Chapter 1

Introduction

I have been told that the fundamental job of leaders and managers is to determine strategy. Yet leaders and managers at best practice this as an art, not a science. Those that have gone to business school undoubtedly have learned about the theory of games, not to mention many other theories on how to make strategic decisions. However, I have often heard such managers refer to less scientific or economic books such as *The Art of War* [Sun Tzu (1988)] as the source for their actions rather than any economic theory. Contrast this with a discipline such as Civil Engineering. Not only are the practitioners schooled in a variety of sciences about their craft (Physics, Mathematics and Engineering) but they continue to practice that science throughout their professional careers. Given the relative importance of strategy to leading and managing, I believe any effort to bring discipline to the subject will be welcomed in the business community.

The problems are deep. There is, in fact, no real science; strategy like war *is* an art. There is not even a good language for discussing the subject. The good news is that many people practice this art and so the number of stories in the field is large, reflecting an ever increasing wealth of practical experience. I believe that a rational framework within which to hold such conversations is a contribution to this field. Von Neumann and Morgenstern (1944) provided a significant start at such a framework, though with a static theory. The obvious next step is an inquiry into an extension of their theory to a dynamic theory. There are dynamic theories[1] but none that I know of that make use of the wealth of experience that we have from the physical sciences on how to make such extensions. The problem is how to borrow from those domains. Straight metaphor misses the wealth of experiences with economic behavior[2]. The best borrowings are probably of a mathematical nature, since mathematics is superb at speaking deeply without knowing what it is talking about. For dynamics, I believe the ideal mathematical language is geometry.

My goal is an ambitious one: To create using geometry and the physical sciences, a language appropriate to the discussion of the dynamic behavior of strategies as they occur in the real world. It is not only about rational behavior but also about real world behaviors. The goal is to construct a language, a sufficiently refined language with distinctions that provide insight allowing practitioners to move from an art form to an engineering form[3].

The ***language*** of engineering is mathematics, which consists of marks, strings of characters that brand an idea. Such characters obey rules of their own and can be properly called a language. They form a sharp set of distinctions which I believe will cut through the difficulties of strategic thought. The language of business consists of words. They provide the vernacular, a way to talk publicly about what is observed. I identify such ***distinctions*** in bold–italic in the text that follows and also provide references to them in the index. Both aspects of the language are important, though the business aspect will be more immediately

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[1] I have found the following interesting and representative: Luce and Raiffa (1957), Williams (1966), Dresher (1981), Eatwell, Milgate and Newman (1987), Ordeshook (1986), Shubik (1991) and Osborne and Rubinstein (1994).

[2] This field has undergone significant change in the last decades and has perhaps moved away from the game theory approach. I am indebted to Dr. Mark Satterthwaite for providing the following set of "course" text-books on modern economics: Myerson (1991) and Mas-Colell, Whinston, and Green (1995). Our goal is to be able to predict economic behavior, which I hope will generate a rebirth of interest in the game theory approach.

[3] The idea of constructing a new language through distinctions might be familiar to those thinking about design questions and artificial intelligence. See for example Winograd and Flores (1986). In this regard, I owe a philosophical debt to these authors, as well as to Rorty (1989) and Rorty (1991).

accessible. It is the engineering aspect however that yields the possibility of measurement, prediction and the application of the scientific method.

Thus this monograph can be read at multiple levels. An initial reading might well focus on the business aspects, reading the mathematical symbols purely as ***mathematical trade-marks*** whose deeper meaning can be deferred to a later reading[4]. A second reading might focus on the mathematics as a language in and of itself and this would demonstrate the quantitative power of the analysis. To facilitate this, in the introductory chapter I provide a sketch or executive summary of the monograph and introduce the mathematics as merely a set of signposts without a great deal of explanation about their structure. I focus on the business aspects. In later chapters, I reverse the process, coming back to each of the signposts and focusing more on their mathematical structures with explanations about their importance and less on the business distinctions. The more detailed mathematical explanations though important, are not as accessible to a large audience; I put these in appendices. In the final chapter I consider the open issues.

1.1 Geometry of Economic Games

I start with the assertion that the ***plays*** of an economic game form a *geo-metry,* literally earth-measurement. Such a ***Strategic Geometry*** is formed from the space of strategic possibilities, decisions or ***strategy–choices***, enumerated in terms of the ***pure strategies*** allowed each player in a multi-person game, coupled with a concept of ***measure*** for each strategy. Each play is an event representing a finished game with all ***moves*** completed and is represented as a point in space; a sequence of plays in a game describes a curve in this strategic geometry space. A sequence of plays of the game suggests a concept of "time" separating successive plays and some concept of relatedness between neighboring plays. I call this dimension ***time–choice***. To describe the relatedness, I introduce the notion of "elasticity" or "connectedness" between "neighboring" games. It is important to note that such elasticity

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[4] I thank Ms. Liisa M. Thomas for the legal meaning of trade-marks; I have made no attempt however to strictly adhere to that legal meaning in this monograph.

represents forces that are normally outside of game theory: They include social behavior such as learning and psychological behavior such as burnout. Curves in the ***choice–space*** of strategy–choices and time–choice reflect movement in the geometry and the dynamics of the elasticity or connectedness. A dynamic theory of games is a theory that provides rules for computing the possible curves. An outcome of the theory will be a set of distinctions that characterize the dynamic behavior of games and hence lead to a deterministic view of the flow of decisions. The theory is expected to evolve through the mechanism of the scientific method.

To introduce the notion of strategic geometry, I extend the game–theory concept of strategy as introduced by Von Neumann and Morgenstern (1944). Game–theory enumerates ***pure strategies*** each of which is a complete plan that specifies what a player will do for each ***move*** under every contingency of what the other players could do and under every chance event that might happen. In general, players do not achieve the optimal payoff by choosing pure strategies; they must choose mixed strategies. Game–theory defines mixed strategies as a weighted sum of these pure strategies, subject to the constraint that the weights add up to unity: The interpretation is that the player chooses the pure strategies randomly, with a probability or frequency for each pure strategy set by the weight for that strategy: the odds. Though the odds may change over time, the static theory identifies the limiting strategy or set of ***equilibrium strategies*** which are both stable and fixed. If the game is played over a long period of time, players that adopt the equilibrium strategy will suffer the minimum loss. This framework of game theory is generally believed to be part of any economic theory, though for games with three persons or more, there is no generally agreed identification of the stable and fixed equilibrium behaviors.

I extend the concept of an equilibrium mixed strategy to a time-dependent strategy–choice, which I also call a ***scalar strategy*** by the assertion that the strategy choices or odds change with time and are ***deterministic***, obeying equations that I will specify. The equations will depend on forces that govern the change and will include both game theory forces and non-game theory forces. The possibility will exist for both stable and unstable behaviors, in addition to the static fixed and stable equilibrium behaviors. A scalar strategy can now be thought of as a set of ***strategy–choices*** for each pure strategy: One chooses an amount that is between zero and infinity for each pure strategy. The relative strategy–choices between different pure strategies set the odds. The strategy–choices, normalized by the sum of all strategy–choices, can be interpreted as determining the probability or frequency one chooses for the play. This normalized strategy–choice makes the connection to the static game–theory mixed strategy.

The geometry is described by this set of scalar strategies $\{x^1, \cdots, x^s\}$, along with the time–choice or ***scalar time*** $t = x^0$ that articulates the meaning of ***causality***: "Events that happen later will not impact events that happen now". We shall see that the concept of "***scalar***" is itself an important distinction that describes how different observers describe the same situation. A ***dynamic play*** consists of a scalar strategy with non-negative strategy–choices associated with each pure strategy. A ***dynamic game*** is a time–sequence of dynamic plays and defines a curve in this choice–space. The points along the curve can be labeled by another scalar, the ***strategic distance*** s. A ***strategic geometry*** is thereby defined by the distance between neighboring points.

The idea of distance or measurement is not treated lightly. In geometry, the power comes from measurement. In the normal field of geometry we have thousands of years of experience documenting measurements. In the field of economics and other related disciplines such as social behavior, project management and organizational dynamics, we have relatively few years of experience measuring. The distinctions introduced here are therefore provisional.

The geometry of games is specified by a metric[5]:

$$ds^2 = g_{ab} dx^a dx^b . \tag{1.1}$$

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[5] I use a notation similar to Hawking and Ellis (1973). I find their text readable and many of their results helpful in the game theory context here. I recommend their exposition as additional reading. I provide a word of caution however. In this monograph I have maintained a consistent set of conventions that in fact differs from theirs. Hence in some cases results quoted here will differ slightly from their results. Other references may also use different conventions, hence the importance of sticking to a single convention in this monograph.

The metric is a set of rules for measuring the length between neighboring points or plays whose ***mathematical trade-mark*** is the ***line element***. (*Line element* is so named because it is a short segment of the line or curve of interest between dynamically related plays.) The line element reflects the common notion about very short distances being *Euclidean*, *i.e.* distances are laid out on a flat space. Even on flat or Euclidean space, the coordinates need not be at right angles (orthogonal). For non-orthogonal coordinates, a ***metric*** can be specified that is a set of ***measures*** at each point. With these measures specified at every point, it is possible to compute the ***geodesic***, the shortest distance between any two points. I draw from the literature the information needed.

The value of what has been created so far is the notion that successive plays of an economic game are separated by a computable distance determined by the line element. It becomes possible to discuss dynamic behaviors that converge to equilibrium and those that don't. In fact equilibrium ceases to be the main interest, just as it ceases to be in physical models for phenomena such as turbulence and other non-linear behaviors. The metric, the set of measures at each point, will determine the nature of the behavior.

The idea of a ***metric*** determining dynamics is well known in geometry, since dynamics is the statement that things move along the geodesic. What physics adds to the discussion of geometry is the experience that motion or ***dynamics*** determines the metric through the set of ***sources*** given by a new mathematical trade-mark:

$$R_{ab} - \tfrac{1}{2} R g_{ab} = -\kappa T_{ab} . \tag{1.2}$$

Most of us are familiar with Newton's idea that matter is the source of gravitational attraction. We may also be familiar with Maxwell's idea that charged matter is the source of electro-magnetic forces. (I propose below that this force is analogous to the economic force.) Einstein (1952) refined both ideas within the context of geometry to state that the energy-momentum of charged and uncharged matter (represented by the mathematical trade-mark T_{ab}) is the source of gravitational attraction (which Einstein represented by the mathematical trade-mark $R_{ab} - \tfrac{1}{2} R g_{ab}$). The refinement was consistent with the theories of mechanics by Newton and Maxwell to the extent that those theories had been verified[6].

Gravity as described by Newton is a force that is felt instantly at any distance from a given source. The theory of Einstein and the theory of Maxwell however forbid any force to travel faster than the speed of light. In particular, the transformation properties of space-time and the sources of matter dictate the propagation properties of the forces. The result of his analysis is that sources of matter can only propagate their forces to other parts of space-time by means of a ***long-range*** messenger field whose trade-mark is $R_{ab} - \frac{1}{2} R g_{ab}$. The analogous statement of Maxwell is that the sources of charged matter can only propagate their forces to other parts of space-time by means of another long-range messenger field whose trade-mark is the electro-magnetic field F_{ab}. The messengers are long-range in the sense that they propagate at the speed of light. Furthermore, the theory determines the ***messenger field*** for matter entirely in terms of the metric trade-mark g_{ab}. Despite the major conceptual difference, Einstein's theory yields the theories of Newton and Maxwell as a good approximation except for astronomically large masses.

The set of measures that specify the distance between neighboring points, the metric g_{ab}, is thus determined by the motion of the sources of matter. From a purely geometric perspective, the messenger field is determined by the trade-mark R_{ab}, which can be identified with the ***curvature*** of space-time. The curvature and the derived trade-mark $R = g^{ab} R_{ab}$ are both functions of the metric. I adopt the ***economic Einstein's equation***, Eq. (1.2) that *the motion of the sources determines the choice-space-time structure and the choice-space-time structure determines the motion of the sources.*

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[6] For specific results from physics, the interested reader might consult the excellent volumes of Feynman (1963) for a quite readable description of any of the classical topics of physics, including fluids, geometry, mechanics and electro–magnetism.

1.2 Market Fluid

It might appear that I have introduced an unnecessary complication into the language of the dynamic theory of games. I recall some real world examples:

- A CEO has a goal of adding value to a product or service. To succeed in commerce, his or her company competes with others for a scarce or limited resource for the consumer.
- A project manager manages resources with a goal of delivering their project on time, within cost and of sufficient quality to satisfy their customer.
- An environmental activist hopes to make better use of the world's resources to achieve a safe and sustainable habitat.

In each of these examples, there is a battle between opposing sides for scarce or limited resources. Alternatively, there is a question of which choice to make, including, though not necessarily, the possibility of taking some risk.

The traditional Theory of Games provides a static language for each of these examples with multiple ***plays*** in a game with two or more ***players*** and a payoff for each possible choice of player ***strategy***. I call this language the ***rules-of-the-game***. The rules-of-the-game extracts the essential battle between opposing sides. I believe any theory of social behavior must include these forces and I will discuss these forces in greater detail in Section 1.4. These forces must be added to the metric forces described above to make a complete ***deterministic*** theory.

However, the rules-of-the-game approach supposes that all other real-world effects average out. In this sense the theory is static (not unlike the study of statics in mechanics as applied, for example, to the construction of a bridge). In certain circumstances, these ignored real-world effects might be important and not ignorable. (In the analogy from mechanics, this might correspond to soldiers marching in-step across the bridge.)

How do I include such real world forces and which ones do I include? One approach to computing dynamic behavior in real world behaviors has been with the tools of systems dynamics using sophisticated computer modeling. Systems dynamics models real world behaviors by interactive systems of positive and negative feedback loops. Such models provide information about the small variations away from expected behavior.

There are many examples that might be so modeled.

- A CEO may have a winning attitude, pursue corporate strategies aggressively and communicate clear values to shareholders and employees.
- To improve the world environment, an activist must consider the political forces as well as the management of the world's resources.
- A project manager must be concerned with the effects of burnout and degree of training of the staff, the impact of management overhead and the effect product–defects have on quality and schedule.

In each case, these new attributes describe *how* the games are played and are widely understood to represent forces that drive games towards or away from equilibrium. These forces influence the time attributes of the game, which is extremely important. A CEO has only so much time to achieve success before funding runs out; an environmental activist must achieve success before political forces exhaust critical resources; and a project manager fails in his or her duty if the project significantly misses the deadline.

I assert that these real world attributes are contained in the sources T_{ab} and describe the ***short-range*** connectedness or elasticity of the "medium" or "material" that is one source of the dynamic aspect of games. These short-range forces are distinct from the messenger forces described in the previous Section. I am led to consider the codification of a ***behavioral medium*** for games. I shift from geometry (thinking about how far distinct points are from each other) to medium (how much stuff there is that generates the geometry). This has been extraordinarily helpful in physics. I believe this will be helpful in economics.

What does this mean for game theory? The theory of games as originally formulated describes a static equilibrium choice, one that characterizes rational behavior. If all parties in the game behave rationally, there is a unique set of strategies for each player. In the current view, this behavior is a single rational direction in space. In reality, games are played in many ways and not all choices correspond to this rational direction. There is some connection between these points, some communication that occurs that lies outside the domain of traditional game theory. This communication is the manifestation of the behavior of the players, not the rules-of-the-game. The role of the new behavioral medium is to represent the short-range aspect of this communication. It influences if and how rational behavior comes about.

Computer models of real world behavior that have been constructed are typically complicated and opaque. Some simplification is necessary to characterize or capture significant attributes of real world behaviors. The above thought picture representing as connected points in space multiple games being played simultaneously is a picture of a ***market fluid***. The multiple games are thought of as market "stuff". A small volume $\mathbf{V}$ or cell in choice–space is filled with a given amount of this stuff. The inverse of this volume per unit stuff is the density of stuff per unit volume, ρ, I call the ***market density***. This cell moves in the choice–space. The cell can be defined so that we follow a given amount of this market stuff, being careful to account for whatever stuff enters or leaves the cell. By definition then, the stuff is neither created nor destroyed. The accounting rules that keep track of any increase or decrease in stuff by the flows into or out of the cell, is called the ***conservation of market density***.

Market density endows a market fluid with inertia, an important attribute in describing how easy or difficult it might be to get the fluid to move. Such properties in physics are influenced by gravity and create gravity. If charged, such fluids create and are influenced by electromagnetic fields. However, this fluid is not like any real world fluid for many reasons, not the least of which is that it resides in a higher dimensional space. As such its properties need not be familiar. A starting point will be to generalize the concept of a "perfect fluid" to this multidimensional space, characterizing the fluid by a small number of distinctions that include mass and charge. The language that results will have more generality than this simple model.

The purpose of this monograph is to articulate in some detail both the consequences that follow from this starting point, as well as indicating how this starting point might be extended and in the process indicate that the approach is quite general.

A "perfect fluid" is a behavioral medium with a market density ρ. Such a medium is called ***elastic*** and has additional properties derived from its ***elasticity*** ε that is a given function of the market density: The fluid has a ***pressure*** p which can be thought of as ***power*** or ***control***, an ***energy density*** per unit volume μ and a ***flow*** V_a. Note that each of the business distinctions comes with a mathematical trade-mark, which obey certain rules. For example, these various mathematical trade-marks are related to each other by rules:

$$\mu = \rho(1+\varepsilon)$$
$$p = \rho^2 \frac{d\varepsilon}{d\rho} \quad . \tag{1.3}$$

The rules determine the form of the source for such fluids and are extended to the multi-dimensional choice–space. The sources for this elastic market fluid determine the energy-momentum trade-mark:

$$T_{ab} = (\mu + p)V_a V_b - p g_{ab} . \tag{1.4}$$

As asserted, the sources depend on the pressure p, the density μ and the flow V^a. It is possible to determine the elasticity and market density in terms of the energy density and pressure. I reduce the discussion of possible non-rational and hence dynamic behaviors, to a discussion of the flow of play along particular strategic directions and the ease or sluggishness of that flow dependant on the pressure and energy density of the behavioral medium in that neighborhood. The key result is that there will be a dynamic force proportional to the gradient of the fluid flow which is like its physics counterpart of acceleration.

How does this model represent the more usual business picture? The thought picture of multiple simultaneous games being played is represented by points in the choice–space. The concept that the points have an energy density which results from a market density is a sharpening of the concept that certain strategies have a behavioral ***capital***, a weight, ***strategic-mass*** or confidence, which makes them more important than other choices independent of their game–theory value in determining economic equilibrium. Such effects can be expected to be long-range like the game-theory forces. The strategic-mass might be an articulation of a CEO's stubbornness in pursuing a certain direction considered to be a winning approach. It might be the environmental activist's acknowledgment that certain choices are forced due to political realities. It might help support project manager's belief that output is influenced by behavioral factors, as well as unit productivity.

I believe the ***market density*** distinction is a powerful one; I shall use the term ***strategic-mass*** to represent how much market density is in a region of strategy-space. Capital suggests the accumulation of value based on specific means of production and is particularly useful in a traditional economic context. Strategic-mass is a more general concept with applications to more general decision-making processes. Market density will be more useful in the discussions of the rules relating the mathematical trade-marks and in extracting results from the published literature.

The dynamic theory of games has in addition to forces that depend on the motion or ***flow*** of ***strategic-mass*** a force that depends on the behavioral or ***political pressure*** or what might be properly called management ***control*** or focus. I find this force to be short-range and therefore somewhat different than the other forces. However, the concept is a natural extension of the ideas and thought picture proposed. I show that the economic Einstein equations that govern the flow of strategic-mass yield a form that has both the long-range forces of the messenger fields and this short-range force associated with control.

1.3 Thermodynamics of Games

An interesting observation follows from the rules that hold with the above mathematical distinctions and suggests some additional derived business distinctions. The specification of sources T_{ab} consists of a set of numbers at each point in the space of strategies at each instance of time. One particular set of numbers is called the energy $E = T_{00}$ associated with the medium. It is a meaningful and insightful way of talking about the properties of the media and typically is thought of being either ***mechanical*** or ***thermal***. Based on the mathematical rules, the mechanical energy is a property of the motion of the media moving collectively. The thermal energy is a property associated with some internal ***short-range*** structure of the medium whose exact nature is not necessary for the general discussion.

The mathematical rules that generate these distinctions generalize to behavioral media, so these two types of properties can be distinguished in this media as well. I summarize that the energy can be decomposed into thermal and mechanical components:

$$dE = TdS - pd\mathbf{V} . \qquad (1.5)$$

The total energy dE in a small region of space is composed of a ***mechanical*** part $-pd\mathbf{V}$ and a ***thermal*** part TdS. The mechanical part is determined by the pressure and density represented here as the reciprocal $\mathbf{V} = \rho^{-1}$. The mechanical part is therefore derived from the distinctions introduced so far.

The thermal part represents the energy that is not accounted for by the purely mechanical collective motion of the fluid. It is described by two new distinctions: ***entropy*** S and ***temperature*** T. They characterize the internal motion. We are familiar with this distinction between collective and internal behavior in markets. A collective move is when all the games being played move towards a new set of strategies. This is the mechanical motion. Internal moves are small variations or fluctuations that occur between different games, though all about a common point. Such small fluctuations are a type of noise. This is the thermal motion. One reflection of such thermal noise is the number of resources necessary to produce something needed for the market; the more noise, the more bodies or resources are required. Thus for initial consideration, temperature can be thought to be the ***resource*** needed to create a unit of something.

Why are two distinctions needed? In common parlance, what we have introduced is the concept of heat (as a manifestation of thermal energy), determined by both entropy and temperature. A small pan filled with boiling water and a large pan filled with boiling water are at the same temperature but have significantly different amounts of heat (thermal energy). This follows from the mathematical rules for these trade-marks. At each point of space the ***behavioral entropy*** $dS = dE/T + pd\mathbf{V}/T$ is determined by the energy and density. As a result of applying the mathematical rules this relationship determines a unique ***behavioral temperature***.

Though I am led to these concepts by analogy to physics, I prove these concepts from the mathematical rules alone [*Cf.* Appendix A]. This is an important point since the distinctions need to stand on their own in the field of games and be supported by observations of phenomena in that field. I believe these new concepts do make sense in the theory of games. I think an indecisive CEO is indicative of thermal motion. An environmental activist unclear on the correct approach may flip–flop on direction. A project manager recognizes that poor communication within an organization causes un-recoverable loss in terms of output and schedule. There are clearly thermal as well as mechanical behaviors; internal as well as collective.

1.4 Rules-of-the-game

I believe that sources are an important aspect of the dynamic theory. I claim that the motion of sources is determined by the choice-space-time structure and the choice-space-time structure determines the motion of the sources. I distinguished those sources from the static aspects of game theory with the caveat that both types of forces are long-range. Where do the static aspects of game theory come from? Are their similarities related in a deep way to a geometric notion?

At the very least it is generally agreed that a key attribute of the rules-of-the-game is the notion that for each individual, there is a payoff or value based on the individual's choice and the choices of all the other players. Equilibrium strategies that differ by an overall scale change are equivalent and so the existence of an equilibrium strategy implies the existence of a line or direction in choice–space. The rules-of-the-game determines a force based on the payoff for some equilibrium strategy. In terms of theories of the physical world, I show that such a force resembles more that of a magnetic field than a scalar force such as Newton's gravity. A magnetic field sets a direction around which charged particles move, rather than attracting them to or repelling them from a specific point. A necessary consequence of this is that the market fluid is ***charged***.

To approach this and ultimately to justify my assertions, I return to a discussion of geometry and observe that it would be elegant to derive the dynamic theory totally from geometry without the need to introduce any additional sources; the hope would be that sources are determined by geometry. This hope is not unfounded and in particular I will use the idea to generate the sources that describe the rules-of-the-game and demonstrate that the resultant forces are consistent with the Von Neumann's ideas of a static theory.

The foundation for this hope is based on an old concept in physics but not one generally known, that familiar notions of space, time and matter might all have their origins within geometry [Weyl (1922)]. In such a theory, gravitational and magnetic forces are unified. Many investigators have used these ideas to create an interesting "Theory of Everything" that unifies both the short-range and long-range forces. It is a theory that has current popularity[7]. In economic theory, the origin of the geometry (the decision space) is actually clearer than the origin of the sources (market fluid). There is no already known underlying theory for sources that we can rely on. Things are therefore quite different from physics. I turn the thinking around and look more intently at the geometry to see if there is a useful and helpful theory of dynamics that can be inferred from that geometry. This inquiry provides the necessary insight for introducing the appropriate magnetic field.

There is still a need for a behavioral medium to represent the short-range forces. These also contribute as sources for the unified geometry. At the current level of maturity of this approach, it is perhaps too big a jump to hope that all sources can be eliminated into some super-unified geometry.

I turn to geometry and use the fact that unification comes about because certain dimensions of the unified space are *hidden*. I assert that

[7] An early version of this theory is by Kaluza and Klein, reviewed in Supplemental Note 23 in Pauli (1958). This work is closer in spirit to what we need than the modern theory of strings or the theory of everything. A survey of the modern theory was given on the BBC and is nicely presented in book form by Davies and Brown (1988). A systematic exposition can be found in Green, Schwarz & Witten (1987).

they might be hidden because of symmetries of the unified metric and certain metric components might masquerade as the unknown sources. If the space has symmetries (such as the earth which is idealized as a spherical ball), then distances between points depend on fewer dimensions (the distance between two points on the earth is the one dimensional length of the great circle, specified by the arc length or angle as measured from the center of earth). The simplification does not change the definition of the distance as expressed by Eq. (1.1). The line element depends on a set of numbers, the metric components g_{ab}. When the rules for the line element mathematical trade-mark are applied, the number of such metric components stays the same.

However, it is possible to consider an equivalent geometry in fewer dimensions, the "small" space rather than the original "large" space. The rules for the number of metric components determine there will be fewer metric components in the "small" space than the "large" space. The missing set of components needs to be accounted for: The rules allow one to introduce sources to replace metric components on a one–for–one basis in such a way as to replicate the original geometry. This is an extraordinary result, though one we shall have to accept for now as fact since the mathematical argument is quite involved. For specific symmetries, we need to extract the nature of these additional sources. Though the general result is of some interest, I focus here on the sources that determine the ***rules-of-the-game***.

I enumerate the total number of dimensions of the space. There is one dimension for time–choice. For each player j there will be n_j dimensions corresponding to the number of pure strategies available to that player. To reproduce the static theory, I introduce one additional but natural and hidden dimension for each player that represents the ***value–choice*** for that player. Each player attempts to maximize their perception of value, which does not influence the metric for the game. I translate this requirement to the ***Value–Choice Hypothesis***: **The metric does not depend on the players' value–choices.**

The consequence of this hypothesis is that the theory is equivalent to a theory with one less dimension for each player, one in which for each player j there is a ***market source***, A_a^j. I note that the associated electro-magnetic field Eq. (1.6) is in fact a subset of the unified curvature field. There is a reason why these messenger fields have similar properties.

In addition, there are ***scalar sources*** γ_{jk} which provide ***structural coupling*** of the market to the players[8]. The mathematical trade-mark–rules relate these sources to the underlying geometry[9] and in particular identify these sources with components of the underlying metric. These source and structural coupling components account for all the unified metric components.

The missing ingredient is why I relate the rules-of-the-game forces to the electro-magnetic field. To relate this result to what one would expect from traditional game theory, I summarize that for any game whatsoever, game theory determines a ***payoff matrix*** that specifies payoffs between pairs of players depending on pure strategies chosen by each player. In this context, a pure strategy for each player is a plan for what that player would do, its ***move***, given what is known at the time of that move, for all possible choices of prior moves or chance moves by all the players up to that point. The payoff matrix determines the payments at the end of a ***play*** of the game after all moves have been made. It is far from obvious that every game can be put into this form and even less obvious that most of what we call economic or social behavior indeed has the form of such a game. I refer the interested reader to the extensive literature [Cf. Footnote 1] for details and to Von Neumann and Morgenstern (1944) for the original proofs. These proofs provide a basis for the theory proposed here.

The product, formed from the payoff matrix and the ***strategy vector*** of one player, determines the payoff to the other player. For such a game at equilibrium, the game may or not be fair and the total sum of payoffs may or may not sum to zero. For a general non-zero sum game, each player has a view of the payoffs and therefore of the payoff matrix. A non-zero sum game can be converted to a zero-sum game by adding an

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[8] I borrow the term structural coupling from Winograd and Flores (1986).

[9] The notation and results are provided in a later chapter. There I indicate that the larger space with symmetries has metric components $\gamma_{\mu\nu}$ using Greek letters and the smaller space has metric components g_{ab} using Latin letters. The mathematical rules relate the smaller-space metric components to the larger-space metric components.

artificial player that contributes nothing strategic to the game. We can thus deal with zero-sum games.

I will show below that it is always possible to reframe any zero-sum game as a symmetric and hence necessarily fair game. It is done by creating two artificial players whose strategies consist of the total set of the initial player strategies and one new "hedge" player with a single strategy. It is a theorem that a symmetric game always is represented by an anti-symmetric matrix which is the trade-mark for an electro-magnetic field. For such a zero-sum game I specify the symmetric player's ***game matrix*** F_{ab}^{j} by its components and the mathematical trade-mark rules that the marks $\{a, \quad b\}$ denote the strategy–choices for the collection of all players. The game matrix will influence the behavior of other games but should not do it instantaneously. General arguments requiring the effects of the game matrix to propagate smoothly to distant parts of space lead to the following trade-mark for the game matrix:

$$F_{ab}^{j} = \partial_a A_b^j - \partial_b A_a^j . \tag{1.6}$$

Moreover, these general arguments are consistent with identifying A_a^j with the market source defined above from the "large" unified space.

Though the details need to be made clear and indeed will be made clear later, the general consequences should be clear: The game matrix determines the rules-of-the-game. In fact there are ***economic Maxwell equations*** that say that *the game matrix is determined by the motion of the charged fluid and the motion of the charged fluid is determined by the game matrix.*

Of course this will be modified by the economic Einstein equation, Eq. (1.2). I have source components T_{ab} with contributions from the behavioral fluid (determined by the strategic-mass, pressure (or control) and flow of the fluid) and contributions from the game matrix. These contributions are strictly determined by the rules and will be presented in later chapters. Conceptually, we have a complete theory.

1.5 Economic Justification

There are loose ends or questions readers may still have that require more technical detail about the mathematical trade-marks. First, the static theory of games has been alluded to, though no examples presented. Second, I have alluded to the natural relationship between the symmetric game matrix and the electro-magnetic field but without any details. Third, the relationship between the game matrix and the symmetrized payoff matrix has been alluded to but not specified in any detail. I deal with these now.

I start with an example of a ***symmetric*** game that is representative of the many examples available from the literature [See footnote 1]. I consider two armies fighting each other: the Red and the Blue. Both armies have a symmetric set of choices: Fight from the high ground, fight from the low ground or fight as archers. The one that chooses high ground, defeats the one that chooses low ground but will be vulnerable to attack by archers some of the time. The one that picks low ground defeats the archers by hand to hand combat but will be defeated if the enemy chooses high ground. And the one that chooses archers will defeat the enemy on high ground some of the time but will be defeated some of the time by the enemy on low ground.

The following is a possible (common) payoff matrix reflecting these choices:

$$G=\left(\begin{array}{c|ccc:c} \text{Blue/Red} & \text{High} & \text{Low} & \text{Archers} & \\ \hline \text{High} & 0 & 100 & -25 & 2/7 \\ \text{Low} & -100 & 0 & 50 & 1/7 \\ \text{Archers} & 25 & -50 & 0 & 4/7 \\ \hdashline & 2/7 & 1/7 & 4/7 & \end{array}\right).$$

The game is zero-sum, symmetric and fair, with the payoffs lying between –25 and 25 units of value. No pure strategy is optimal but a mixed set of strategies yields the equilibrium payoff of zero.

The margin shows the optimal mixed strategy vector for each player:

$$\begin{aligned}\text{Blue} &= \{2/7 \quad 1/7 \quad 4/7\} \\ \text{Red} &= \{2/7 \quad 1/7 \quad 4/7\}\end{aligned}.$$

The mixed strategies for each player are the same. If Blue chooses high ground every time, Red is capable of choosing archers every time and Blue would lose 25 points per game. This is the worst that could happen to Blue. On repeated playing of the battle/game, choosing the three strategies at random using the above strategy vectors for each player to specify the probabilities gives the best outcome for each player independent of what the other player chooses. By changing the payoffs, the payoff matrix can be changed so that one player gets the advantage. For such generalizations, the mixed strategies for each player need not be the same.

Without the choice of archers, the game would express one of the maxims from the Art of War: Always pick the high ground. The presence of the archers–strategy fundamentally changes the nature of the battle, so much so that picking the high ground is not even the most favored strategy. What is favored is the "new" technology of the archer (this could be guerilla warfare or a variety of modern equivalents). The distinction is the concept of ***mixed strategies***, which take full account of the capabilities of the enemy.

This example highlights the distinctions that result from the rules-of-the-game: payoff matrix, strategy vectors, pure strategies, mixed strategies, fair games and zero-sum and non-zero sum games. Games that involve more than two players evoke additional distinctions: coalitions, discrimination, majority, *etc.* These distinctions are consequences of the payoff matrix and the identification of pure strategies. They reflect some but not all the potential forces that may enter in a dynamic game. I assert that a dynamic theory of games is determined by the forces that result from the rules-of-the-game as specified here, the choice-space-time forces and the short range behavioral forces.

The concept from the static theory of games is that every game can be analyzed, pure strategies and a game matrix identified and a rule specified that determines the mixed strategies. Thus each player has some number of pure strategies to choose from, which dictate behavior under every conceivable move of that player or random chance event during the game. Based on the pure strategy choice for each player, there will be a payoff. In general, the player succeeds not by sticking to a single pure strategy but by picking the strategies at random according to determined frequencies. The resultant mixed strategy produces their optimal strategy. I generalize this mixed strategy in the dynamic theory: Successive choices will now follow deterministically as a result of specific forces.

To go one level deeper requires specifying how the equilibrium strategies are obtained from the game matrix G_{ab}. For a mixed strategy X^a for player one and a mixed strategy Y^a for player two, the value of the game will be the trade-mark $X^a G_{ab} Y^b$. Each player optimizes his or her outcome by varying their strategy. The mathematical result of Von Neumann is that if the first player looks for a maximum and the second player then looks for the minimum, the two will be equal to the same numerical value if the order is reversed.

$$v = \max_X \min_Y \left(X^a G_{ab} Y^b \right) = \min_Y \max_X \left(X^a G_{ab} Y^b \right).$$

The expression determines the game value v and is valid for any zero-sum two-person game matrix with any number of pure strategies. Moreover, the result is equivalent to simultaneously maximizing the following two sets of equations with the game value specified above:

$$\begin{aligned} X^a G_{ab} &\geq v \\ -G_{ab} Y^b &\geq -v \end{aligned} . \tag{1.7}$$

The result is not obvious and the proof was given by Von Neumann and Morgenstern (1944). The first equation says that for player two, the worst that can happen is player one maximizes the outcome, so that the best player two can do is pick a pure strategy that minimizes the outcome; player two need not pick a mixed strategy. The second equation is analogous for player one.

Given any game and the numerical values of the game matrix, the above prescription determines the strategies and game value. If both players have two pure strategies, the prescription is not complicated and the interested reader can find solutions in the references provided in Footnote 1. When players have three or more strategies however, the prescription is more complicated. There are algorithms for solving such game theories, one of which is called linear programming which can be applied to any game and is effective in finding solutions even when the number of strategies is large.

Luce and Raiffa (1957) provide an appendix with a good summary of the method. The essential idea is to insure that the game has a positive value by adding a constant to the game matrix so that all terms are positive and then dividing the unknown value in Eq. (1.7) so that new variables appear on the left:

$$\begin{array}{ll} x^a \geq 0 & x^a G_{ab} \geq 1 \\ y^a \geq 0 & -G_{ab} y^b \geq -1 \end{array}.$$

These are constraint equations for a dual set of optimization problems: Subject to the above constraints the first seeks to minimize $\sum x^a$ by varying the vector x^a and the second seeks to maximize $\sum y^a$ by varying the vector y^a. The strategies are determined by normalizing the resultant vectors so that the sum of each is unity.

The solution is simpler for ***symmetric games*** in which each player sees the same game matrix. Since the game is zero-sum, this means that the game matrix is an anti-symmetric game matrix $G_{ab} = -G_{ba}$. For such games the game value is zero, thus Eq. (1.7) simplifies.

For symmetric games there is also a solution using differential equations due to Von Neumann and summarized by Luce and Raiffa (1957) in one of their appendices:

$$\varphi^a = G_{ab} X_b \geq 0$$
$$X = \sum_a X^a \quad \varphi = \sum_a \varphi^a .$$
$$\frac{dX^a}{dt} = \varphi^a - \varphi X^a$$

Equation (1.7) provides the first inequality and the other trade-marks serve to define the differential equation. The equation was constructed to provide a numerical solution for the strategies for any symmetric game and hence for any anti-symmetric game matrix. For large values of the independent variable t, the sum of the strategies X approaches unity and the sum of the payoffs φ approaches zero. Since all payoffs are non-negative, this provides an algorithm for determining the strategies.

My interest in this equation is that it is a prototype for a dynamic equation, with the static solution obtained as a limiting case. The key ingredient of the equation is that equilibrium strategy is a fixed point of the equation: Both sides of the equation vanish. The equation certainly contains the forces due to the rules-of-the-game. Without a full theory however, there is nothing unique about this equation.

I consider other similar equations with the idea of isolating the economic force. I note that the game matrix appears with the term φ^a, which is the product of the game matrix with the strategy vector. To highlight this ingredient, I propose an equally good equation:

$$\begin{aligned} &\frac{df^a}{dt} = f\varphi^a - \varphi f^a \\ &\varphi^a = G_{ab} f^b / f \\ &f = \sum_a f^a \end{aligned} \quad .$$

In this case the sum of the strategies f is a constant of the motion and so this looks similar to Von Neumann's equation with $X^a = f^a/f$. I transform this equation by introducing a new strategy vector y^a:

$$\frac{dy^a}{dt} = y\varphi^a .$$

The sum of these new strategies y is neither constant nor constrained to be unity. It can be shown however that the new strategies determine the same frequencies as f^a:

$$\frac{y^a}{y} = \frac{f^a}{f} .$$

The virtue of the choice is that the final form shows clearly the form of the interaction between strategies and the game matrix:

$$\frac{dy^a}{dt} = G_{ab} y^b .$$

This equation is linear in the strategy, so that it also holds for the velocity or flow $V^a = dy^a/dt$:

$$\frac{dV^a}{dt} = G_{ab}V^b .$$

I assert that this equation captures the essence of the economic or ***rules-of-the-game*** forces. The equation admits a solution that grows with time along the equilibrium direction, as well as other rotating solutions that are bounded. The frequency y^a/y approaches the equilibrium strategy. Thus the above equation trade-mark has the key characteristic of Von Neumann's equation. The static solution corresponds to the stable point where the velocity is constant on the left and the vanishing on the right results when the velocity is proportional to the equilibrium solution. The above trade-mark also describes the motion of a charged particle in a magnetic field given by the antisymmetric matrix G_{ab}. This demonstrates the assertion made at the outset that removing the restriction that the strategy vector components add to unity exhibits a rules-of-the-game force that is analogous to that generated by a magnetic field.

The result for the rules-of-the-game holds for any symmetric game. To create a satisfactory theory, this result needs to be generalized to games that are not symmetric. Again, Luce and Raiffa (1957) provide an appropriate appendix due to von Neumann that shows that any game can be put into a symmetric form..

I summarize their argument in terms of the payoff matrix[10] and strategy vectors and formulate the static theory in terms of a symmetric game and hence antisymmetric ***game matrix***. I do this for two players engaged in an arbitrary zero-sum game, with m and n pure strategies respectively. The $m \times n$ ***payoff matrix*** for player one is G. The reduced space (*i.e.* the strategy–choice and time–choice space) will have $n \oplus m \oplus 1$ dimensions. In general, the game matrix will have a ***game value*** v and ***defensive strategies*** represented by (generalized) strategy

[10] I consider the payoff matrix for one player, noting that the argument is to be repeated for each player. If all players share the same sense of value-choice, then an approximation is to represent the value-choice dimensions with a single common value-choice dimension.

vectors X and Y for each player that ensure for each respective player a best case payoff as determined in Eq. (1.7) by the product of the payoff matrix and the respective strategy vector.

I reformulate this two-person game with $m \times n$ strategies into an equivalent symmetric game in $n \oplus m \oplus 1$ dimensions with a composite payoff matrix formed from the original payoff matrix:

$$\left(F_{ab}\right)=\left(\begin{array}{c|ccc} & n & m & \text{time}=\text{"0"} \\ \hline n & 0 & -G^T & m_0^{-1}G^T X \\ m & G & 0 & -m_0^{-1}GY \\ \text{"0"} & -m_0^{-1}X^T G & m_0^{-1}Y^T G^T & 0 \end{array}\right). \tag{1.8}$$

I note some differences and similarities between this and von Neumann's formulation. In both cases there is an additional player with what is sometimes called a *hedge* strategy. I identify the hedge strategy with *time*. Von Neumann assumes that the hedge payoffs are unity. They can in general be a multiple of unity and in fact I assume that the multiple is itself proportional to the value of the game. The strategies that occur above are equilibrium strategies, so in general $G^T X$ will be a column vector whose every component is equal or exceeds the game value. A similar statement holds for GY.

The key ingredient for dynamics is that the equilibrium strategy be in the null space of the payoff matrix. In other words the equilibrium strategy times the payoff matrix is zero. For the above form of the payoff matrix, the associated null vector must be:

$$\left(\sigma^b\right)=\left(Y \quad X \quad m_0\right). \tag{1.9}$$

In addition to the game value and equilibrium strategies, there is a parameter m_0 which I am free to specify outside of the rules-of-the-game. It is not a property of the static solution.

The rules that form the composite symmetric game can be articulated by considering component parts. The composite game consists of two symmetric-players, each with all the choices of both players, as well as a choice in time. Symmetric-player one, choosing a strategy from among the m strategies of player one, plays against time and symmetric-player two, choosing a strategy from among the n strategies of player two.

Symmetric-player one will receive the amount determined from the payoff matrix G and a constant value associated with time, with each player having made the indicated choice. Thus symmetric-player one and symmetric-player two act like player one and two respectively. The payoffs to time account for the fact that the original game need not have a zero game value.

A similar argument shows that there is also the possibility that symmetric-player one and symmetric-player two reverse roles and act like player two and player one respectively. In this case, the payoff is opposite in sign. No payoffs occur when symmetric-player one and symmetric-player two make choices from m (or equivalently both from n). The game consists of two embedded versions of the same game in such a way that the composite game is symmetric, having zero expected value.

Because it is an equivalent game, the defensive strategies for the symmetric-players are determined by the strategies of the original game. Because the game is symmetric, the composite defensive strategies for each player are identical. Finally, if there are only two players, the defensive strategies are static equilibrium strategies. The form for the ***equilibrium strategy*** flow σ^b meets these conditions. Equilibrium is indicated by the fact that the payoff when either symmetric-player picks this strategy is zero: $F_{ab}\sigma^b = 0$. The equilibrium strategy depends on the strategies for player one and player two, plus a new distinction, a ***time scale*** m_0, which determines the weight of the time–choice.

1.6 Dynamic Games

I go from a static game determined by fixed mixed equilibrium strategies to a dynamic theory that is ***deterministic*** with time-dependent mixed strategies. I assume that for paths in the space of strategies that are not along the equilibrium strategy flow, there is a defined value for each player payoff matrix F^j_{ab} as well as a defined value for the flow V^b. Even if the payoff matrix were constant over the whole space, non-zero payoffs $F^j_{ab}V^b$ result when the flow is distinct from the equilibrium strategy. I propose that such non-zero payoffs in fact determine the magnitude of the forces generated by the rules-of-the-game [See Eq. (1.10) below].

The argument is extended to any number of players. With two players, the ***equilibrium strategy*** is also a ***defensive strategy***. For N players, there will be a defensive strategy even if there is no clear notion of what constitutes a stable set of equilibrium strategies. For such games, each player is defensive if he plays the game he sees (using his payoff matrix F^j_{ab}) as if the other players join together in a ***coalition*** against him. With three or more players, many such coalitions are possible: There will be effective two-person games according to each partitioning of the players into disjoint coalitions. An analysis of such coalitions was used by Von Neumann and Morgenstern to determine one definition of stable behavior.

The problem to resolve for games with more than two players will be the form of the game matrix. Although the payoff matrix G between any two players might be constant over space, the choice of strategy vectors that appear in the game matrix depends on the coalition formed, if any. Each coalition specifies a different path in space, which suggests that the game matrix has a different value along each of these paths. The full game matrix is then an interpolation between these known values. There is support for the idea introduced by Von Neumann and Morgenstern (1944) that these more complicated games have a stable pattern of behavior as opposed to a single equilibrium strategy. This pattern of behavior is determined by the behavior along the various coalitions. What I see in addition is that stable equilibrium, if it exists, need not be the set of defensive positions, nor correspond to the stable behavior patterns of von Neumann and Morgenstern.

With these arguments and caveats, I am led to a dynamic theory given by the ***economic Einstein's equation***, Eq. (1.2) that *the motion of the sources determines the choice-space-time structure and the choice-space-time structure determines the motion of the sources.* In particular, the theory determines the motion of the sources and hence the rate at which the flow changes in terms of previously discussed trade-marks:

$$\frac{DV^a}{\partial s} = g^{ab} V_j F^j_{bc} V^c + \left(g^{ab} - V^a V^b\right) \frac{\partial_b p}{\mu + p} - \tfrac{1}{2} g^{ab} V_j V_k \partial_b \gamma^{jk} \,. \quad (1.10)$$

In this monograph, I demonstrate that this form comes from a more technical exposition and articulation of the trade-mark rules of Eq. (1.2). I see the possibility that the rules lead to a quantitative relationship between the flow of the behavioral strategic-mass in terms of the metric field, the rules-of-the-game, fluid dynamic forces based on pressure gradients and structural coupling sources.

The static theory of games is a special case in which the flow of strategic-mass is constant. The dynamic theory of games corresponds to the case that the flow is not constant and its acceleration (the rate of change $DV^a/\partial s$ of the flow) is determined by forces corresponding to rules-of-the-game notions, as well as notions of behavior and possibly additional hidden sources. Though this has the sound of a metaphor from physics, it is not; rather it is a strict result of the rules applied to the relationship between the sources and the metric in choice–space.

I sketch the argument for the special case that recovers the static theory where the rules-of-the-game provide no external forces that act upon the strategic-mass. The static theory of games ignores the influence of hidden sources. All static effects are assumed to be captured by the rules-of-the-game. Still, behavioral sources are reasonably expected to move the flow of strategic-mass if some strategy choices are subject to more pressure (or control) than others: Thus the static theory of games assumes no such pressure differences. Similarly the static theory assumes no differences associated with changes in the hidden sources.

It is then plausible that the rules for the static theory relate the movement of strategic-mass only to the game matrix:

$$\frac{DV^a}{\partial s} = g^{ab} V_j F^j_{bc} V^c .$$

This is the reformulation of the Von Neumann model in Section 1.5. This is the equation for a charged fluid moving in an electro-magnetic field in choice–space, justifying the comment made earlier that the motion looks like that of a particle in a magnetic field. For the special case that the flow takes on the equilibrium value, there are no forces (right-hand side) and there is no acceleration (left-hand side).

I obtain a ***deterministic*** generalization of the static theory. There is acceleration due to non-zero payoffs as seen by each player, generated by the rules-of-the-game. The strength of each player's contribution is determined by the "charge" or ***value–scale*** V_j. Along the equilibrium strategy flow σ^b, the payoff $F^j_{ab}\sigma^b = 0$ vanishes. For a zero-sum game the equilibrium flows for each player are equal. If the equilibrium payoff flow is also constant, then the rule above is satisfied. Of course I need to say much more about the rules that govern the trade-marks to elevate this argument to something other than a sketch. I will illuminate properties of Eq. (1.10) in later chapters.

The acceleration of strategic-mass provides a generalization of the equilibrium strategy: A dynamic strategy is associated with any flow satisfying Eq. (1.10). Moreover, the acceleration of strategic-mass provides a means of calculating the behavior near equilibrium if one exists or more generally near any ***fixed point*** defined as one having zero acceleration and whose sum of forces vanishes. It satisfies the equation for all time and so obeys the rule that the flow is constant and not acted upon by an external force. For the behavior to be ***stable***, I require in addition that for a sufficiently small change away from this point, the resultant flow stays "close by" for all time. If this is not possible, then the behavior is ***unstable***. For example, in a game with three or more persons, the behavior near defensive or coalition points might be unstable. Even for this case however there could be regions with an analogous stable property: Anything that starts near such a region remains nearby. This would provide an example of the ***stable pattern of behavior*** of Von Neumann and Morgenstern (1944).

The acceleration of strategic-mass provides deterministic non-rational behaviors or curves that are reasonable extensions of rational behavior. I take as "reasonable" those extensions that approach rational behavior if the set of strategies exhibit stable behavior. The form above based on the rules-of-the-game forces was introduced by Von Neumann as a "trick" or practical proposal for calculating equilibrium strategies (See Section 1.5). My inquiry was initiated to see if this "trick" could be developed into an interesting theory of dynamics with the fixed points describing the static equilibrium. Such a theory would carry out the next phase of Von Neumann's program. In fact I do obtain an interesting theory, one in which strategic-mass is forced to move not only by rules-of-the-game forces but by behavioral and metric forces.

1.7 Nature of Time

I conclude this introductory chapter with a few comments about the nature of time in a theory of games. I would like time to correspond to the usual notion and imply some type of ***causality*** of events. Things that happen later do not impact things that happen earlier. What is usual? To say that time corresponds to the usual notion ignores the fact that controversies have historically surrounded time. The most recent accepted notions of time come from physics. In this arena, there are two conflicting categories corresponding to alternative geometries, those in which the resultant geometry is named Euclidean and those in which the resultant geometry is non-Euclidean, named Riemannian. I see two fundamental categories occurring for the dynamic theory of games as well. My analysis suggests a preference for what is called Riemannian and rests on the analogy of the magnetic field in physics with the game matrix generating the rules-of-the-game. The magnetic field generates stable behavior for charged particles moving along the field when they are sufficiently nearby. It is precisely this behavior I expect for the flow of strategic-mass near a stable defensive path for a two-person game.

This choice determines the form of the metric. The pure strategies form a space that is locally Euclidean. This means that locally I could choose an orthogonal coordinate system in which the scales are the same for the space (non-time) components. I label the scale with -1. The question raised above about Euclidean or Riemannian geometries is whether the scale for the time component has the same label or one that is the same with an opposite sign. That choice determines what is technically called the ***signature*** of the geometry and is an attribute that is true at every point in space and is the same for every possible observer. It is fundamental to the theory and not an attribute of any specific model calculation. A Euclidean signature is one in which the time component has the same sign as the space components. I choose the opposite sign, namely $+1$ corresponding to a Riemannian signature[11]. With this choice,

[11] An equivalent choice is that the space components are $+1$ and the time component is -1. The Riemannian distinction is when the choices between the space and time components are opposite. My choice is an arbitrary convention that leads to no observable consequence. Indeed, the literature in physics is split on this convention.

the game matrix is analogous to the electro–magnetic field in physics. The choice has to be viewed however as provisional, subject to change as more experience is gained with the theory.

1.8 Outline

I have provided a business sketch of a dynamic theory of games focusing more on the business distinctions than the mathematical trade-mark distinctions. As I have remarked, the rules governing the mathematical trade-marks require elaboration to further illuminate the theory and make possible quantitative predictions as the theory is ***deterministic***. I do this in subsequent chapters. There I provide an exposition of the theory and the distinctions related to economic behavior. In other words I fill in the sketch. With economic justifications, I borrow from geometry a richly constructed language and rules and borrow from physics insight and short-cuts in understanding these rules. Such short-cuts provide additional distinctions that can be used in the economic theory.

I suggest that this monograph be read at a number of levels. One level is designed to introduce a new theory that provides an understanding of economics: a dynamic theory of games. This introduction provides that level. On a second level I wish to impart to interested readers sufficient tools for them to compute behaviors from the theory. In the chapters that follow in the main body of the monograph, I have taken an approach long adopted by mathematicians: Try to make the ideas plausible to interested and trusting readers yet provide a guide for replicating and verifying the ideas for those less trusting souls like me who need to verify that the derivations are correct before accepting them as insightful. In the body of the monograph, I acknowledge that for a small number of readers, I may not have provided enough of a guide as to the correctness of the mathematical arguments, even to those with the necessary mathematical sophistication. For those readers I have provided a third level in the appendices.

The goal in this monograph is to advance our understanding of the field of economic interactions by providing a new language based on a new set of distinctions inspired in part by a science metaphor and in part by observations in the field. The power of creating language however is to come to an understanding of "what we don't know we don't know"[12]. I believe all three levels contribute to that goal.

I have organized the monograph generally along the lines of the introduction. In Chapter 2, I define the geometry of games. I introduce some common geometric notions and show how they are expressed in the language of differential geometry. In Chapter 3, I obtain a first form for the acceleration of strategic-mass. An important question of whether thermodynamics make sense in such a theory is answered in the affirmative in Appendix A. In Chapter 4, I inquire into the symmetries of a game and identify key symmetries. I obtain a more usable form for the acceleration of strategic-mass. The discussion relies on notions from differential geometry which are presented in Appendix B. A general and useful class of symmetries, called central symmetries, is defined and expanded upon in Appendix C. I provide a soluble model of the full economic Einstein equations using these ideas in Appendix D, with a numerical example for a fair game in Appendix E. In Chapter 5 I analyze the properties expected for general solutions. I describe the basic behavior when only the rules-of-the-game apply and provide an analysis of the general case with a graphical presentation in Chapter 6. A useful insight into the solutions and their existence is provided by streamlines, presented in Appendix F. I describe a generalization of the perfect fluid in Appendix G. In Chapter 7 I return to the main ideas raised in the introduction about language and suggest applications and open problems that might be treated by the theory.

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[12] "But if we could ever become reconciled to the idea that most of reality is indifferent to our descriptions of it, and that the human self is created by the use of a vocabulary rather than being adequately or inadequately expressed in a vocabulary, then we should at last have assimilated what was true in the Romantic idea that truth is made rather than found. What is true about this claim is just that "Languages" are made rather than being found, and truth is a property of linguistic entities, of sentences."—Rorty (1989).