SOME COREFLECTIVE CATEGORIES OF TOPOLOGICAL MODULES

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There are essential differences between the theory of topological algebraic systems with a discrete signature and the theory of topological algebraic systems with a continuous signature (see [C]). A topological module is a classic example of a topological algebraic system with a continuous signature.

It is well known that the class of all topological P-groups (P-rings) is a coreflective subcategory of the category of all topological groups (topological rings). For some results in coreflective subcategories of the category of topological abelian groups see in [GT], [HH].

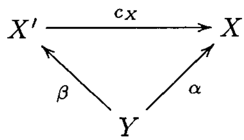
In this context Professor M. Choban posed the following question: Is the class of all P-modules of *TopMod* a coreflective subcategory of *TopMod*?

We give in this paper a partial answer to this question; the general case remains open.

Notation

All topological rings are assumed associative, Hausdorff and with identity. All topological modules are assumed unitary and Hausdorff. For a topological ring R with identity *TopMod* denotes the category of all topological left R-modules. If M is a left R-module, $X \subseteq R$ and $Y \subseteq M$, set $X \cdot Y = \{rm : r \in X, m \in Y\}$.

Recall that a subcategory $\mathfrak{B}$ of a category $\mathfrak{A}$ is said to be *coreflective* provided for each $X \in \mathfrak{A}$ there is $X' \in \mathfrak{B}$ and a morphism $c_X : X' \to X$ such that for each $Y \in \mathfrak{B}$ and each morphism $\alpha : Y \to X$ there is a unique morphism $\beta : Y \to X'$ in $\mathfrak{B}$ such that the following diagram is commutative,



i.e., $\alpha = c_X \circ \beta$.

We will introduce a new cardinal invariant for topological rings.

Definition 1. Let R be a topological ring. Denote by $h(R)$ the minimal infinite regular cardinal number for which there exists a neighborhood V of zero which can be represented as a reunion of a family of cardinality $< h(R)$ of compact subsets.

Remark 2. We note that if $\mathfrak{m}$ is an arbitrary infinite cardinal number then there exist regular cardinal numbers $\mathfrak{m}'$ such that $\mathfrak{m} \leq \mathfrak{m}'$.

Indeed, let $\mathfrak{m} = \aleph_\alpha$, where α is an ordinal. Then $\aleph_\alpha < \aleph_{\alpha+1}$ and $\aleph_{\alpha+1}$ is a regular cardinal number. We obtain that $h(R)$ is defined for any topological ring.

Remark 3. For a topological ring R, $h(R) = \aleph_0$ if and only if R is locally compact.

Definition 4. A topological ring R is called locally σ-compact provided $h(R) = \aleph_1$.

Being locally σ-compact means that there exists a base of neighborhoods of zero consisting of subsets which are σ-compact, i.e., countable unions of compact sets.

Definition 5 (see, e.g., [AGM], p. 275)**.** Let $\mathfrak{m}$ be an infinite cardinal. We will say that a topological space $(X, \mathcal{T})$ is a $\mathfrak{m}$-*space* provided any intersection of a collection of cardinality $< \mathfrak{m}$ of open subsets is open.

We recall the following construction from general topology: Let $(X, \mathcal{T})$ be a topological space and $\mathfrak{m}$ an infinite regular cardinal number. Consider the topology $\mathcal{T}_\mathfrak{m}$ on X having a base consisting of intersections of collections of cardinality $< \mathfrak{m}$ of open sets of $(X, \mathcal{T})$. Then $\mathcal{T}_\mathfrak{m} \geq \mathcal{T}$ and $\mathcal{T}_\mathfrak{m}$ is called the $\mathfrak{m}$-*modification* of $\mathcal{T}$.

If $(X, \mathcal{T})$ is a topological group (ring), then $(X, \mathcal{T}_\mathfrak{m})$ is a topological group (ring), too.

We will outline the proof of this assertion only for topological groups (for topological rings the proof is analogous): Let $x, y \in X$ and $xy^{-1} \in$

$W \in \mathcal{T}_{\mathfrak{m}}$. There exists a collection $\{W_i\}_{i<\mathfrak{m}}$ of open sets of $(X, \mathcal{T})$ such that $W = \cap_{i<\mathfrak{m}} W_i$. Choose for each $i < \mathfrak{m}$, U_i, $V_i \in \mathcal{T}$ such that $x \in U_i$, $y \in V_i$ and $U_i \cdot V_i^{-1} \subseteq W_i$. Set $U = \cap_{i<\mathfrak{m}} U_i$ and $V = \cap_{i<\mathfrak{m}} V_i$ Then $x \in U \in \mathcal{T}_{\mathfrak{m}}$, $y \in V \in \mathcal{T}_{\mathfrak{m}}$ and $U \cdot V^{-1} \subseteq W$.

An important case is when $\mathfrak{m} = \aleph_1$. The spaces $(X, \mathcal{T}_{\aleph_1})$ and $(X, \mathcal{T})$ have the same collections of G_δ-sets.

Recall that a topological space $(X, \mathcal{T})$ is called a *P-space* provided $\mathcal{T} = \mathcal{T}_{\aleph_1}$ ([GJ], p. 62–63). A topological module or ring whose underlying topological space is a P-space is called briefly a *P-module*, or a *P-ring*, respectively.

Lemma 6. *Let $M \in TopMod$ and K be a compact subset of R. Then for every neighborhood W of zero there exists a neighborhood U of zero such that $K \cdot U \subseteq W$.*

Proof. For every $k \in K$ there exist a neighborhood W_k of k and a neighborhood $U^{(k)}$ of zero such that $W_k \cdot U^{(k)} \subseteq W$. By compactness of K there exist $k_0, \ldots, k_n \in K$ such that $K \subseteq W_{k_0} \cup \cdots \cup W_{k_n}$. Then, evidently, $K \cdot U \subseteq W$, where $U = U^{(k_0)} \cap \cdots \cap U^{(k_n)}$. □

Lemma 7. *Let R be a topological ring and $h(R) \leq \mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal number. Let $(M, \mathcal{T}) \in TopMod$ and V be a 0_R-neighborhood, $V = \cup_{\alpha\in\Omega} K_\alpha$, $|\Omega| < \mathfrak{m}$, where each K_α is a compact subset. Then for every 0-neighborhood W of $(M, \mathcal{T}_{\mathfrak{m}})$ there exists a 0-neighborhood U of $(M, \mathcal{T}_{\mathfrak{m}})$ such that $V \cdot U \subseteq W$.*

Proof. We can assume without loss of generality that $W = \cap_{\alpha\in\Omega} W_\alpha$, where each W_α is a 0-neighborhood of $(M, \mathcal{T})$. Fix $\alpha \in \Omega$; then by Lemma 6 for every $\beta \in \Omega$ there exists a 0-neighborhood H_β of $(M, \mathcal{T})$ such that $K_\beta \cdot H_\beta \subseteq W_\alpha$. Put $H^{(\alpha)} = \cap_{\beta\in\Omega} H_\beta$; then $V \cdot H^{(\alpha)} \subseteq W_\alpha$. It follows that $V \cdot (\cap_{\alpha\in\Omega} H^{(\alpha)}) \subseteq \cap_{\alpha\in\Omega} W_\alpha$. Then $U = \cap_{\alpha\in\Omega} H^{(\alpha)}$ is an intersection of a collection of cardinality $< \mathfrak{m}$ of 0-neighborhoods for $(M, \mathcal{T})$, hence it is 0-neighborhood of $(M, \mathcal{T}_{\mathfrak{m}})$. □

Lemma 8. *Let R be a topological ring, $h(R) \leq \mathfrak{m}$, where $\mathfrak{m}$ is an infinite regular cardinal number $\geq \aleph_1$ and $M \in TopMod$ is such that the underlying topological space of M is a $\mathfrak{m}$-space. Then for each $m \in M$ there exists a 0_R-neighborhood V such that $Vm = 0$.*

Proof. Let $m \in M$ and V_0 be a 0_R-neighborhood such that $V_0 = \cup_{\alpha\in\Omega} K_\alpha$, $|\Omega| < \mathfrak{m}$, where K_α are compact subspaces. Then for each $\alpha \in \Omega$, $K_\alpha m \subseteq$

M is a compact subspace of M. Since $\mathfrak{m} \geq \aleph_1$, the topological space M is a P-space. Therefore each $K_\alpha m$ is finite ([GJ], Exercise 4K, 1, p. 63).

It follows that $|V_0 m| < \mathfrak{m}$. Fix an enumeration $\{x_\alpha | \alpha \in \Omega\}$ of $V_0 m \setminus \{0\}$ (it is not assumed that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$). For each x_α there exists a 0_M-neighborhood U_α such that $x_\alpha \notin U_\alpha$. Then $U = \cap_{\alpha\in\Omega} U_\alpha$ is a 0_M-neighborhood and $V_0 m \cap U = 0$. There exists a 0_R-neighborhood V such that $V \subseteq V_0$, $Vm \subseteq U$. It follows that $Vm \subseteq V_0 m \cap U = 0$, hence $Vm = 0$. □

Remark 9. The element x of a topological R-module M for which there exists a neighborhood V of 0 in R such that $Vx = 0$ are precisely those having open annihilator (or left annihilator if the module M is the ring R itself).

Theorem 10. *Let R be a topological ring and $h(R) \leq \mathfrak{m}$, where $\mathfrak{m}$ is a regular cardinal number $\geq \aleph_1$. Then the subcategory of TopMod consisting of all topological R-modules whose topology is a $\mathfrak{m}$-topology is coreflective.*

Proof. Let $(M, \mathcal{T}) \in$ *TopMod*. Consider the submodule M' of M consisting of elements with open annihilator.

We affirm that $(M', \mathcal{T}_\mathfrak{m}|M')$ is a topological R-module.

Obviously, $\mathcal{T}_\mathfrak{m}$ is a group topology on M. By Lemma 7 the mapping $R \times M' \to M'$, $(r, m') \mapsto rm'$ is continuous at $(0,0)$ with respect to the topology $\mathcal{T}_\mathfrak{m}|M'$. By definition of M', for each $m' \in M'$ the mapping $R \to M'$, $r \mapsto rm'$ is continuous with respect to the topology $\mathcal{T}_\mathfrak{m}|M'$. It is obviously that for each $r \in R$ the mapping $M' \to M'$, $m' \mapsto rm'$ is continuous with respect to the topology $\mathcal{T}_\mathfrak{m}|M'$. Therefore $(M', \mathcal{T}_\mathfrak{m}|M')$ is a topological R-module.

Denote by $i : M' \to M$, $i(m') = m'$, $m' \in M'$. Let $(M'', \mathcal{T}'')$ be a R-module whose topology is a $\mathfrak{m}$-topology and $\alpha : M'' \to M$ a continuous homomorphism. We affirm that $\alpha(M'') \subseteq M'$.

By Lemma 8 for each $m'' \in M''$ there exists a 0_R-neighborhood V' such that $V'm'' = 0$. Then $V'\alpha(m'') = \alpha(V'm'') = 0$, i.e., $\alpha(m'') \in M'$.

Put $\widehat{\alpha}(m'') = \alpha(m'')$ for any $m'' \in M''$. Then $\widehat{\alpha}$ is a continuous homomorphism of M'' in M and $\alpha = i \circ \widehat{\alpha}$.

The uniqueness of $\widehat{\alpha}$ is evident. □

Corollary 11. *Let R be a fixed locally σ-compact ring. Then the subcategory of all P-modules is coreflective in the category of all topological R-modules.*

Remark 12. If R is a P-ring, then the subcategory of *TopMod* consisting of all P-modules over R is coreflective.

Indeed, if $(M, \mathcal{T}) \in$ *TopMod*, consider the abelian topological group $(M', \mathcal{T}^\delta)$, where $M' = M$ and $\mathcal{T}^\delta$ is the $\aleph_1$-modification of $\mathcal{T}$. Since R is a P-ring, $\mathcal{T}^\delta$ is a R-module topology. Consider the mapping $i : M' \to M$, $m' \mapsto m'$. We affirm that $(M', \mathcal{T}^\delta)$ is the coreflection of $(M, \mathcal{T})$ in the category of all P-modules. Indeed, let $(M'', \mathcal{T}'')$ be a P-module and $\alpha : M'' \to M$ a continuous homomorphism. Put $\widehat{\alpha} : M'' \to M'$, $m'' \mapsto \alpha(m'')$. Then $\widehat{\alpha}$ is a continuous homomorphism of M'' in M and $\alpha = i \circ \widehat{\alpha}$. The uniqueness of $\widehat{\alpha}$ is evidently.

Remark 13 ([GJ], Exercise 4K, 8, p. 63)**.** Every P-space X is zero-dimensional, i.e., has a base consisting closed and open subsets.

We note that if R is a connected topological ring and ${}_RM = M$ is a P-module, then $M = 0$. Indeed, by Remark 13, M is zero-dimensional. If $x \in M$, then $Rx \subseteq M$ and Rx is connected. Therefore $Rx = 0$ and so $M = 0$.

We obtained that if R is a connected topological ring then the subcategory of *TopMod* consisting of all P-modules is coreflective.

We noted that the $\aleph_1$-modification of a topology of a topological group (ring) is a group (ring) topology. We give here examples of topological modules $(M, \mathcal{T})$ for which the $\aleph_1$-modification of $\mathcal{T}$ is not a module topology:

Example 14. Let R be any nonzero connected topological ring. Consider R as a left topological R-module with the multiplication as a module operation. Then the $\aleph_1$-modification is not a R-module topology.

Another example of this kind is the following.

Example 15. Let p be any prime number and $\mathbb{Z}_p$ be the ring of p-adic integers with the natural compact topology $\mathcal{T}$, $M = \mathbb{Z}_p$. Then the pair $((\mathbb{Z}_p, \mathcal{T}), (M, \mathcal{T}))$ is a compact left $\mathbb{Z}_p$-module. The $\aleph_1$-modification of $\mathcal{T}$ is the discrete topology $\mathcal{T}_d$ on M. Obviously, the pair $((\mathbb{Z}_p, \mathcal{T}), (M, \mathcal{T}_d))$ is not a topological $\mathbb{Z}_p$-module.

We give here an example of a non-discrete P-module over a compact ring.

We will identify the set of all natural numbers with all ordinals $< \omega$, where ω is the first infinite ordinal. As usual, ω_1 denotes the first uncountable ordinal.

Example 16. Let $\mathbb{F}_2$ be the field consisting of two elements and $R = \mathbb{F}_2^\omega$ be the topological product of ω copies of $\mathbb{F}_2$. Put S the subset of R consisting of elements with open annihilator.

Claim. S is a dense ideal of R.

Indeed, let $r_1, r_2 \in S$, then there exist neighborhoods V_1, V_2 of zero such that $V_1 r_1 = 0$, $V_2 r_2 = 0$. Therefore $(V_1 \cap V_2)(r_1 - r_2) = 0 \to r_1 - r_2 \in S$. Let $r \in S$ and $m \in R$, then there exists V a neighborhood of zero such that $Vr = 0$. We have $Vrm = 0 \to rm \in S$. Evidently, $S \supseteq \oplus_{i\in\omega}(\mathbb{F}_2)_i$, hence S is dense.

Now, we consider the group $M = \oplus_{\alpha\in\omega_1} S_\alpha$, where $S_\alpha = S$ (a direct sum of ω_1 copies of R-module S).

For any $\beta \in \omega_1$ put $M_\beta = \{x : x \in M,\ pr_\alpha(x) = 0 \text{ for every } \alpha \leq \beta\}$. Then $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_\alpha \supseteq M_{\alpha+1} \supseteq \cdots$, and $\cap_{\alpha\in\omega_1} M_\alpha = 0$.

Evidently, the family $\{M_\alpha\}_{\alpha\in\omega_1}$ gives a group topology $\mathcal{T}$ on M and that $(M, \mathcal{T})$ is a P-space.

We note that $(M, \mathcal{T})$ is a topological R-module.

Indeed, $RM_\alpha \subseteq M_\alpha$ for each $\alpha \in \omega_1$. Let $m \in M$; then there exists a neighborhood V of zero of R such that $Vm = 0$. We have proved that $(M, \mathcal{T})$ is a topological R-module.

Acknowledgment

The author is grateful to professors Mitrofan Choban and Mihail Ursul for their constant interest and valuable indications. I am also grateful to the referee for her/his very careful reading and substantial helpful suggestions.

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