $\nu(t),\ t \geq 0$, by

$$\nu(t) = \sup\{n \geq 0 : \tau_n \leq t\},$$

that gives the number of jumps of the Markov renewal process in the time interval $(0, t]$.

Definition 1.20 A stochastic process $x(t),\ t \geq 0$, defined by the following relation

$$x(t) = x_{\nu(t)}, \quad t \geq 0,$$

is called a *semi-Markov process*, associated to the Markov renewal process $x_n, \tau_n,\ n \geq 0$.

Remark 1.1. Markov jump processes are special cases of semi-Markov processes with semi-Markov kernel

$$Q(x, B, t) = P(x, B)[1 - e^{-q(x)t}].$$

Let $\theta_n := \tau_n - \tau_{n-1}$, that is $\tau_n = \tau_0 + \sum_{k=1}^{n} \theta_k$. The random variable $\theta_x, x \in E$, will denote the *sojourn time* in state x. The process $x_n, \theta_n, n \geq 0$, will be called also a *Markov renewal process.*

It is worth noticing that jump Markov and semi-Markov processes consider in this book are *regular*, that is [127,56],

$$\mathbb{P}(\nu(t) < \infty) = 1,$$

for every $t \geq 0$.

1.3.2 *Markov Renewal Equation and Theorem*

Let Q_1 and Q_2 be two semi-Markov kernels on $(E, \mathcal{E})$. Then their convolution, denoted by $Q_1 \star Q_2$, is defined by

$$(Q_1 \star Q_2)(x, B, t) = \int_E \int_0^t Q_1(x, dy, ds) Q_2(y, B, t - s), \qquad (1.34)$$

where $x \in E$, $t \in \mathbb{R}_+$, $B \in \mathcal{E}$.

The function $Q_1 \star Q_2$ is also a semi-Markov kernel. For a semi-Markov kernel Q on $(E, \mathcal{E})$, we set by induction

$$Q^1 = Q, \quad Q^{n+1} = Q \star Q^n, \quad n \geq 0, \qquad (1.35)$$

and

$$Q^0(x, B, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \mathbf{1}_B(x) & \text{if } t > 0. \end{cases} \qquad (1.36)$$

We prove easily that

$$Q^{m+n} = Q^m \star Q^n. \qquad (1.37)$$

Note that

$$Q^n(x, B, t) = \mathbb{P}(x_n \in B, \tau_n \leq t \mid x_0 = x).$$

Let us now consider two real-valued functions $U(x, t)$ and $V(x, t)$ defined on $E \times \mathbb{R}_+$.

Definition 1.21 The *Markov renewal equation* is defined as follows

$$U(x, t) - \int_E \int_0^t Q(x, dy, ds) U(y, t - s) = V(x, t), \quad x \in E, \qquad (1.38)$$

where U is an unknown function and V a given function.

The above Markov renewal equation can be also written as follows

$$U(t) - \int_0^t Q(ds) U(t - s) = V(t),$$

which is the usual form in the scalar case of the classical renewal equation on the half-real line $t \geq 0$.

By using convolution $\star$, this equation can be written as follows

$$[I - Q] \star U = V \quad \text{or} \quad U = V + Q \star U,$$

where I is the identity operator: $I \star U = U \star I = U$.

Theorem 1.3 *(Markov renewal theorem [159]) Let the following conditions hold:*

C1: *the stochastic kernel $P(x, B) = Q(x, B, \infty)$ induces an irreducible ergodic Markov chain with the stationary distribution ρ,*

C2: *the mean sojourn times are uniformly bounded, that is:*

$$m(x) := \int_0^\infty \overline{F}_x(t)dt \leq C < +\infty,$$

and

$$m := \int_E \rho(dx)m(x) > 0,$$

C3: *the distribution functions $F_x(t) := Q(x, E, t)$, $x \in E$, are non arithmetic (that is, not concentrated on a set $\{na : n \in \mathbb{N}\}$, where $a > 0$, is a constant; the largest a is called the span of distribution).*

C4: *the nonnegative function $V(x, t)$ is direct Riemann integrable[47] on $\mathbb{R}_+$, so*

$$\int_E \rho(dx) \int_0^\infty V(x, t)dt < +\infty.$$

Then Equation (1.38) has a unique solution $U(x, t)$, and the following limit result holds

$$\lim_{t\to\infty} U(x, t) = \int_E \rho(dx) \int_0^\infty V(x, t)dt/m.$$

Let us apply the above theorem in order to obtain the limit distribution of the semi-Markov process.

The transition probabilities of the semi-Markov process

$$P_t(x, B) = \mathbb{P}(x(t) \in B \mid x(0) = x), \tag{1.39}$$

satisfy the following Markov renewal equation

$$P_t(x, B) - \int_E \int_0^t Q(x, dy, ds)P_{t-s}(y, B) = \mathbf{1}_B(x)\overline{F}_x(t). \tag{1.40}$$

Now, applying the renewal limit theorem to the above equation, we get

$$\pi(B) := \lim_{t\to\infty} P_t(x,B) = \int_B \rho(dx)m(x)/m, \quad B \in \mathcal{E}.$$

The *limit distribution* $\pi(B)$ is said to be the *stationary distribution* of the semi-Markov process $x(t)$.

1.3.3 *Auxiliary Processes*

The following auxiliary processes will be used:

$$\begin{aligned} &\tau(t) = \tau_{\nu(t)}, \quad \tau_+(t) = \tau_{\nu_+(t)}, \quad \nu_+(t) = \nu(t) + 1, \\ &\gamma(t) = t - \tau(t), \quad \gamma_+(t) = \tau_+(t) - t. \end{aligned} \tag{1.41}$$

The distributions of the auxiliary processes (1.41) satisfy the Markov renewal equation (1.38) with certain function $V(x,t)$. Let us consider the distribution function of the remaining sojourn time $\gamma_+(t)$, which will be used in the phase merging principle (see Chapter 4),

$$\Phi_x(u,t) := \mathbb{P}(\gamma_+(t) \leq u \mid x(0) = x).$$

The right-hand side of the Markov renewal equation (1.38) is calculated as follows:

$$\begin{aligned} V_x(u,t) : &= \mathbb{E}[\mathbf{1}_{(\gamma_+(t)\leq u, \tau_1 > t)} \mid x(0) = x] \\ &= \mathbb{E}[\mathbf{1}_{(\tau_1 - t \leq u, \tau_1 > t)} \mid x(0) = x] \\ &= \mathbb{E}[\mathbf{1}_{(t < \tau_1 \leq t+u)} \mid x(0) = x] \\ &= F_x(t+u) - F_x(t) \\ &= \overline{F}_x(t) - \overline{F}_x(t+u). \end{aligned}$$

Now, the Markov renewal theorem yields the following limit result:

$$\begin{aligned} \lim_{t\to\infty} \Phi_x(u,t) &= \int_E \rho(dx) \int_0^\infty [\overline{F}_x(t) - \overline{F}_x(t+u)]dt/m \\ &= \int_E \rho(dx) \left[\int_0^\infty \overline{F}_x(t)dt - \int_u^\infty \overline{F}_x(t)dt \right] /m \\ &= \int_E \rho(dx) \int_0^u \overline{F}_x(t)dt/m \\ &= \int_0^u \overline{F}(t)dt/m, \end{aligned}$$

where, by definition,

$$\overline{F}(t) := \int_E \rho(dx)\overline{F}_x(t).$$

It is worth noticing that the limit remaining sojourn time γ_+ distribution, naturally called the *stationary renewal time distribution*, is defined by

$$F_+(u) := \mathbb{P}(\gamma_+ \leq u) = \int_0^u \overline{F}(t)dt/m,$$

with density

$$f_+(u) = \overline{F}(u)/m.$$

1.3.4 *Compensating Operators*

The compensating operator is a basic important device in our analysis of stochastic systems switched by semi-Markov processes.

Let us consider a Markov renewal process

$$x_n, \quad \tau_n, \quad n \geq 0, \tag{1.42}$$

where $x_n = x(\tau_n)$, and $\tau_{n+1} = \tau_n + \theta_{n+1}$, $n \geq 0$, and

$$\mathbb{P}(\theta_{n+1} \leq t \mid x_n = x) = F_x(t) = \mathbb{P}(\theta_x \leq t).$$

Definition 1.22 (^{174}) The *compensating operator* $\mathbb{L}$ of the Markov renewal process (1.42) is defined by the following relation

$$\mathbb{L}\varphi(x_0, \tau_0) = q(x_0)\mathbb{E}[\varphi(x_1, \tau_1) - \varphi(x_0, \tau_0) \mid \mathcal{F}_0], \tag{1.43}$$

where $q(x) = 1/m(x)$, $m(x) = \mathbb{E}\theta_x = \int_0^\infty \overline{F}_x(t)dt$, and

$$\mathcal{F}_t := \sigma(x(s), \tau(s);\ 0 \leq s \leq t).$$

Of course, by the homogeneity property of the Markov renewal process, we have

$$q(x)\mathbb{E}[\varphi(x_{n+1}, \tau_{n+1}) - \varphi(x_n, \tau_n) \mid \mathcal{F}_n] = \mathbb{L}\varphi(x_n, \tau_n). \tag{1.44}$$

Proposition 1.3 *The compensating operator (1.43) of the Markov renewal process (1.42) can be defined by the relation*

$$\begin{aligned}\mathbb{L}\varphi(x,t) &= q(x)\int_0^\infty \int_E Q(x,dy,ds)\,[\varphi(y,t+s)-\varphi(x,t)] \\ &= q(x)\left[\int_0^\infty F_x(ds)\int_E P(x,dy)\varphi(y,t+s)-\varphi(x,t)\right]. \quad (1.45)\end{aligned}$$

If $q(x)=0$, *then* $\mathbb{L}\varphi(x,t)=0$.

The claim of Proposition 1.3 follows directly from Definition 1.22.

1.3.5 *Martingale Characterization of Markov Renewal Processes*

Let $x(t), t \geq 0$, be a semi-Markov process and let $x_n, \tau_n, n \geq 0$, be the corresponding Markov renewal process, and $\mathbb{L}$ be the compensating operator. Define the process $\xi_n, n \geq 0$, by

$$\xi_n := \varphi(x_n,\tau_n) - \sum_{i=1}^{n}(\tau_i-\tau_{i-1})\mathbb{L}\varphi(x_{i-1},\tau_{i-1}), \quad n \geq 0,$$

and $\mathcal{G}_n := \sigma(x_k, \tau_k; k \leq n), n \geq 0$.

Proposition 1.4 *([174]) The process* $\xi_n, n \geq 0$, *is a* $\mathcal{G}_n$*-martingale sequence for any function* $\varphi \in \mathbf{B}(E \times \mathbb{R}_+)$, *such that* $\mathbb{E}_x\,|\varphi(x_1,\tau_1)| < +\infty$.

1.4 Semimartingales

Let us consider an adapted stochastic process $x(t)$, $t \geq 0$, on the stochastic basis $\Im$, with trajectories in the space $\mathbf{D}[0,+\infty)$. Set $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

Definition 1.23 ([70]) The stochastic process $x(t)$, $t \geq 0$, is said to be:

(1) a *semimartingale*, if it has the following representation

$$x(t) = x_0 + \mu(t) + \alpha(t), \quad t \geq 0, \tag{1.46}$$

where $x_0 = x(0)$ is a finite $\mathcal{F}_0$-measurable random variable, $\mu(t)$ is a local martingale, with $\mu_0 = 0$, and $\alpha(t)$ is a bounded variation process, with $\alpha(0)=0$;

(2) a *special semimartingale*, if it has the representation (1.46) with $\alpha(t)$ a *predictable process.*[70]

Then we note $x \in \mathcal{S}$ and $x \in \mathcal{S}_p$ respectively.

The representation (1.46), for a special semimartingale is unique. A semimartingale with bounded jumps is a special semimartingale. While the representation (1.46) for a semimartingale is not unique, the continuous martingale part is unique.

Let h be a *truncation function*, that is, $h : \mathbb{R}_0^d \to \mathbb{R}_0^d$, bounded with compact support and $h(x) = x$ in a neighborhood of 0. Let $x(t)$, $t \geq 0$, be a d-dimensional semimartingale.

For a fixed truncation function h, let us consider the processes:

$$\check{x}_h(t) = \sum_{s \leq t} [\Delta x(s) - h(\Delta x(s))] = \int_0^t \int_{\mathbb{R}_0} (u - h(u))\mu(ds, du), \quad t \geq 0$$

$$x_h(t) = x(t) - \check{x}_h(t), \quad t \geq 0,$$

where $\Delta x(s) := x(s) - x(s-)$, and $\mu(ds, du)$ is the measure of jumps of $x(t)$.

Since $\check{x}_h(t)$ is of bounded variation, the process $x_h(t)$ is a semimartingale and has bounded jumps, consequently, it is a special semimartingale, and has the canonical representation

$$x_h(t) = x_0 + M_h(t) + B_h(t), \quad t \geq 0, \tag{1.47}$$

where $M_h(t)$ is a local martingale, with $M_h(0) = 0$ and $B_h(t)$ is a predictable process of bounded variation.

Definition 1.24 ([70]) For a fixed truncation function h, we call a triple of *predictable characteristics*, with respect to h, the triplet $T = (B, C, \nu)$, of which the semimartingale $x(t)$, $t \geq 0$, has the following representation 70,132.

$$x(t) = x_0 + B_h(t) + x^c(t) + \int_0^t \int_{\mathbb{R}_0} h(u)(\mu(ds, du) - \nu(ds, du))$$
$$+ \int_0^t \int_{\mathbb{R}^*} (u - h(u))\mu(ds, du),$$

where:

- $B = B_h$, is the predictable process in (1.47),
- $C = (c^{ij})$ is a continuous process of bounded variation, with $c^{ij} = < x^{ic}, x^{jc} >$, which is the predictable process of (x^{ic}, x^{jc}), that is, the process $x^{ic}x^{jc} - < x^{ic}, x^{jc} >$ is a local martingale;

- ν is the *compensator of the measure* μ of jumps of $x(t)$, that is a predictable measure on $\mathbb{R}_+ \times \mathbb{R}_0^d$.

It is convenient to introduce the second modified characteristic $\tilde{C} = (\tilde{c}^{ij})$, by

$$(\tilde{c}^{ij}) =< M_h^i, M_h^j > .$$

We will use the semimartingales as a tool in order to establish Poisson approximation results (see Chapter 7).

▷ **Example 1.7.** *Brownian motion.* Let $w(t), t \geq 0$ be a Wiener process with $w(0) = 0$. This is a local martingale with $\langle w, w \rangle_t = \sigma^2(t)$. Its predictable characteristics are $(B, C, \nu) = (0, \sigma^2(t), 0)$.

▷ **Example 1.8.** *Gaussian process.* Let $x(t), t \geq 0$, be a Gaussian process. We have $(B, C, \nu) = (\mathbb{E}x(t), \mathbb{E}(x(t) - \mathbb{E}x(t))^2, 0)$.

▷ **Example 1.9.** *Generalized diffusion* (^{62}). Let us consider Borel functions $a \geq 0$ and b defined on $\mathbb{R}_+ \times \mathbb{R}$, and a family of transition kernels K_t, $t \geq 0$, on $(\mathbb{R}, \mathcal{B})$, satisfying the following conditions:

$$K_t(x, \{0\}) = 0,$$
$$\int_{\mathbb{R}} (1 \wedge y)^2 K_t(x, dy) < +\infty.$$

Let $x(t), t \geq 0$, be a semimartingale with predictable characteristics (B, C, ν) given by:

$$B_h(t) = \int_0^t b(s, x(s))ds,$$
$$C(t) = \int_0^t a(s, x(s))ds,$$
$$\nu(dt, dx) = K_t(x(t), dx)dt.$$

In that case, the semimartingale $x(t), t \geq 0$, is said to be a *generalized diffusion.* If $a(t,x)$, $b(t,x)$ and $K_t(x, B)$ do not depend upon t, them it is called a *time-homogeneous generalized diffusion* (compare with PLII).

For a time-homogeneous generalized diffusion $x(t), t \geq 0$, the infinitesimal generator $\mathbb{L}$ acts on functions $\varphi \in C^2(\mathbb{R})$, as follows

$$\begin{aligned}\mathbb{L}\varphi(x) &= b(x)\varphi'(x) + \frac{1}{2}a(x)\varphi''(x) \\ &\quad + \int_{\mathbb{R}} K(x, dy)[\varphi(x+y) - \varphi(x) - h(y)\varphi'(y)]. \qquad (1.48)\end{aligned}$$

The triplet (b, a, K) is called the infinitesimal characteristics of the generalized time-homogeneous diffusion.

▷ **Example 1.10.** *Processes with stationary independent increments.* For the processes with stationary independent increments given in Section 1.2.4, with cumulant function $\psi(\lambda)$, given in (1.16), we have $(B_t, C_t, \nu_t(dx)) = (at, \sigma^2 t, tH(dx))$.

1.5 Counting Markov Renewal Processes

In this section, we consider counting processes as semimartingales.

Let $x_n, \theta_n, n \geq 0$, be a Markov renewal process taking values in $E \times [0, +\infty)$, and defined by the semi-Markov kernel

$$Q(x, B, t) = P(x, B)F_x(t).$$

So, the components x_{n+1} and θ_{n+1} are conditionally independent

$$\begin{aligned}&\mathbb{P}(x_{n+1} \in B, \theta_{n+1} \leq t \mid x_n = x) \\ &= \mathbb{P}(x_{n+1} \in B \mid x_n = x)\mathbb{P}(\theta_{n+1} \leq t \mid x_n = x) \\ &= P(x, B)F_x(t).\end{aligned}$$

The renewal moments are defined by

$$\tau_n = \sum_{k=1}^{n} \theta_k, \quad n \geq 1, \quad \tau_0 = 0.$$

The counting process is defined by

$$\nu(t) = \max\{n \geq 1 : \tau_n \leq t\}.$$

Definition 1.25 (see, e.g. [21,70,133]) An integer-valued *random measure* for the Markov renewal process $x_n, \tau_n, n \geq 0$, is defined by the relation

$$\mu(dx, dt) = \sum_{n \geq 1} \delta_{(x_n, \tau_n)}(dx, dt) \mathbf{1}_{(\tau_n < +\infty)}, \tag{1.49}$$

where δ_a is the *Dirac measure* concentrated at point a.

It is worth noticing that the random measure (1.49) defines the multivariate point process $x_n, \tau_n, n \geq 0$, (see, e.g. [70]). By Theorem III.1.26 [70], there exists a unique predictable random measure $\overline{\nu}(dx, dt)$ which is the compensator of the measure $\mu(dx, dt)$, that is, for any nonnegative continuous function $w(x, t)$,

$$\int_0^t \int_E w(x, s)[\mu(dx, dt) - \overline{\nu}(dx, ds)]$$

is a local martingale (Section 1.2.6) with respect to the natural filtration

$$\mathcal{F}_t^x := \sigma(x(s), s \leq t), \quad t \geq 0, \tag{1.50}$$

of the corresponding semi-Markov process $x(t), t \geq 0$.

Certainly, there exists a unique predictable random measure $\overline{\nu}(t)$ which is a compensator of the counting process $\nu(t), t \geq 0$, that is, the process $\mu(t) = \nu(t) - \overline{\nu}(t), t \geq 0$, is a local martingale with respect to the filtration (1.50). Note that

$$\overline{\nu}(dt) = \int_E \overline{\nu}(dx, dt). \tag{1.51}$$

Moreover, it is possible to give the constructive representation of the compensator (1.51) for the Markov renewal process, at any rate, where the family of distributions $F_x(t), x \in E$, is absolutely continuous, with respect to the Lebesgue measure on $\mathbb{R}_+$, and has the following representation

$$\overline{F}_x(t) := 1 - F_x(t) = \exp[-\Lambda(x, t)], \quad \Lambda(x, t) = \int_0^t \lambda(x, s) ds. \tag{1.52}$$

Proposition 1.5 *([118]) The compensator $\overline{\nu}(t), t \geq 0$, for the counting process $\nu(t), t \geq 0$, of a Markov renewal process can be represented as follows*

$$\overline{\nu}(t) = \int_0^t \lambda(x(s), \gamma(s)) ds,$$

where $\gamma(s) := s - \tau(s), \tau(s) := \tau_{\nu(s)}$.

It is worth noticing that the compensator of the counting Markov renewal process is a *stochastic integral functional* of the Markov process $x(t), \gamma(t), t \geq 0$ (see Section 2.2).

PROOF. Introduce the conditional distributions of the Markov renewal process $x_n, \tau_n, n \geq 0$:

$$\begin{aligned} F_n(dx, dt) &= \mathbb{P}(x_{n+1} \in dx, \tau_{n+1} \in dt \mid \mathcal{F}_n) \qquad (1.53) \\ &= \mathbb{P}(x_{n+1} \in dx, \tau_{n+1} \in dt \mid x_n, \tau_n) \\ &= P(x_n, dx) F_{x_n}(dt - \tau_n). \qquad (1.54) \end{aligned}$$

By Theorem III.1.33 [70], the compensating measure of the multivariate point process (1.49) can be represented as follows

$$\overline{\nu}(dx, dt) = \sum_{n \geq 0} \mathbf{1}_{(\tau_n < t \leq \tau_{n+1})} F_n(dx, dt) / \overline{H}_n(t),$$

where

$$\overline{H}_n(t) = \int_t^\infty \int_E F_n(dx, ds) = \overline{F}_{x_n}(t - \tau_n).$$

Therefore, the compensator of the counting process $\nu(t), t \geq 0$, is represented by

$$\overline{\nu}(t) = \sum_{n \geq 0} \int_{\tau_n}^{t \wedge \tau_{n+1}} H_n(ds) / \overline{H}_n(s).$$

Now, calculate

$$\begin{aligned} \int_{\tau_n}^{\tau_{n+1}} H_n(ds) / \overline{H}_n(s) &= \int_{\tau_n}^{\tau_{n+1}} F_{x_n}(ds - \tau_n) / \overline{F}_{x_n}(s) \\ &= \int_0^{\theta_{n+1}} F_{x_n}(ds) / \overline{F}_{x_n}(s) \\ &= \int_0^{\theta_{n+1}} \lambda(x_n, s) ds, \quad \text{(by using (1.52))} \\ &= \Lambda(x_n, \theta_{n+1}). \end{aligned}$$

Similarly, for $\tau_n < t \leq \tau_{n+1}$, we get

$$\int_{\tau_n}^{t} H_n(ds) / \overline{H}_n(s) = \Lambda(x_n, t - \tau_n) = \Lambda(x_n, \gamma(t)).$$

Hence,

$$\begin{aligned}\overline{\nu}(t) &= \sum_{n=0}^{\nu(t)} \Lambda(x_n, \theta_{n+1}) + \Lambda(x(t), \gamma(t)) \\ &= \int_0^t \lambda(x(s), \gamma(s))ds.\end{aligned}$$

□

Corollary 1.2 *The compensator of the counting process for a Markov jump process with the intensity function* $q(x), x \in E$, *is represented as follows*

$$\overline{\nu}(t) = \int_0^t q(x(s))ds.$$

1.6 Reducible-Invertible Operators

Let $\mathbf{B}$ be the Banach space of real-valued measurable functions, with $\|\cdot\|$ the sup-norm, defined on the state space E.

Let $Q\colon \mathbf{B} \to \mathbf{B}$ be a linear operator acting on $\mathbf{B}$, and denote by

$$\mathcal{D}_Q := \{\varphi\colon \varphi \in \mathbf{B}, Q\varphi \in \mathbf{B}\}, \quad \text{the domain of } \mathbf{Q},$$

by

$$\mathcal{R}_Q := \{\psi\colon \psi = Q\varphi, \varphi \in \mathbf{B}\}, \quad \text{the range of } \mathbf{Q},$$

and by

$$\mathcal{N}_Q := \{\varphi\colon Q\varphi = 0, \varphi \in \mathbf{B}\}, \quad \text{the null-space kernel of } \mathbf{Q}.$$

An operator Q is said to be bounded if there exist constants $C > 0$, such that

$$\|Q\varphi\| \le C\,\|\varphi\|, \quad \varphi \in \mathcal{D}_Q.$$

The least of these constants is called the norm of the operator Q, and is denoted by $\|Q\|$. The operator norm is

$$\|Q\| = \sup_{\varphi \in \mathbf{B}} \frac{\|Q\varphi\|}{\|\varphi\|}. \tag{1.55}$$

Let $Q\colon \mathbf{B} \to \mathbf{B}$ be a linear operator that maps $\mathcal{D}_Q$ to $\mathcal{R}_Q$ one-to-one. Thus a linear operator Q^{-1} is defined as a map $\mathcal{R}_Q$ onto $\mathcal{D}_Q$, which satisfies the following conditions:

$$Q^{-1}Q\varphi = \varphi, \quad \varphi \in \mathcal{D}_Q, \quad QQ^{-1}\psi = \psi, \quad \psi \in \mathcal{R}_Q.$$

The operator Q^{-1} is defined uniquely and is called an *inverse operator.*

Definition 1.26 (^{116})

(1) An operator Q is said to be *densely defined* if its domain of definition is dense in $\mathbf{B}$, that is, $\overline{\mathcal{D}}_Q = \mathbf{B}$, ($\overline{\mathcal{D}}_Q$ is the closure of $\mathcal{D}_Q$).

(2) An operator Q is said to be *closed* if for every convergent sequence $x_n \to x$, and $Qx_n \to y$, as $n \to \infty$, it follows that $x \in \mathcal{D}_Q$ and $Qx = y$.

(3) A bounded linear operator Q is said to be *reducible-invertible* if the Banach space $\mathbf{B}$ can be decomposed in a direct sum of two subspaces, that is

$$\mathbf{B} = \mathcal{N}_Q \oplus \mathcal{R}_Q, \tag{1.56}$$

where the null-space has nontrivial dimension, $\dim\mathcal{N}_Q \geq 1$.

(4) A densely defined operator $Q\colon \mathbf{B} \to \mathbf{B}$ is said to be *normally solvable* if its range of values $\mathcal{R}_Q$ is closed.

Remark 1.2.

1) A reducible-invertible operator is normally solvable.

2) The decomposition (1.56) generates the *projector* Π on the subspace $\mathcal{N}_Q$

$$\Pi\varphi := \begin{cases} \varphi, \ \varphi \in \mathcal{N}_Q, \\ 0, \ \ \varphi \in \mathcal{R}_Q. \end{cases}$$

The operator $I - \Pi$ is the projector on the subspace $\mathcal{R}_Q$

$$(I - \Pi)\varphi := \begin{cases} 0, \ \ \varphi \in \mathcal{N}_Q, \\ \varphi, \ \varphi \in \mathcal{R}_Q. \end{cases}$$

where I is the *identity operator* in $\mathbf{B}$.

Lemma 1.3 (^{100}) *If the linear operator Q is normally resolvable, then the operator $Q + \Pi$ has an inverse.*

PROOF. Applying the projector Π to both sides of the equation

$$[Q+\Pi]\varphi=\psi, \tag{1.57}$$

and since $\Pi Q\varphi = Q\Pi\varphi = 0$, we get

$$\Pi\varphi=\Pi\psi.$$

On the other hand, rewriting (1.57), we have

$$Q\varphi=[I-\Pi]\psi\in\mathcal{R}_Q.$$

This equation, as Q is a normally solvable operator, has a solution which is the solution of (1.57). □

Definition 1.27 (^{116}) Let Q be a reducible-invertible operator. The operator

$$R_0:=\Pi-(Q+\Pi)^{-1} \tag{1.58}$$

is called the *potential operator* or the *potential* of Q.

It is easy to check that the potential can also written as follows

$$R_0=(\Pi-Q)^{-1}-\Pi. \tag{1.59}$$

Proposition 1.6 *The following equalities hold:*

$$QR_0 = R_0Q = \Pi - I, \tag{1.60}$$
$$\Pi R_0 = R_0\Pi = 0, \tag{1.61}$$
$$QR_0^n = R_0^nQ = R_0^{n-1}, \quad n\geq 1, \tag{1.62}$$
$$\|R_0\| = \left\|Q_0^{-1}\right\|. \tag{1.63}$$

Equation (1.60) is called *Poisson equation.* The right hand side of this equation is sometimes defined as $I-\Pi$ (see, e.g. ^{116}).

Proposition 1.7 *Let* $Q\colon \mathbf{B}\to\mathbf{B}$ *be a reducible-invertible operator. Then the equation*

$$Q\varphi=\psi, \tag{1.64}$$

under the solvability condition $\Pi\psi = 0$, *has a general solution with representation*

$$\varphi=-R_0\psi+\varphi_0,\quad \varphi_0\in\mathcal{N}_Q.$$

If moreover the condition $\Pi\varphi = 0$ *holds, Equation (1.64) has a unique solution represented by*

$$\varphi = -R_0\psi. \tag{1.65}$$

For a *uniformly ergodic Markov chain*, the operator $Q := P - I$ (where I is the identity operator), is *reducible-invertible* ^{100}, that is,

$$\mathbf{B} = \mathcal{N}_Q \oplus \mathcal{R}_Q,$$

with $dim\mathcal{N}_Q = 1$.

For a uniformly ergodic Markov process with generator Q, and semigroup $P_t, t \geq 0$, the potential R_0 is a bounded operator defined by

$$R_0 = \int_0^\infty (P_t - \Pi)dt, \tag{1.66}$$

where the projector operator Π is defined as follows

$$\Pi\varphi(u) = \int_E \rho(dx)\varphi(x)\mathbf{1}(x),$$

with $\rho(B), B \in \mathcal{E}$, the stationary distribution of the Markov chain and the indicator function $\mathbf{1}(x) = 1$, for any $x \in E$.

Definition 1.28 Let $Q_\varepsilon \colon \mathbf{B} \longrightarrow \mathbf{B}$, $\varepsilon > 0$, be a family of linear operators. We say that it *converges in the strong sense*, as $\varepsilon \longrightarrow 0$, to the operator Q, if

$$\lim_{\varepsilon \longrightarrow 0} \|Q_\varepsilon\varphi - Q\varphi\| = 0 \quad \varphi \in \mathbf{B}.$$

And we note $s - \lim_{\varepsilon \longrightarrow 0} Q_\varepsilon = Q$.