

Chapter 1

Euclidean Geometry and Vectors

1.1 Euclidean Geometry

1.1.1 *The Postulates of Euclid*

The two Greek roots in the word **geometry**, *geo* and *metron*, mean “earth” and “a measure,” respectively, and until the early 19th century the development of this mathematical discipline relied exclusively on our visual, auditory, and tactile perception of the space in our immediate vicinity. In particular, we believe that our space is **homogeneous** (has the same properties at every point) and **isotropic** (has the same properties in every direction). The abstraction of our intuition about space is **Euclidean geometry**, named after the Greek mathematician and philosopher EUCLID, who developed this abstraction around 300 B.C.

The foundations of Euclidean geometry are five postulates concerning points and lines. A **point** is an abstraction of the notion of a position in space. A **line** is an abstraction of the path of a light beam connecting two nearby points. Thus, any two points determine a unique line passing through them. This is Euclid’s **first postulate**. The **second postulate** states that a line segment can be extended without limit in either direction. This is rather less intuitive and requires an imaginative conception of space as being infinite in extent. The **third postulate** states that, given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center, thereby recognizing the special importance of the circle and the use of straight-edge and compass to construct planar figures. The **fourth postulate** states that all right angles are equal, thereby acknowledging our perception of perpendicularity and its uniformity. The **fifth and final postulate** states that if two lines are drawn in the plane to intersect a third line in such a way that the sum of the

inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the **parallel postulate**, stating that, given a line and a point not on the line, there exists *one and only one straight line in the same plane* that passes through the point and never intersects the first line, no matter how far the lines are extended. For more information about the parallel postulate, see the book *Gödel, Escher, Bach: An Eternal Golden Braid* by D. R. Hofstadter, 1999. The parallel postulate is somewhat contrary to our physical perception of distance perspective, where in fact two lines constructed to run parallel seem to converge in the far distance.

While any geometric construction that does not exclusively rely on the five postulates of Euclid can be called non-Euclidean, the two basic **non-Euclidean geometries**, **hyperbolic** and **elliptic**, accept the first four postulates of Euclid, but use their own versions of the fifth. Incidentally, Euclidean geometry is sometimes called **parabolic**. For more information about the non-Euclidean geometries, see the book *Euclidean and Non-Euclidean Geometries: Development and History* by M. J. Greenberg, 1994.

The parallel postulate of Euclid has many implications, for example, that the sum of the angles of a triangle is 180° . Not surprisingly, this and other implications do not hold in non-Euclidean geometries. *Classical (Newtonian) mechanics assumes that the geometry of space is Euclidean*. In particular, our physical space is often referred to as the **three-dimensional Euclidean space** \mathbb{R}^3 , with \mathbb{R} denoting the set of the real numbers; the reason for this notation will become clear later, see page 7.

The development of Euclidean geometry essentially relies on our intuition that every line segment joining two points has a length associated with it. Length is measured as a multiple of some chosen unit (e.g. meter). A famous theorem that can be derived in Euclidean geometry is the theorem of Pythagoras: the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides. Exercise 1.1.4 outlines one possible proof. This theorem leads to the distance function, or **metric**, in Euclidean space when a cartesian coordinate system is chosen. The metric gives the distance between any two points by the familiar formula in terms of their coordinates (Exercise 1.1.5).

1.1.2 Relative Position and Position Vectors

Our intuitive conception and observation of position and motion suggest that the position of a point in space can only be specified relative to some other point, chosen as a reference. Likewise, the motion of a point can only be specified relative to some reference point.

The view that only relative motion exists and no meaning can be given to absolute position or absolute motion has been advocated by many prominent philosophers for many centuries. Among the famous proponents of this relativistic view were the Irish bishop and philosopher GEORGE BERKELEY (c.1685–1753), and the Austrian physicist and philosopher ERNST MACH (1836–1916). An opposing view of absolute motion also had prominent supporters, such as Sir ISAAC NEWTON (1642–1727). In 1905, the German physicist ALBERT EINSTEIN (1879–1955) and his theory of special relativity seemed to resolve the dispute in favor of the relativists (see Section 2.4 below).

Let us apply the idea of the relative position to points in the Euclidean space \mathbb{R}^3 . We choose an arbitrary point O as a reference point and call it an **origin**. Relative to O , the position of every point P in \mathbb{R}^3 is specified by the directed line segment $\mathbf{r} = \overrightarrow{OP}$ from O to P . This line segment has length $\|\mathbf{r}\| = |OP|$, the distance from O to P , and is called the **position vector** of P relative to O (the Latin word *vector* means “carrier”). Conversely, any directed line segment starting at O determines a point P . This description does not require a coordinate system to locate P .

In what follows, we denote vectors by bold letters, either lower or upper case: \mathbf{u} , \mathbf{R} . Sometimes, when the starting point O and the ending point P of the vector must be emphasized, we write \overrightarrow{OP} to denote the corresponding vector.

The position vectors, or simply **vectors**, can be added and multiplied by real numbers. With these operations of addition and multiplication, the set of all vectors becomes a *vector space*. Because of the special geometric structure of \mathbb{R}^3 , two more operations on vectors can be defined, the *dot product* and the *cross product*, and this was first done in the 1880s by the American scientist JOSIAH WILLARD GIBBS (1839–1903). We will refer to the study of the four operations on vectors (addition, multiplication by real numbers, dot product, cross product) as **vector algebra**. By contrast, **vector analysis** (also known as **vector calculus**) is the *calculus* on \mathbb{R}^3 , that is, differentiation and integration of vector-valued functions of one or

several variables. Vector algebra and vector analysis were developed in the 1880s, independently by Gibbs and by a self-taught British engineer OLIVER HEAVISIDE (1850–1925). In their developments, both Gibbs and Heaviside were motivated by applications to physics: many physical quantities, such as position, velocity, acceleration, and force, can be represented by vectors.

All constructions in vector algebra and analysis are not tied to any particular coordinate system in \mathbb{R}^3 , and do not rely on the interpretation of vectors as position vectors. Nevertheless, it is convenient to depict a vector as a line segment with an arrowhead at one end to indicate direction, and think of the length of the segment as the magnitude of the vector.

Remark 1.1 *Most of the time, we will identify all the vectors having the same direction and length, no matter the starting point. Each vector becomes a representative of an equivalence class of vectors and can be moved around by parallel translation. While this identification is convenient to study abstract properties of vectors, it is not always possible in certain physical problems (Figure 1.1.1).*

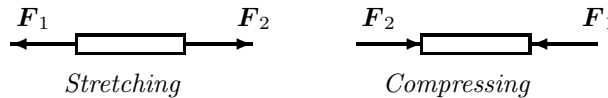


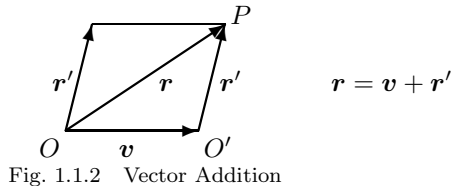
Fig. 1.1.1 Starting Point of a Vector Can Be Important!

1.1.3 Euclidean Space as a Linear Space

Consider the Euclidean space \mathbb{R}^3 and choose a point O to serve as the origin. In mechanics this is sometimes referred to as choosing a *frame of reference*, or *frame* for short. As was mentioned in Remark 1.1, we assume that all the vectors can be moved to the same starting point; this starting point defines the frame. Accordingly, in what follows, the word **frame** will have one of the three meanings:

- A fixed point;
- A fixed point with a fixed coordinate system (not necessarily Cartesian);
- A fixed point and a **vector bundle**, that is, the collection of *all* vectors that start at that point.

Let \mathbf{r} be the position vector for a point P . Consider another frame with origin O' . Let \mathbf{r}' be the position vector of P relative to O' . Now, let \mathbf{v} be the position vector of O' relative to O . The three vectors form a triangle $OO'P$; see Figure 1.1.2. This suggests that we write $\mathbf{r} = \mathbf{v} + \mathbf{r}'$. To get from O to P we can first go from O to O' along \mathbf{v} and then from O' to P along \mathbf{r}' . This can be depicted entirely with position vectors at O if we move \mathbf{r}' parallel to itself and place its initial point at O . Then \mathbf{r} is a diagonal of the parallelogram having sides \mathbf{v} and \mathbf{r}' , all emanating from O . This is called the **parallelogram law** for vector addition. It is a geometric definition of $\mathbf{v} + \mathbf{r}'$. Note that the same result is obtained by forming the triangle $OO'P$.



Now, consider three position vectors, $\mathbf{u}, \mathbf{v}, \mathbf{w}$. It is easy to see that the above definition of vector addition obeys the following algebraic laws:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} && \text{(commutativity)} \\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) && \text{(associativity; see Figure 1.1.3)} \quad (1.1.1) \\ \mathbf{u} + \mathbf{0} &= \mathbf{0} + \mathbf{u} = \mathbf{u} \end{aligned}$$

The zero vector $\mathbf{0}$ is the only vector with zero length and no specific direction.

Next consider two *real* numbers, λ and μ . In vector algebra, real numbers are called **scalars**. The vector $\lambda \mathbf{u}$ is the vector obtained from \mathbf{u} by multiplying its length by $|\lambda|$. If $\lambda > 0$, then the vectors \mathbf{u} and $\lambda \mathbf{u}$ have the same direction; if $\lambda < 0$, then the vectors have opposite directions. For example, $2\mathbf{u}$ points in the same direction as \mathbf{u} but has twice its length, whereas $-\mathbf{u}$ has the same length as \mathbf{u} and points in the opposite direction (Figure 1.1.4).

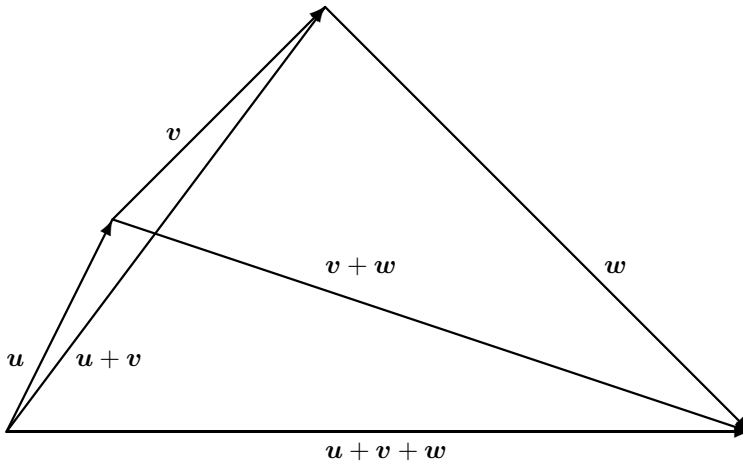


Fig. 1.1.3 Associativity of Vector Addition

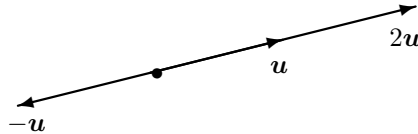


Fig. 1.1.4 Multiplication by a Scalar

Multiplication of a vector by a scalar is easily seen to obey the following algebraic rules:

$$\begin{aligned}
 \lambda(\mathbf{u} + \mathbf{v}) &= \lambda\mathbf{u} + \lambda\mathbf{v} && \text{(distributivity over vector addition)} \\
 (\lambda + \mu)\mathbf{u} &= \lambda\mathbf{u} + \mu\mathbf{u} && \text{(distributivity over real addition)} \\
 (\lambda\mu)\mathbf{u} &= \lambda(\mu\mathbf{u}) && \text{(a mixed associativity of multiplications)} \\
 1 \cdot \mathbf{u} &= \mathbf{u}.
 \end{aligned}
 \tag{1.1.2}$$

In particular, two vectors are parallel if and only if one is a scalar multiple of the other.

Definition 1.1 A (real) **vector space** is any abstract set of objects, called vectors, with operations of vector addition and multiplication by (real) scalars obeying the seven algebraic rules (1.1.1) and (1.1.2).

Note that, while the set of position vectors is a vector space, the concepts of vector length and the angle between two vectors are not included in the general definition of a vector space. A vector space is said to be *n-dimensional* if the space has a set of *n* vectors, $\mathbf{u}_1, \dots, \mathbf{u}_n$ such that any vector \mathbf{v} can be represented as a linear combination of the \mathbf{u}_i , that is, in the form,

$$\mathbf{v} = x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n, \quad (1.1.3)$$

and the scalar **components** x_1, \dots, x_n , are *uniquely* determined by \mathbf{v} . An *n-dimensional* real vector space is denoted by \mathbb{R}^n ; with \mathbb{R} denoting the set of real numbers, this notation is quite natural.

We say that the vectors \mathbf{u}_i , $i = 1, \dots, n$, form a **basis** in \mathbb{R}^n . Notice that nothing is said about the length of the basis vectors or the angles between them: in an abstract vector space, these notions do not exist. The uniqueness of representation (1.1.3) implies that the basis vectors are **linearly independent**, that is, the equality $x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n = \mathbf{0}$ holds if and only if *all the numbers* x_1, \dots, x_n are equal to zero. It is not difficult to show that a vector space is *n dimensional* if and only if the space contains *n linear independent vectors*, and every collection of *n + 1 vectors* is linear dependent; see Problem 1.7, page 411.

In the space \mathbb{R}^3 of position vectors, we do have the notions of length and angle. The standard basis in \mathbb{R}^3 is the **cartesian basis** $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, consisting of the origin O and three mutually perpendicular vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ of unit length with the common starting point O . In a cartesian basis, every position vector $\mathbf{r} = \overrightarrow{OP}$ of a point P is written in the form

$$\mathbf{r} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}}; \quad (1.1.4)$$

the numbers (x, y, z) are called the **coordinates** of the point P with respect to the **cartesian coordinate system** formed by the lines along $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. In the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, the vectors $x \hat{\mathbf{i}} + y \hat{\mathbf{j}}$ form a two-dimensional vector space \mathbb{R}^2 . With some abuse of notation, we sometimes write $\mathbf{r} = (x, y, z)$ when (1.1.4) holds and the coordinate system is fixed.

The word “cartesian” describes everything connected with the French scientist RENÉ DESCARTES (1596–1650), who was also known by the Latin version of his last name, *Cartesius*. Beside the coordinate system, which he introduced in 1637, he is famous for the statement “I think, therefore I am.”

Much of the power of the vector space approach lies in the freedom from any choice of basis or coordinates. Indeed, many geometrical concepts and results can be stated in vector terms without resorting to coordinate systems. Here are two examples:

- (1) The line determined by two points in \mathbb{R}^3 can be represented by the position vector function

$$\mathbf{r}(s) = \mathbf{u} + s(\mathbf{v} - \mathbf{u}) = s\mathbf{v} + (1 - s)\mathbf{u}, \quad -\infty < s < +\infty, \quad (1.1.5)$$

where \mathbf{u} and \mathbf{v} are the position vectors of the two points. More generally, a line passing through the point P_0 and having a **direction vector** \mathbf{d} consists of the points with position vectors $\mathbf{r}(s) = \overrightarrow{OP_0} + s\mathbf{d}$.

- (2) The plane determined by the three points having position vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is represented by the position vector function

$$\begin{aligned} \mathbf{r}(s, t) &= \mathbf{u} + s(\mathbf{v} - \mathbf{u}) + t(\mathbf{w} - \mathbf{u}) \\ &= s\mathbf{v} + t\mathbf{w} + (1 - s - t)\mathbf{u}, \quad -\infty < s, t < +\infty. \end{aligned} \quad (1.1.6)$$

EXERCISE 1.1.1.^B Verify that equations (1.1.5) and (1.1.6) indeed define a line and a plane, respectively, in \mathbb{R}^3 .

EXERCISE 1.1.2.^B Let L_1 and L_2 be two parallel lines in \mathbb{R}^3 . A line intersecting both L_1 and L_2 is called a **transversal**.

(a) Let L be a transversal perpendicular to L_1 . Prove that L is perpendicular to L_2 . Hint: If not, then there is a right triangle with L as one side, the other side along L_1 and the hypotenuse lying along L_2 . (b) Prove that the alternate angles made by a transversal are equal. Hint: Let A and B be the points of intersection of the transversal with L_1 and L_2 respectively. Draw the perpendiculars at A and B . They form two congruent right triangles.

EXERCISE 1.1.3.^B Use the result of Exercise 1.1.2(b) to prove that the sum of the angles of a triangle equals a straight angle (180°). Hint: Let A, B, C be the vertices of the triangle. Through C draw a line parallel to side AB .

EXERCISE 1.1.4.^A Let a, b be the lengths of the sides of a right triangle with hypotenuse of length c . Prove that $a^2 + b^2 = c^2$ (Pythagorean Theorem). Hint: See Figure 1.1.5 and note that the acute angles A and B are complementary: $A + B = 90^\circ$.

EXERCISE 1.1.5.^C Use the result of Exercise 1.1.4 to derive the Euclidean distance formula: $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}$

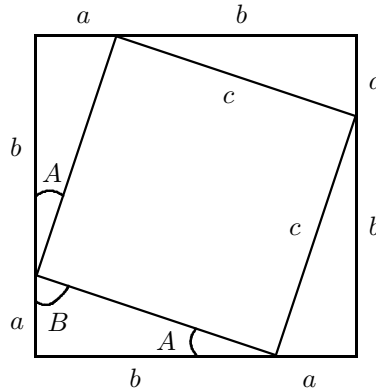


Fig. 1.1.5 Pythagorean Theorem

EXERCISE 1.1.6.^A Prove that the diagonals of a parallelogram intersect at their midpoints. Hint: let the vectors \mathbf{u} and \mathbf{v} form the parallelogram and let \mathbf{r} be the position vector of the point of intersection of the diagonals. Argue that $\mathbf{r} = \mathbf{u} + s(\mathbf{v} - \mathbf{u}) = t(\mathbf{u} + \mathbf{v})$ and deduce that $s = t = 1/2$.

1.2 Vector Operations

1.2.1 Inner Product

Euclidean geometry and trigonometry deal with lengths of line segments and angles formed by intersecting lines. In abstract vector analysis, lengths of vectors and angles between vectors are defined using the axiomatically introduced notions of norm and inner product.

In \mathbb{R}^3 , where the notions of angle and length already exist, we use these notions to *define* the inner product $\mathbf{u} \cdot \mathbf{v}$ of two vectors. We denote the length of vector \mathbf{u} by $\|\mathbf{u}\|$. A **unit vector** is a vector with length equal to one. If \mathbf{u} is a non-zero vector, then $\mathbf{u}/\|\mathbf{u}\|$ is the unit vector with the same direction as \mathbf{u} ; this unit vector is often denoted by $\hat{\mathbf{u}}$. More generally, a hat $\hat{}$ on top of a vector means that the vector has unit length. With the dot \cdot denoting the inner product of two vectors, we will sometimes write $a.b$ to denote the product of two real numbers a, b .

Definition 1.2 Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^3 . The **inner product** of \mathbf{u} and \mathbf{v} , denoted $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta, \tag{1.2.1}$$

where θ is the angle between \mathbf{u} and \mathbf{v} , $0 \leq \theta \leq \pi$ (see Figure 1.2.1), and the notation $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$ means the usual product of two numbers. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

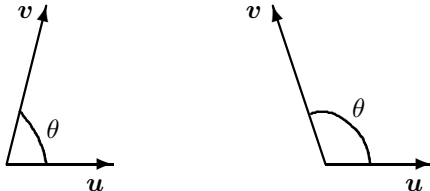


Fig. 1.2.1 Angle Between Two Vectors

Alternative names for the inner product are **dot product** and **scalar product**.

If \mathbf{u} and \mathbf{v} are non-zero vectors, then $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\theta = \pi/2$. In this case, we say that the vectors \mathbf{u} and \mathbf{v} are **orthogonal** or **perpendicular**, and write $\mathbf{u} \perp \mathbf{v}$. Notice that

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \geq 0. \quad (1.2.2)$$

In \mathbb{R}^3 , a set of three unit vectors that are mutually orthogonal is called an **orthonormal set** or **orthonormal basis**. For example, the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ of a cartesian coordinate system make an orthonormal basis. Indeed, $\hat{\mathbf{i}} \perp \hat{\mathbf{j}}$, $\hat{\mathbf{i}} \perp \hat{\mathbf{k}}$ and $\hat{\mathbf{j}} \perp \hat{\mathbf{k}}$, $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$, and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$.

The word “orthogonal” comes from the Greek *orthogonios*, or “right-angled”; the word “perpendicular” comes from the Latin *perpendicularum*, or “plumb line”, which is a cord with a weight attached to one end, used to check a straight vertical position. The Latin word *norma* means “carpenter’s square,” another device to check for right angles.

The dot product simplifies the computations of the angles between two vectors. Indeed, if \mathbf{u} and \mathbf{v} are two unit vectors, then $\mathbf{u} \cdot \mathbf{v} = \cos \theta$. More generally, for two non-zero vectors \mathbf{u} and \mathbf{v} we have

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} \right), \quad (1.2.3)$$

The notion of the dot product is closely connected with the **ORTHOGONAL PROJECTION**. If \mathbf{u} and \mathbf{v} are two non-zero vectors, then we can write $\mathbf{u} = \mathbf{u}_v + \mathbf{u}_p$, where \mathbf{u}_v is parallel to \mathbf{v} and \mathbf{u}_p is perpendicular to \mathbf{v} (see Figure 1.2.2).

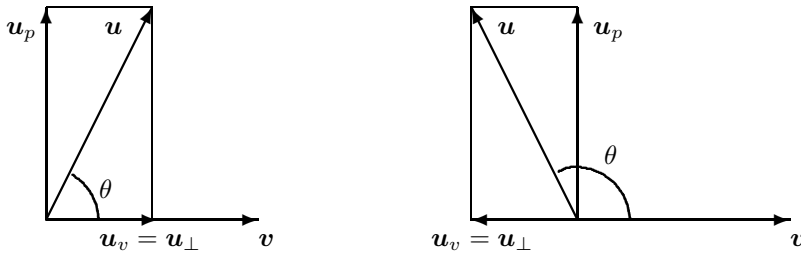


Fig. 1.2.2 Orthogonal Projection

It follows from the picture that $\|\mathbf{u}_v\| = \|\mathbf{u}\| \cdot |\cos \theta|$ and \mathbf{u}_v has the same direction as \mathbf{v} if and only if $0 < \theta < \pi/2$. Comparing this with (1.2.1) we conclude that

$$\mathbf{u}_v = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}. \tag{1.2.4}$$

The vector \mathbf{u}_v is called the **orthogonal projection** of \mathbf{u} on \mathbf{v} , and is denoted by \mathbf{u}_\perp ; the number $\mathbf{u} \cdot \mathbf{v} / \|\mathbf{v}\|^2$ is called the **component** of \mathbf{u} in the direction of \mathbf{v} ; note that $\mathbf{v} / \|\mathbf{v}\|^2$ is a unit vector. The verb “to project” comes from Latin “to through forward.” *Let us emphasize that the orthogonal projection of a vector is also a vector.*

Let us now use the idea of the orthogonal projection to establish the PROPERTIES OF THE INNER PRODUCT.

Consider two non-zero vectors \mathbf{u} and \mathbf{w} and a unit vector \mathbf{v} . Then $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v}$ is the projection of $\mathbf{u} + \mathbf{w}$ on \mathbf{v} . From Figure 1.2.3, we conclude that $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$.

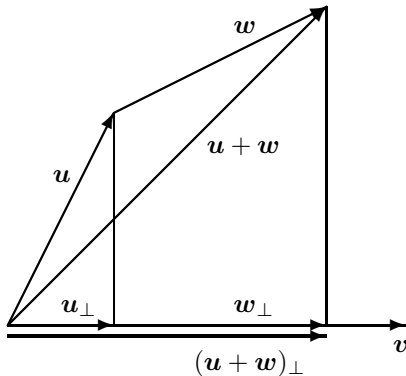


Fig. 1.2.3 Orthogonal Projection of Two Vectors

Furthermore, if λ is any real scalar, then $(\lambda\mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v})$. For example $(2\mathbf{u}) \cdot \mathbf{v} = 2(\mathbf{u} \cdot \mathbf{v})$. Also $(-\mathbf{u}) \cdot \mathbf{v} = -(\mathbf{u} \cdot \mathbf{v})$, since the angle between $-\mathbf{u}$ and \mathbf{v} is $\pi - \theta$ and $\cos(\pi - \theta) = -\cos \theta$. These observations are summarized by the formula

$$(\lambda\mathbf{u} + \mu\mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{u} \cdot \mathbf{w}) + \mu(\mathbf{v} \cdot \mathbf{w}), \quad (1.2.5)$$

where λ and μ are any real scalars.

Note that these properties of inner product are independent of any coordinate system.

Next, we will find an expression for the inner product in terms of the components of the vectors in cartesian coordinates. Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be an orthonormal set forming a cartesian coordinate system. Any position vector $\mathbf{x} = \overrightarrow{OP}$ can be expressed as $\mathbf{x} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}}$, where $x_1 = \mathbf{x} \cdot \hat{\mathbf{i}}$, $x_2 = \mathbf{x} \cdot \hat{\mathbf{j}}$ and $x_3 = \mathbf{x} \cdot \hat{\mathbf{k}}$ are the cartesian coordinates of the point P . If \mathbf{y} is another vector, then $\mathbf{y} = y_1\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + y_3\hat{\mathbf{k}}$ and by (1.2.5) and the orthonormal property of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, we get

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3. \quad (1.2.6)$$

This formula expresses $\mathbf{x} \cdot \mathbf{y}$ in terms of the coordinates of \mathbf{x} and \mathbf{y} . Together with (1.2.3), we can use the result for computing the angle between two vector with known components in a given cartesian coordinate system.

In linear algebra and in some software packages, such as MATLAB, vectors are represented as **column vectors**, that is, as 3×1 matrices; for a summary of linear algebra, see page 451. If \mathbf{x} and \mathbf{y} are column vectors, then the transpose \mathbf{x}^T is a **row vector** (1×3 matrix) and, by the rules of matrix multiplication, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.

EXERCISE 1.2.1.^C Let \mathbf{x}, \mathbf{y} be column vectors and, A a 3×3 matrix. Show that $A\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^T A\mathbf{x} = \mathbf{x}^T A^T \mathbf{y} = A^T \mathbf{y} \cdot \mathbf{x}$. Hint: $(AB)^T = B^T A^T$.

We can now summarize the main properties of the inner product:

- (I1) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
- (I2) $(\lambda\mathbf{u} + \mu\mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{u} \cdot \mathbf{w}) + \mu(\mathbf{v} \cdot \mathbf{w})$, where λ, μ are real numbers.
- (I3) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- (I4) $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u} \perp \mathbf{v}$.

Property (I4) includes the possibility $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, because, by convention, the zero vector $\mathbf{0}$ does not have a specific direction and is therefore

orthogonal to any vector. This is consistent with (I2): taking $\lambda = \mu = 1$ and $\mathbf{v} = \mathbf{0}$, we also find $\mathbf{w} \cdot \mathbf{0} = 0$ for every \mathbf{w} .

EXERCISE 1.2.2.^C Prove the law of cosines: $a^2 = b^2 + c^2 - 2bc \cos \theta$, where a, b, c are the sides of a triangle and θ is the angle between b and c . Hint: Let $c = \|\mathbf{r}_1\|$, $b = \|\mathbf{r}_2\|$. Then $a^2 = \|\mathbf{r}_2 - \mathbf{r}_1\|^2 = (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)$.

We now discuss some APPLICATIONS OF THE INNER PRODUCT. We start with the EQUATION OF A LINE IN \mathbb{R}^2 . Choose an origin O and drop the perpendicular from O to the line L ; see Figure 1.2.4.

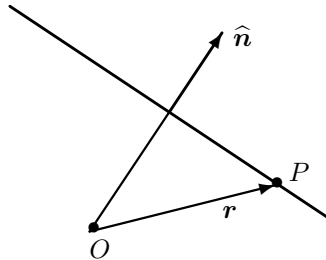


Fig. 1.2.4 Line in The Plane

Let \mathbf{n} be a unit vector lying on this perpendicular. For any point P on L , the position vector \mathbf{r} satisfies

$$\mathbf{r} \cdot \mathbf{n} = d, \quad (1.2.7)$$

where $|d|$ is the distance from O to L ; indeed, $|\mathbf{r} \cdot \mathbf{n}|$ is the length of the projection of \mathbf{r} on \mathbf{n} . In a cartesian coordinate system (x, y) , $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, and equation (1.2.7) becomes $ax + by = d$, where $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. More generally, every equation of the form $a_1x + a_2y = a_3$, with real numbers a_1, a_2, a_3 , defines a line in \mathbb{R}^2 .

Similar arguments produce the EQUATION OF A PLANE IN \mathbb{R}^3 . Let \mathbf{n} be a unit vector perpendicular to the plane. For any point P in the plane, the equation (1.2.7) holds again; Figure 1.2.4 represents the view in the plane spanned by the vectors $\hat{\mathbf{n}}$ and \mathbf{r} and containing points O, P . In a cartesian coordinate system (x, y, z) , $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and equation (1.2.7) becomes $ax + by + cz = d$, where $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$. More generally, every equation of the form $a_1x + a_2y + a_3z = a_4$ defines a plane in \mathbb{R}^3 with a (not necessarily unit) normal vector $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$. For alternative ways to represent a line and a plane see equations (1.1.5) and (1.1.6) on page 8.

EXERCISE 1.2.3.^C Using equation (1.2.7), write an equation of the plane that is 4 units from the origin and has the unit normal $\hat{\mathbf{n}} = (2, -1, 2)/3$. How many such planes are there?

EXERCISE 1.2.4.^C Let $2x - y + 2z = 12$ and $x + y - z = 1$ be the equations of two planes. Find the cosine of the angle between these planes.

Yet another application of the dot product is to computing the WORK DONE BY A FORCE. Let \mathbf{F} be a force vector acting on a mass m and moving it through a displacement given by vector \mathbf{r} . The work W done by \mathbf{F} moving m through this displacement is $W = \mathbf{F} \cdot \mathbf{r}$, since $\|\mathbf{F}\| \cos \theta$ is the magnitude of the component of \mathbf{F} along \mathbf{r} and $\|\mathbf{r}\|$ is the distance moved.

We will see later that, beside the position and force, many other mechanical quantities (acceleration, angular momentum, angular velocity, momentum, torque, velocity) can be represented as vectors.

To conclude our discussion of the dot product, we will do some ABSTRACT VECTOR ANALYSIS. The properties (I1)–(I3) of the inner product can be taken as axioms defining an inner product operation in any vector space. In other words, an inner product is a rule that assigns to any pair \mathbf{u} , \mathbf{v} of vectors a real number $\mathbf{u} \cdot \mathbf{v}$ so that properties (I1)–(I3) hold. With this approach, the definition and properties of the inner product are independent of coordinate systems.

Consider the vector space \mathbb{R}^n with a basis $\mathfrak{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$; see page 7. We can represent every element \mathbf{x} of \mathbb{R}^n as an n -tuple (x_1, \dots, x_n) of the components of \mathbf{x} in the fixed basis. Clearly, for $\mathbf{y} = (y_1, \dots, y_n)$ and $\lambda \in \mathbb{R}$,

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n), \quad \lambda \mathbf{x} = (\lambda x_1, \dots, \lambda x_n).$$

We then *define*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n = \sum_{i=1}^n x_i y_i. \quad (1.2.8)$$

It is easy to verify that this definition satisfies (I1)–(I3). For $n = 3$ with a Cartesian basis, equation (1.2.6) is a special case of (1.2.8).

If an inner product is defined in a vector space, then in view of property (I1) we can define a **norm** or **length** of a vector by

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2}. \quad (1.2.9)$$

While an inner product defines a norm, other norms in \mathbb{R}^n exist that are not inner product-based; see Problem 1.8 on page 411.

An **orthonormal basis** in \mathbb{R}^n is a basis consisting of pair-wise orthogonal vectors of unit length.

EXERCISE 1.2.5.^C Verify that, under definition (1.2.8), the corresponding basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ is necessarily orthonormal. Hint: argue that the basis vector \mathbf{u}_k is represented by an n -tuple with zeros everywhere except the position k .

EXERCISE 1.2.6.^B Prove the **parallelogram law**:

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2. \quad (1.2.10)$$

Show that in \mathbb{R}^3 this equality can be stated as follows: in a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the sides (hence the name “parallelogram law”).

Theorem 1.2.1 The norm defined by (1.2.9) satisfies the **triangle inequality**

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \quad (1.2.11)$$

and the **Cauchy-Schwartz inequality**

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|. \quad (1.2.12)$$

Proof. We first show that (1.2.11) follows from (1.2.12). Indeed,

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

To prove (1.2.12), first suppose \mathbf{u} and \mathbf{v} are unit vectors. By properties (I1)–(I3) of the inner product, for any scalar λ ,

$$0 \leq (\mathbf{u} + \lambda\mathbf{v}) \cdot (\mathbf{u} + \lambda\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\lambda\mathbf{u} \cdot \mathbf{v} + \lambda^2\mathbf{v} \cdot \mathbf{v} = 1 + 2\lambda\mathbf{u} \cdot \mathbf{v} + \lambda^2.$$

Now, take $\lambda = -(\mathbf{u} \cdot \mathbf{v})$. Then $0 \leq 1 - 2(\mathbf{u} \cdot \mathbf{v})^2 + (\mathbf{u} \cdot \mathbf{v})^2 = 1 - (\mathbf{u} \cdot \mathbf{v})^2$. Hence, $|\mathbf{u} \cdot \mathbf{v}| \leq 1$. On the other hand, for every non-zero vectors \mathbf{u} and \mathbf{v} , $\mathbf{u} = \|\mathbf{u}\| \cdot \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{v} = \|\mathbf{v}\| \cdot \mathbf{v}/\|\mathbf{v}\|$. Since $\mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{v}/\|\mathbf{v}\|$ are unit vectors, we have $|\mathbf{u} \cdot \mathbf{v}|/(\|\mathbf{u}\| \|\mathbf{v}\|) \leq 1$, and so $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$.

If either \mathbf{u} or \mathbf{v} is a zero vector, then (1.2.12) trivially holds. Theorem 1.2.1 is proved. \square

Remark 1.2 Analysis of the proof of Theorem 1.2.1 shows that equality in either (1.2.11) or (1.2.12) holds if and only if one of the vectors is a

scalar multiple of the other: $\mathbf{u} = \lambda\mathbf{v}$ or $\mathbf{v} = \lambda\mathbf{u}$ for some real number λ ; we have to write two conditions to allow either \mathbf{u} or \mathbf{v} , or both, to be the zero vector.

EXERCISE 1.2.7.^C Choose a Cartesian coordinate system (x, y, z) with the corresponding unit basis vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. Let P, Q , be points with coordinates $(1, -3, 2)$ and $(-2, 4, -1)$, respectively. Define $\mathbf{u} = \overrightarrow{OP}$, $\mathbf{v} = \overrightarrow{OQ}$.

(a) Compute $\overrightarrow{QP} = \mathbf{u} - \mathbf{v}$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$. Compute the angle between \mathbf{u} and \mathbf{v} . Verify the Cauchy-Schwartz inequality and the triangle inequality.

(b) Let $\mathbf{w} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$. Check that the associative law holds for $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

(c) Suppose \mathbf{u} is a force vector. Compute the component of \mathbf{u} in the \mathbf{v} direction. Suppose \mathbf{v} is the displacement of a unit mass acted on by the force \mathbf{u} . Compute the work done.

Inequality (1.2.12) is also known as the Cauchy-Bunyakovsky-Schwartz inequality, and all three possible combinations of any two of these three names can also refer to the same or similar inequality. This inequality is extremely useful in many areas of mathematics, and all three, Cauchy, Bunyakovsky, and Schwartz, certainly deserve to be mentioned in connection with it. The Russian mathematician VIKTOR YAKOVLEVICH BUNYAKOVSKY (1804–1889) and the German mathematician HERMANN AMANDUS SCHWARZ (1843–1921) discovered a version of (1.2.12) for the integrals:

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b f^2(x)dx \right)^{1/2} \left(\int_a^b g^2(x)dx \right)^{1/2}; \quad (1.2.13)$$

Bunyakovsky published it in 1859, Schwartz, most probably unaware of Bunyakovsky's work, in 1884. The French mathematician AUGUSTIN LOUIS CAUCHY (1789–1857) has his name attached not just to (1.2.12) but to many other mathematical results. There are two main reasons for that: he was the first to introduce modern standards of rigor in the mathematical proofs, and he published a lot of papers (789 to be exact, some exceeding 300 pages), covering most areas of mathematics. We will be mentioning Cauchy a lot during our discussion of complex analysis. Throughout the rest of our discussions, we will refer to (1.2.12) and all its modifications as the **Cauchy-Schwartz inequality**.

EXERCISE 1.2.8.^A (a) Use the same arguments as in the proof of (1.2.12) to establish (1.2.13). (b) Use the same arguments as in the proof of (1.2.12)

to establish the following version of the Cauchy-Schwartz inequality:

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2 \right)^{1/2}. \quad (1.2.14)$$

In both parts (a) and (b), assume all the necessary integrability and convergence.

We conclude this section with a brief discussion of *transformations* of a linear vector space. We will see later that a mathematical model of the motion of an object in space is a special transformation of \mathbb{R}^3 .

Definition 1.3 A transformation A of the space \mathbb{R}^n , $n \geq 2$, is a rule that assigns to every element \mathbf{x} of \mathbb{R}^n a unique element $A(\mathbf{x})$ from \mathbb{R}^n . When there is no danger of confusion, we write $A\mathbf{x}$ instead of $A(\mathbf{x})$.

A transformation A is called an **isometry** if it preserves the distances between points: $\|A\mathbf{x} - A\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ for all \mathbf{x}, \mathbf{y} in \mathbb{R}^n .

A transformation A is called **linear** if $A(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda A(\mathbf{x}) + \mu A(\mathbf{y})$ for all \mathbf{x}, \mathbf{y} from \mathbb{R}^n and all real numbers λ, μ .

A transformation is called **orthogonal** if it is both a linear transformation and an isometry.

The two Latin roots in the word “transformation,” *trans* and *forma*, mean “beyond” and “shape,” respectively. The two Greek roots in the word “isometry”, *isos* and *metron*, mean “equal” and “measure.” We know from linear algebra that, in \mathbb{R}^n with a fixed basis, every linear transformation is represented by a square matrix; see Exercise 8.1.4, page 453, in Appendix.

EXERCISE 1.2.9^A (a) Show that if A is a linear transformation, then $A(\mathbf{0}) = \mathbf{0}$. Hint: use that $\mathbf{0} = \lambda \mathbf{0}$ for all real λ .

(b) Show that the transformation A is orthogonal if and only if it preserves the inner product: $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all \mathbf{x}, \mathbf{y} from \mathbb{R}^n . Hint: use the parallelogram law (1.2.10).

1.2.2 Cross Product

In the three-dimensional vector space \mathbb{R}^3 , we use the Euclidean geometry and trigonometry to define the inner product of two vectors. This definition easily extends to every \mathbb{R}^n , $n \geq 2$. In \mathbb{R}^3 , and only in \mathbb{R}^3 , there exists another product of two vectors, called the **cross product**, or **vector product**.

Definition 1.4 Let \mathbf{u} and \mathbf{v} be two vectors in \mathbb{R}^3 . Let θ be the angle between \mathbf{u} and \mathbf{v} ($0 \leq \theta \leq \pi$, see Figure 1.2.1). The cross product, $\mathbf{u} \times \mathbf{v}$, is the vector having magnitude $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin \theta$ and lying on the line perpendicular to \mathbf{u} and \mathbf{v} and pointing in the direction in which a right-handed screw would move when \mathbf{u} is rotated toward \mathbf{v} through angle θ .

Sometimes, the symbol \odot is used to represent a vector perpendicular to the plane and coming out of the plane toward the observer, while the symbol \otimes represents a similar vector, but going away from the observer; see Figure 1.2.6.

The triple $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$ forms a *right-handed triad* (Figure 1.2.5). More generally, we say that an ordered triplet of vectors $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ with a common origin in \mathbb{R}^3 is a **right-handed triad** (or right-handed triple) if the vectors are not in the same plane and the shortest turn from \mathbf{u} to \mathbf{v} , as seen from the tip of \mathbf{w} , is *counterclockwise*.

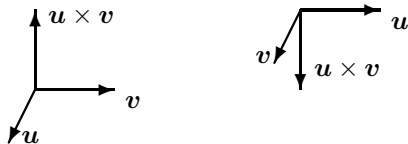


Fig. 1.2.5 The Cross Product I

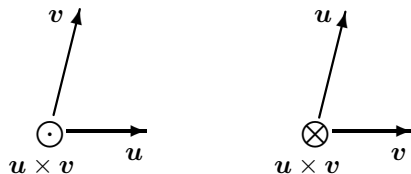


Fig. 1.2.6 The Cross Product II

An important application of cross-product in mechanics is the moment of a force about a point O . Suppose an object located at a point P is subjected to a force vector \mathbf{F} , applied at P . Let \mathbf{r} be the position vector of P . The force \mathbf{F} tends to rotate the object around O and exerts a **torque**, or **moment**, \mathbf{T} around O . (The Latin verb *torquere* means “to twist.”) The magnitude of the torque \mathbf{T} is $\|\mathbf{T}\| = \|\mathbf{r}\| \cdot \|\mathbf{F}\| \sin \theta$, where θ is the angle between \mathbf{r} and \mathbf{F} ; recall that $a \cdot b$ denotes the usual product of two numbers

a, b . The quantity $\|\mathbf{F}\| \sin \theta$ is the magnitude of the component of \mathbf{F} perpendicular to \mathbf{r} . (The component of \mathbf{F} along \mathbf{r} has no rotational effect.) The magnitude $\|\mathbf{r}\|$ is called the **moment arm**. Our experience with levers convinces us that the torque magnitude is proportional to the moment arm and the magnitude of force applied perpendicular to the arm. Hence, we define the **torque** of \mathbf{F} around O to be the vector $\mathbf{T} = \mathbf{r} \times \mathbf{F}$, where \mathbf{r} is the position at which \mathbf{F} is applied. The direction of \mathbf{T} is perpendicular to \mathbf{r} and \mathbf{F} and $(\mathbf{r}, \mathbf{F}, \mathbf{T})$ is a right-handed triad.

PROPERTIES OF THE CROSS PRODUCT. From the definition it follows immediately that the vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ has the following three properties:

- (C1) $\|\mathbf{w}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin \theta$.
 (C2) $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v} = 0$.
 (C3) $-\mathbf{w} = \mathbf{v} \times \mathbf{u}$.

A fourth property captures the geometry of the right-handed screw in algebraic terms. Choose any right-handed cartesian coordinate system given by three orthonormal vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Suppose the components of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} = \mathbf{u} \times \mathbf{v}$ in the basis $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are, respectively, $(u_1, u_2, u_3), (v_1, v_2, v_3),$ and (w_1, w_2, w_3) . Then

$$(C4) \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} > 0,$$

where \det is the determinant of the matrix; a brief review of linear algebra, including the determinants, is in Appendix. To prove (C4), choose $\hat{\mathbf{k}}' = \mathbf{w}/\|\mathbf{w}\|$, $\hat{\mathbf{j}}' = \mathbf{v}/\|\mathbf{v}\|$, and select a unit vector $\hat{\mathbf{i}}'$ orthogonal to both $\hat{\mathbf{k}}'$ and $\hat{\mathbf{j}}'$ to make $(\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}')$ a right-handed triad. In this new coordinate system, property (C4) becomes

$$\det \begin{pmatrix} u'_1 & u'_2 & 0 \\ 0 & \|\mathbf{v}\| & 0 \\ 0 & 0 & \|\mathbf{w}\| \end{pmatrix} = u'_1 \|\mathbf{v}\| \cdot \|\mathbf{w}\| > 0. \quad (1.2.15)$$

Since $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a right-handed triad, the choice of $\hat{\mathbf{i}}'$ implies that $u'_1 > 0$, and (1.2.15) holds. For the system $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ with the same origin as $(\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}')$, consider an orthogonal transformation that moves the basis vectors $\hat{\mathbf{i}}, Vcj, \hat{\mathbf{k}}$ to the vectors $\hat{\mathbf{i}}', Vcj', \hat{\mathbf{k}}'$, respectively. If B is the matrix representing this transformation in the basis $(\hat{\mathbf{i}}, Vcj, \hat{\mathbf{k}})$, then $\det B = 1$,

and the two matrices, A in (C4) and A' in (1.2.15) are related by $A' = BAB^T$. Hence, $\det A = \det A' > 0$ and (C4) holds.

EXERCISE 1.2.10.^C Verify that $A' = BAB^T$. Hint: see Exercise 8.1.4 on page 453 in Appendix. Pay attention to the basis in which each matrix is written.

The following theorem shows that the properties (C1), (C2), and (C4) define a unique vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

Theorem 1.2.2 For every two non-zero, non-parallel vectors \mathbf{u}, \mathbf{v} in \mathbb{R}^3 , there is a unique vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ satisfying (C1), (C2), (C4). If (u_1, u_2, u_3) and (v_1, v_2, v_3) are the components of \mathbf{u} and \mathbf{v} in a cartesian right-handed system $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, then the components w_1, w_2, w_3 of $\mathbf{u} \times \mathbf{v}$ are

$$w_1 = u_2v_3 - u_3v_2, \quad w_2 = u_3v_1 - u_1v_3, \quad w_3 = u_1v_2 - u_2v_1. \quad (1.2.16)$$

Conversely, the vector with components defined by (1.2.16) has Properties (C1), (C2), and (C4).

Proof. Let \mathbf{w} be a vector so that $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot \mathbf{v} = 0$, that is, \mathbf{w} is orthogonal to both \mathbf{u} and \mathbf{v} . By the geometry of \mathbb{R}^3 , there is such a vector. Choose a \mathbf{w} with magnitude $\|\mathbf{w}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$, satisfying (C1). By (C2),

$$u_1w_1 + u_2w_2 + u_3w_3 = 0, \quad (1.2.17)$$

$$v_1w_1 + v_2w_2 + v_3w_3 = 0. \quad (1.2.18)$$

Multiply (1.2.17) by v_3 and (1.2.18) by u_3 and subtract to get

$$\overbrace{(u_1v_3 - u_3v_1)}^a w_1 = \overbrace{(u_3v_2 - u_2v_3)}^b w_2. \quad (1.2.19)$$

Similarly, multiply (1.2.17) by v_1 and (1.2.18) by u_1 and subtract to get

$$\overbrace{(v_2u_1 - v_1u_2)}^c w_2 = \overbrace{(u_3v_1 - u_1v_3)}^{-a} w_3. \quad (1.2.20)$$

Abbreviating, let $a = u_1v_3 - u_3v_1$, $b = u_3v_2 - u_2v_3$ and $c = u_1v_2 - v_1u_2$. Then (1.2.19) and (1.2.20) yield

$$w_1 = (b/a)w_2; \quad w_3 = (-c/a)w_2. \quad (1.2.21)$$

Hence, $\|\mathbf{w}\|^2 = (b/a)^2w_2^2 + w_2^2 + (c/a)^2w_2^2$ and

$$\|\mathbf{w}\|^2 = (1 + (b^2 + c^2)/a^2)w_2^2 = (a^2 + b^2 + c^2)(w_2^2/a^2). \quad (1.2.22)$$

Now, by simple algebra,

$$\begin{aligned}
 a^2 + b^2 + c^2 &= (u_1 v_3 - u_3 v_1)^2 + (u_3 v_2 - u_2 v_3)^2 + (u_1 v_2 - v_1 u_2)^2 \\
 &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\
 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) \\
 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta.
 \end{aligned} \tag{1.2.23}$$

Applying Property (C1), we get $a^2 + b^2 + c^2 = \|\mathbf{w}\|^2$. Using (1.2.22), $w_2^2/a^2 = 1$. Hence, $w_2 = \pm a$ and by (1.2.21), $w_1 = \pm b$ and $w_3 = \mp c$. To determine the signs, consider the special case $\mathbf{u} = \hat{\mathbf{i}}$ and $\mathbf{v} = \hat{\mathbf{j}}$. Then $u_1 = 1, u_2 = 0$ and $v_1 = 0, v_2 = 1$ and $c = 1 \cdot 1 - 0 \cdot 0 = 1$. On the other hand, the determinant in Property (C4) for this choice of \mathbf{u} and \mathbf{v} is

$$\det \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pm b & \pm a & \mp c \end{vmatrix} = \mp c,$$

depending on whether $w_3 = -1$ or $w_3 = 1$. Since the determinant must be positive, we must take $w_2 = -a$, in order to make $w_3 = c = 1$ in this case. This implies $w_1 = -b$. Therefore, \mathbf{w} is uniquely determined and has components $w_1 = -b, w_2 = -a, w_3 = c$. In other words, there exists a unique vector with the properties (C1), (C2), and (C4), and its components are given by (1.2.16).

Conversely, let \mathbf{w} be a vector with components given by (1.2.16). Then direct computations show that \mathbf{w} has the properties (C2) and (C4). After that, we repeat the calculations in (1.2.23) to establish Property (C1). The details of this argument are the subject of Problem 1.3 on page 410.

Theorem 1.2.2 is proved. \square

Remark 1.3 Formula (1.2.16) can be represented symbolically by

$$\mathbf{u} \times \mathbf{v} = \det \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{1.2.24}$$

and expanding the determinant by co-factors of the first row. Together with properties of the determinant, this representation implies Property (C3) of the cross product. Also, when combined with (C1), formula (1.2.24) can be used to compute the angle between two vectors with known components. Still, given the extra complexity of evaluating the determinant, the inner

product formulas (1.2.3) and (1.2.6) are usually more convenient for angle computations.

Remark 1.4 From (1.2.16) it follows that $(\lambda \mathbf{u}) \times \mathbf{v} = \lambda(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \lambda \mathbf{v}$ for any scalar λ . Another consequence of (1.2.16) is the distributive property of the cross product:

$$\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}. \quad (1.2.25)$$

Still, the cross product is not associative; instead, the following identity holds:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}. \quad (1.2.26)$$

EXERCISE 1.2.11.^A Prove that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \quad (1.2.27)$$

Then use the result to verify (1.2.26). Hint: A possible proof of (1.2.27) is as follows (fill in the details). Choose an orthonormal basis $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ so that $\hat{\mathbf{i}}$ is parallel to \mathbf{w} and $\hat{\mathbf{j}}$ is in the plane of \mathbf{w} and \mathbf{v} . Then $\mathbf{w} = w_1 \hat{\mathbf{i}}$ and $\mathbf{v} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}}$ and

$$\mathbf{v} \times \mathbf{w} = \det \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & 0 \\ w_1 & 0 & 0 \end{vmatrix} = -v_2 w_1 \hat{\mathbf{k}};$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \det \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ 0 & 0 & -v_2 w_1 \end{vmatrix} = -u_2 v_2 w_1 \hat{\mathbf{i}} + u_1 v_2 w_1 \hat{\mathbf{j}};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = u_1 w_1 (v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}}) - (u_1 v_1 + u_2 v_2) w_1 \hat{\mathbf{i}} = -u_2 v_2 w_1 \hat{\mathbf{i}} + u_1 v_2 w_1 \hat{\mathbf{j}}.$$

While the properties (C1)–(C4) of the cross product are independent of the coordinate system, the definition does not generalize to \mathbb{R}^n for $n \geq 4$ because in dimension $n \geq 4$ there are too many vectors orthogonal to two given vectors.

Property (C1) implies that $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram generated by the vectors \mathbf{u} and \mathbf{v} . Accordingly, we have $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if one of the vectors is a scalar multiple of the other. If P_1, P_2, P_3 are three points in \mathbb{R}^3 , these points are **collinear** (lie on the same line) if and only if

$$\overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3} = \mathbf{0}, \quad (1.2.28)$$

where $\overrightarrow{P_i P_j} = \overrightarrow{OP_j} - \overrightarrow{OP_i}$. If (x_i, y_i, z_i) are the cartesian coordinates of the point P_i , then the criterion for collinearity (1.2.28) becomes

$$\det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = \mathbf{0}. \quad (1.2.29)$$

In the following three exercises, the reader will see how the mathematics of vector algebra can be used to solve problems in physics.

EXERCISE 1.2.12.^C Suppose two forces $\mathbf{F}_1, \mathbf{F}_2$ are applied at P ; $\mathbf{r} = \overrightarrow{OP}$. Show that the total torque at P is $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, where $\mathbf{T}_1 = \mathbf{r} \times \mathbf{F}_1$ and $\mathbf{T}_2 = \mathbf{r} \times \mathbf{F}_2$.

EXERCISE 1.2.13.^A Consider a rigid rod with one end fixed at the origin O but free to rotate in any direction around O (say by means of a ball joint). Denote by P the other end of the rod; $\mathbf{r} = \overrightarrow{OP}$. Suppose a force \mathbf{F} is applied at the point P . The rod will tend to rotate around O .

(a) Let $\mathbf{r} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\mathbf{F} = \hat{i} + \hat{j} + \hat{k}$. Compute the torque \mathbf{T} . (b) Let $\mathbf{r} = 2\hat{i} + 4\hat{j}$ and $\mathbf{F} = \hat{i} + \hat{j}$, so that the rotation is in the (\hat{i}, \hat{j}) plane. Compute \mathbf{T} . In which direction will the rod start to rotate?

EXERCISE 1.2.14.^A Suppose a rigid rod is placed in the (\hat{i}, \hat{j}) plane so that the mid-point of the rod is at the origin O , and the two ends P and P_1 have position vectors $\mathbf{r} = \hat{i} + 2\hat{j}$ and $\mathbf{r}_1 = -\hat{i} - 2\hat{j}$. Suppose the rod is free to rotate around O in the (\hat{i}, \hat{j}) plane. Let $\mathbf{F} = \hat{i} + \hat{j}$ and $\mathbf{F}_1 = -\hat{i} - \hat{j}$ be two forces applied at P and P_1 , respectively. Compute the total torque around O . In which direction will the rod start to rotate?

1.2.3 Scalar Triple Product

The scalar triple product $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of three vectors is defined by

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Using (1.2.24) it is easy to see that, in cartesian coordinates,

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

From the properties of determinants it follows that

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -(\mathbf{v}, \mathbf{u}, \mathbf{w}) = (\mathbf{v}, \mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{u}, \mathbf{v}).$$

Thus,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}. \quad (1.2.30)$$

In other words, the scalar triple product does not change under cyclic permutation of the vectors or when \cdot and \times symbols are switched.

EXERCISE 1.2.15.^C Verify that the ordered triplet of non-zero vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a right-handed triad if and only if $(\mathbf{u}, \mathbf{v}, \mathbf{w}) > 0$.

Recall that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \sin \theta$ is the area of the parallelogram formed by \mathbf{v} and \mathbf{w} . Therefore, $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped formed by \mathbf{u}, \mathbf{v} , and \mathbf{w} . Accordingly, $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$ if and only if the three vectors are *linearly dependent*, that is, one of them can be expressed as a linear combination of the other two. Similarly, four points $P_i, i = 1, \dots, 4$ are **co-planar** (lie in the same plane) if and only if

$$(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P_4}) = 0, \quad (1.2.31)$$

where $\overrightarrow{P_iP_j} = \overrightarrow{OP_j} - \overrightarrow{OP_i}$. If (x_i, y_i, z_i) are the cartesian coordinates of the point P_i , then (1.2.31) becomes

$$\det \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0. \quad (1.2.32)$$

Notice a certain analogy with (1.2.28) and (1.2.29).

EXERCISE 1.2.16.^C Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (-2, 1, 2)$, $\mathbf{w} = (-1, 2, 1)$. (a) Compute $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{w}$, $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$. (b) Compute the area of the parallelogram formed by \mathbf{u} and \mathbf{v} . (c) Compute the volume of the parallelepiped formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ using the triple product $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

1.3 Curves in Space

1.3.1 Vector-Valued Functions of a Scalar Variable

To study the mathematical kinematics of moving bodies in \mathbb{R}^3 , we need to define the velocity and acceleration vectors. The rigorous definition of these vectors relies on the concept of the derivative of a vector-valued function with respect to a scalar. We consider an idealized object, called a **point mass**, with all mass concentrated at a single point.

Choose an origin O and let $\mathbf{r}(t)$ be the position vector of the point mass at time t . The collection of points $P(t)$ so that $\overrightarrow{OP}(t) = \mathbf{r}(t)$ is the trajectory of the point mass. This trajectory is a **curve** in \mathbb{R}^3 . More generally, a curve \mathcal{C} is defined by specifying the position vector of a point P on \mathcal{C} as a function of a scalar variable t .

Definition 1.5 A curve \mathcal{C} in a frame O in \mathbb{R}^3 is the collection of points defined by a vector-valued function $\mathbf{r} = \mathbf{r}(t)$, for t in some interval I in \mathbb{R} , bounded or unbounded. A point P is on the curve \mathcal{C} if and only if $\overrightarrow{OP} = \mathbf{r}(t_0)$ for some $t_0 \in I$. A curve is called **simple** if it does not intersect or touch itself. A curve is called **closed** if it is defined for t in a bounded closed interval $I = [a, b]$ and $\mathbf{r}(a) = \mathbf{r}(b)$. For a simple closed curve on $[a, b]$, we have $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, $a \leq t_1 < t_2 \leq b$ if and only if $t_1 = a$ and $t_2 = b$.

By analogy with the elementary calculus, we say that the vector function \mathbf{r} is **continuous** at t_0 if

$$\lim_{t \rightarrow t_0} \|\mathbf{r}(t) - \mathbf{r}(t_0)\| = 0. \quad (1.3.1)$$

Accordingly, we say that the curve \mathcal{C} is **continuous** if the vector function that defines \mathcal{C} is continuous.

Similarly, the **derivative** at t_0 of a vector-valued function $\mathbf{r}(t)$ is, by definition,

$$\left. \frac{d\mathbf{r}}{dt} \right|_{t=t_0} \equiv \mathbf{r}'(t_0) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)}{\Delta t}. \quad (1.3.2)$$

We say that \mathbf{r} is **differentiable** at t_0 if the derivative $\mathbf{r}'(t)$ exists at t_0 ; we say that \mathbf{r} is *differentiable on* (a, b) if $\mathbf{r}'(t)$ exists for all $t \in (a, b)$. We say that the curve is **smooth** if the corresponding vector function is differentiable and the derivative is not a zero vector.

Yet another notation for the derivative $\mathbf{r}'(t)$ is $\dot{\mathbf{r}}(t)$, especially when the parameter t is interpreted as time. For a scalar function of time $x = x(t)$, the same notations for the derivative are used:

$$\frac{dx}{dt} \equiv x'(t) \equiv \dot{x}(t).$$

Note that $\mathbf{r}(t + \Delta t) - \mathbf{r}(t) = \Delta\mathbf{r}(t)$ is a vector in the same frame O . The limits in (1.3.1) and (1.3.2) are defined by using the distance, or metric, for vectors. Thus, $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$ means that $\|\mathbf{r}(t) - \mathbf{r}(t_0)\| \rightarrow 0$ as $t \rightarrow t_0$. The derivative $\mathbf{r}'(t)$, being the limit of the difference quotient $\Delta\mathbf{r}(t)/\Delta t$ as $\Delta t \rightarrow 0$, is also a vector.

Given a fixed frame O , the formulas of differential calculus for vector functions in this frame are easily obtained by following the corresponding derivations for scalar functions in ordinary calculus. As in ordinary calculus, there are several rules for computing derivatives of vector-valued functions. All these rules follow directly from the definition (1.3.2).

The derivative of a sum:

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t). \quad (1.3.3)$$

Product rule for multiplication by a scalar: if $\lambda(t)$ is a scalar function, then

$$\frac{d}{dt}(\lambda(t)\mathbf{r}(t)) = \lambda'(t)\mathbf{r}(t) + \lambda(t)\mathbf{r}'(t). \quad (1.3.4)$$

Product rules for scalar and cross products:

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt}, \quad (1.3.5)$$

and

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}. \quad (1.3.6)$$

The chain rule: If $t = \phi(s)$ and $\mathbf{r}_1(s) = \mathbf{r}(\phi(s))$, then

$$\frac{d\mathbf{r}_1}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{d\phi}{ds}. \quad (1.3.7)$$

From the two rules (1.3.3) and (1.3.4), it follows that if $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are constant vectors in the frame O so that $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$, then $\mathbf{r}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}$.

Remark 1.5 *The underlying assumption in the above rules for differentiation of vector functions is that all the functions are defined in the same frame. We will see later that these rules for computing derivatives can fail if the vectors are defined in different frames and the frames are moving relative to each other.*

Lemma 1.1 *If \mathbf{r} is differentiable on (a, b) and $\|\mathbf{r}(t)\|$ does not depend on t for $t \in (a, b)$, then $\mathbf{r}(t) \perp \mathbf{r}'(t)$ for all $t \in (a, b)$. In other words, the derivative of a constant-length vector is perpendicular to the vector itself.*

Proof. By assumption, $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant for all t . By the product rule (1.3.5), $2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ and the result follows. \square

EXERCISE 1.3.1.^A (a) Show that if \mathbf{r} is differentiable at t_0 , then \mathbf{r} is continuous at t_0 , but the converse is not true. (b) Does continuity of \mathbf{r} imply continuity of $\|\mathbf{r}\|$? Does continuity of $\|\mathbf{r}\|$ imply continuity of \mathbf{r} ? (c) Does differentiability of \mathbf{r} imply differentiability of $\|\mathbf{r}\|$? Does differentiability of $\|\mathbf{r}\|$ imply differentiability of \mathbf{r} ?

The complete description of every curve consists of two parts: (a) the set of its points in \mathbb{R}^3 , (b) the ordering of those points relative to the ordering of the parameter set. For some curves, this complete description is possible in purely vector terms, that is, without choosing a particular coordinate system in the frame O . For other curves, a purely vector description provides only the set of points, while the ordering of that set is impossible without the selection of the particular coordinate system. We illustrate this observation on two simple curves: a straight line and a circle.

A straight line is described by $\mathbf{r}(t) = \mathbf{r}_2 - \phi(t)(\mathbf{r}_1 - \mathbf{r}_2)$, $-\infty < t < \infty$, where \mathbf{r}_1 and \mathbf{r}_2 are the position vectors of two distinct points on the line and $\phi(t)$ is a scalar function whose range is all of \mathbb{R} . The function ϕ determines the ordering of the points on the line. For example, if $\phi(t) = t$, then the point $\mathbf{r}(t_2)$ follows $\mathbf{r}(t_1)$ in time if $t_2 > t_1$.

The circle as a set of points in \mathbb{R}^3 is defined by the two conditions, $\|\mathbf{r}(t)\| = R$ and $\mathbf{r}(t) \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ is the unit normal to the plane of the circle. Direct computations show that these conditions do not determine the function $\mathbf{r}(t)$ uniquely, and so do not give an ordering of points on the circle. To specify the ordering, we can, for example, fix one point $\mathbf{r}(t_0)$ on the circle at a reference time t_0 and define the angle between $\mathbf{r}(t)$ and $\mathbf{r}(t_0)$ as a function of t . But this is equivalent to choosing a polar coordinate system in the plane of the circle.

1.3.2 The Tangent Vector and Arc Length

Let $\mathbf{r} = \mathbf{r}(t)$ define a curve in \mathbb{R}^3 . If $\overrightarrow{OP} = \mathbf{r}(t_0)$ and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then, by definition, the unit tangent vector $\hat{\mathbf{u}}$ at P is:

$$\hat{\mathbf{u}}(t_0) = \frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|} \quad (1.3.8)$$

Note that the vector $\Delta\mathbf{r} = \mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)$ defines a line through two points on the curve; similar to ordinary calculus, definition (1.3.2) suggests that the vector $\mathbf{r}'(t_0)$ should be parallel to the tangent line at P .

The equation of the tangent line at point P is

$$\mathbf{R}(s) = \mathbf{r}(t_0) + s\hat{\mathbf{u}}(t_0). \quad (1.3.9)$$

EXERCISE 1.3.2.^C Let \mathcal{C} be a planar curve defined by the vector function $\mathbf{r}(t) = \cos t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}}$, $-\pi < t < \pi$. Compute the tangent vector $\mathbf{r}'(t)$ and the unit tangent vector $\hat{\mathbf{u}}(t)$ as functions of t . Compute $\mathbf{r}'(0)$ and $\hat{\mathbf{u}}(0)$. Draw the curve \mathcal{C} and the vectors $\mathbf{r}'(0)$, $\hat{\mathbf{u}}'(0)$. Verify your results using a computer algebra system, such as MAPLE, MATLAB, or MATHEMATICA.

EXERCISE 1.3.3.^C Let \mathcal{C} be a spatial curve defined by the vector function $\mathbf{r}(t) = \cos t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}} + t \hat{\mathbf{k}}$. Compute the tangent vector $\mathbf{r}'(t)$, the unit tangent vector $\hat{\mathbf{u}}(t)$ and the vector $\hat{\mathbf{u}}'(t)$. Compute $\mathbf{r}'(\pi/2)$. Draw the curve \mathcal{C} for $0 \leq t \leq \pi/2$ and draw $\hat{\mathbf{u}}'(\pi/2)$ at the point $\mathbf{r}(\pi/2)$. Verify your results using your favorite computer algebra system.

Definition 1.6 A curve \mathcal{C} , defined by a vector function $\mathbf{r}(t)$, $a < t < b$, is called **smooth** if the unit tangent vector $\hat{\mathbf{u}} = \hat{\mathbf{u}}(t)$ exists and is a continuous function for all $t \in (a, b)$. If the curve is closed, then, additionally, we must have $\mathbf{r}'(a) = \mathbf{r}'(b)$. The curve is called **piece-wise smooth** if it is continuous and consists of finitely many smooth pieces.

EXERCISE 1.3.4.^A Give an example of a non-smooth curve \mathcal{C} defined by a vector function $\mathbf{r}(t)$, $-1 < t < 1$, so that the derivative vector $\mathbf{r}'(t)$ exists and is continuous for all $t \in (-1, 1)$.

EXERCISE 1.3.5.^C Explain how the graph of a function $y = f(x)$ can be interpreted as a curve in \mathbb{R}^3 . Show that this curve is smooth if and only if the function $f = f(x)$ has a continuous derivative, and show that, at the point $(x_0, f(x_0), 0)$, formula (1.3.9) defines the same line as $y = f(x_0) + f'(x_0)(x - x_0)$, $z = 0$.

Given a curve \mathcal{C} and two points with position vectors $\mathbf{r}(c), \mathbf{r}(d)$, $a < c \leq d < b$, on the curve, we define the distance between the two points *along the curve* using a limiting process. The construction is similar to the definition of the Riemann integral in ordinary calculus.

For each $n \geq 2$, choose points $c = t_0 < t_1 < \cdots < t_n = d$ and form the sums $L_n = \sum_{i=0}^{n-1} \|\Delta \mathbf{r}_i\|$, where $\Delta \mathbf{r}_i = \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)$. Assume that $\max_{0 \leq i \leq n-1} (t_{i+1} - t_i) \rightarrow 0$ as $n \rightarrow \infty$. If the limit $\lim_{n \rightarrow \infty} L_n$ exists for all $a < c < d < b$, and does not depend on the particular choice of the points t_k , then the curve \mathcal{C} is called **rectifiable**. By definition, the distance

$L_{\mathcal{C}}(c, d)$ between the points $\mathbf{r}(c)$ and $\mathbf{r}(d)$ along a rectifiable curve \mathcal{C} is

$$L_{\mathcal{C}}(c, d) = \lim_{n \rightarrow \infty} L_n,$$

Theorem 1.3.1 Assume that $\mathbf{r}'(t)$ exists for all $t \in (a, b)$ and the vector function $\mathbf{r}'(t)$ is continuous. Then the curve \mathcal{C} is rectifiable and

$$L_{\mathcal{C}}(c, d) = \int_c^d \|\mathbf{r}'(t)\| dt. \quad (1.3.10)$$

Proof. It follows from the assumptions of the theorem and from relation (1.3.2) that $\Delta \mathbf{r}_i = \mathbf{r}'(t_i) \Delta t_i + \mathbf{v}_i$, where $\Delta t_i = t_{i+1} - t_i$ and the vectors \mathbf{v}_i satisfy $\max_{0 \leq i \leq n-1} \|\mathbf{v}_i\| / \Delta t_i \rightarrow 0$ as $\max_{0 \leq i \leq n-1} \Delta t_i \rightarrow 0$. Therefore,

$$\begin{aligned} \|\Delta \mathbf{r}_i\| &= \|\mathbf{r}'(t_i)\| \Delta t_i + \varepsilon_i \Delta t_i, \\ \sum_{i=0}^{n-1} \|\Delta \mathbf{r}_i\| &= \sum_{i=0}^{n-1} \|\mathbf{r}'(t_i)\| \Delta t_i + \sum_{i=0}^{n-1} \varepsilon_i \Delta t_i, \end{aligned} \quad (1.3.11)$$

where the numbers ε_i satisfy $\max_{0 \leq i \leq n-1} \varepsilon_i \rightarrow 0$, $n \rightarrow \infty$. Then (1.3.10) follows after passing to the limit. \square

EXERCISE 1.3.6^B (a) Verify (1.3.11). Hint: use the triangle inequality to estimate $\|\mathbf{r}'(t_i) \Delta t_i + \mathbf{v}_i\| - \|\mathbf{r}'(t_i)\| \Delta t_i$. (b) Show that a piece-wise smooth curve is rectifiable. Hint: apply the above theorem to each smooth piece separately, and then add the results.

EXERCISE 1.3.7.^C Interpreting the graph of the function $y = f(x)$ as a curve in \mathbb{R}^3 , and assuming that $f'(x)$ exists and is continuous, show that the length of this curve from $(c, f(c), 0)$ to $(d, f(d), 0)$, as given by (1.3.10), is $\int_c^d \sqrt{1 + |f'(x)|^2} dx$; the derivation of this result in ordinary calculus is similar to the derivation of (1.3.10).

Given a point $\mathbf{r}(c)$ on a rectifiable curve \mathcal{C} , we define the arc length function $\mathfrak{s} = \mathfrak{s}(t)$, $t \geq c$, as

$$\mathfrak{s}(t) = L_{\mathcal{C}}(c, t)$$

It follows that $d\mathfrak{s}/dt = \|\mathbf{r}'(t)\| \geq 0$. We call $d\mathfrak{s} = \|\mathbf{r}'(t)\| dt$ the line element of the curve \mathcal{C} . If $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$, where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ is a cartesian coordinate system at O , then

$$\left(\frac{d\mathfrak{s}}{dt}\right)^2 = \left\|\frac{d\mathbf{r}}{dt}\right\|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2. \quad (1.3.12)$$

If the curve is smooth, then $d\mathfrak{s}/dt > 0$ and \mathfrak{s} is a monotone function of t so that t is a well-defined function of \mathfrak{s} . Hence, $\mathbf{r}(t(\mathfrak{s}))$ is a function of \mathfrak{s} , and is called the **canonical parametrization** of the smooth curve by the arc length. By the rules of differentiation,

$$\frac{d\mathbf{r}}{d\mathfrak{s}} = \frac{d\mathbf{r}}{dt} \frac{dt}{d\mathfrak{s}} = \frac{d\mathbf{r}}{dt} \frac{1}{d\mathfrak{s}/dt} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \hat{\mathbf{u}}(t).$$

EXERCISE 1.3.8.^C Consider the **right-handed circular helix**

$$\mathbf{r}(t) = a \cos t \hat{\mathbf{i}} + a \sin t \hat{\mathbf{j}} + t \hat{\mathbf{k}}, \quad a > 0. \quad (1.3.13)$$

Re-write the equation of this curve using the arc length \mathfrak{s} as the parameter.

1.3.3 Frenet's Formulas

In certain frames, called inertial, the **Second Law of Newton** *postulates* the following relation between the force $\mathbf{F} = \mathbf{F}(t)$ acting on the point mass m and the point's trajectory \mathcal{C} , defined by a curve $\mathbf{r} = \mathbf{r}(t)$:

$$m \frac{d^2 \mathbf{r}(t)}{dt^2} = \mathbf{F}(t). \quad (1.3.14)$$

A detailed discussion of inertial frames and Newton's Laws is below on page 43. When $\mathbf{F}(t)$ is given, the solution of the differential equation (1.3.14) is the trajectory $\mathbf{r}(t)$. However, to get a unique solution of (2.1.1), we must start at some time t_0 and provide two initial conditions $\mathbf{r}'(t_0)$ and $\mathbf{r}(t_0)$ to determine a specific path. In other words, $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ are reference vectors for the motion. At every time $t > t_0$, the vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have a well-defined geometric orientation relative to the initial vectors $\mathbf{r}(t_0)$, $\mathbf{r}'(t_0)$. The three *Frenet formulas* provide a complete description of this orientation. In what follows, we assume that the curve \mathcal{C} is smooth, that is, the unit tangent vector $\hat{\mathbf{u}}$ exists at every point of the curve.

To write the formulas, we need several new notions: *curvature*, *principal unit normal vector*, *unit binormal vector*, and *torsion*. We will use the canonical parametrization of the curve by the arc length \mathfrak{s} measured from some reference point P_0 on the curve.

Let $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathfrak{s})$ be the unit tangent vector at P , where the parameter \mathfrak{s} is the arc length from P_0 to P . By Lemma 1.1 on page 26, the derivative $\hat{\mathbf{u}}'(\mathfrak{s})$ of $\hat{\mathbf{u}}(\mathfrak{s})$ with respect to \mathfrak{s} is orthogonal to $\hat{\mathbf{u}}$. By definition, the **curvature** $\kappa(\mathfrak{s})$ at P is

$$\kappa(\mathfrak{s}) = \|\hat{\mathbf{u}}'(\mathfrak{s})\|;$$

the principal unit normal vector at P is

$$\widehat{\mathbf{p}} = \frac{1}{\kappa} \widehat{\mathbf{u}}'(\mathfrak{s}); \quad (1.3.15)$$

the unit binormal vector at P is

$$\widehat{\mathbf{b}}(\mathfrak{s}) = \widehat{\mathbf{u}}(\mathfrak{s}) \times \widehat{\mathbf{p}}(\mathfrak{s}).$$

EXERCISE 1.3.9.^C *Parameterizing the circle by the arc length, verify that the curvature of the circle of radius R is $1/R$.*

To define the torsion, we derive the relation between $\widehat{\mathbf{b}}'(\mathfrak{s})$ and the vectors $\widehat{\mathbf{u}}, \widehat{\mathbf{p}}, \widehat{\mathbf{b}}$. Using Lemma 1.1 once again, we conclude that $\widehat{\mathbf{b}}'(\mathfrak{s})$ is orthogonal to $\widehat{\mathbf{b}}(\mathfrak{s})$. Next, we differentiate the relation $\widehat{\mathbf{b}}(\mathfrak{s}) \cdot \widehat{\mathbf{u}}(\mathfrak{s}) = 0$ with respect to \mathfrak{s} and use the product rule (1.3.5) to find $\widehat{\mathbf{b}}'(\mathfrak{s}) \cdot \widehat{\mathbf{u}}(\mathfrak{s}) + \widehat{\mathbf{b}}(\mathfrak{s}) \cdot \widehat{\mathbf{u}}'(\mathfrak{s}) = 0$. By construction, the unit vectors $\widehat{\mathbf{u}}, \widehat{\mathbf{p}}, \widehat{\mathbf{b}}$ are mutually orthogonal, and then the definition (1.3.15) of the vector $\widehat{\mathbf{p}}$ implies that $\widehat{\mathbf{b}}(\mathfrak{s}) \cdot \widehat{\mathbf{u}}'(\mathfrak{s}) = 0$. As a result, $\widehat{\mathbf{b}}'(\mathfrak{s}) \cdot \widehat{\mathbf{u}}(\mathfrak{s}) = 0$. Being orthogonal to both $\widehat{\mathbf{u}}(\mathfrak{s})$ and $\widehat{\mathbf{b}}(\mathfrak{s})$, the vector $\widehat{\mathbf{b}}'(\mathfrak{s})$ must then be parallel to $\widehat{\mathbf{p}}(\mathfrak{s})$. We therefore define the torsion of the curve \mathcal{C} at point P as the number $\tau = \tau(\mathfrak{s})$ so that

$$\widehat{\mathbf{b}}'(\mathfrak{s}) = -\tau(\mathfrak{s}) \widehat{\mathbf{p}}(\mathfrak{s}); \quad (1.3.16)$$

the choice of the negative sign ensures that the torsion is positive for the right-handed circular helix (1.3.13).

Note that the above definitions use the canonical parametrization of the curve by the arc length \mathfrak{s} ; the corresponding formulas can be written for an arbitrary parametrization as well; see Problem 1.11 on page 412.

Relations (1.3.15) and (1.3.16) are two of the Frenet formulas. To derive the third formula, note that $\widehat{\mathbf{p}}(\mathfrak{s}) = \widehat{\mathbf{b}}(\mathfrak{s}) \times \widehat{\mathbf{u}}(\mathfrak{s})$. Differentiation with respect to \mathfrak{s} yields $\widehat{\mathbf{p}}' = \widehat{\mathbf{b}} \times \widehat{\mathbf{u}}' + \widehat{\mathbf{b}}' \times \widehat{\mathbf{u}} = \widehat{\mathbf{b}} \times \kappa \widehat{\mathbf{p}} - \tau \widehat{\mathbf{p}} \times \widehat{\mathbf{u}}$, and

$$\widehat{\mathbf{p}}'(\mathfrak{s}) = -\kappa \widehat{\mathbf{u}}(\mathfrak{s}) + \tau \widehat{\mathbf{b}}(\mathfrak{s}). \quad (1.3.17)$$

Different sources refer to relations (1.3.15) – (1.3.17) as either the **Frenet** or the **Frenet–Serret** formulas. In 1847, the French mathematician JEAN FRÉDÉRIC FRENET (1816–1900) derived two of these formulas in his doctoral dissertation. Another French mathematician, JOSEPH ALFRED SERRET (1819–1885), gave an independent derivation of all three formulas, but we could not find the exact time of his work. Of course, neither Frenet nor Serret used the modern vector notations in their derivations.

At every point P of the curve, the vector triple $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{b}})$ is a right-handed coordinate system with origin at P . We will call this coordinate system **Frenet's trihedron** at P . The choice of initial conditions $\mathbf{r}(t_0), \mathbf{r}'(t_0)$ means setting up a coordinate system in the frame with origin at P_0 , where $\overrightarrow{OP_0} = \mathbf{r}(t_0)$. The coordinate planes spanned by the vectors $(\hat{\mathbf{u}}, \hat{\mathbf{p}})$, $(\hat{\mathbf{p}}, \hat{\mathbf{b}})$, and $(\hat{\mathbf{b}}, \hat{\mathbf{u}})$ are called, respectively, the **osculating**, **normal**, and **rectifying (binormal)** planes. The word *osculating* comes from Latin *osculum*, literally, *a little mouth*, which was the colloquial way of saying "a kiss". Not surprisingly, of all the planes that pass through the point P , the osculating plane comes the closest to containing the curve \mathcal{C} .

EXERCISE 1.3.10.^B A curve is called **planar** if all its points are in the same plane. Show that a planar curve other than a line has the same osculating plane at every point and lies entirely in this plane (for a line, the osculating plane is not well-defined).

The curvature and torsion uniquely determine the curve, up to its position in space. More precisely, if $\kappa(\mathbf{s})$ and $\tau(\mathbf{s})$ are given continuous functions of \mathbf{s} , we can solve the corresponding equations (1.3.15)–(1.3.17) and obtain the vectors $\hat{\mathbf{u}}(\mathbf{s}), \hat{\mathbf{p}}(\mathbf{s}), \hat{\mathbf{b}}(\mathbf{s})$ which determine the shape of a family of curves. To obtain a particular curve \mathcal{C} in this family, we must specify initial values $(\hat{\mathbf{u}}(\mathbf{s}_0), \hat{\mathbf{p}}(\mathbf{s}_0), \hat{\mathbf{b}}(\mathbf{s}_0))$ of the trihedron vectors and an initial value $\mathbf{r}(\mathbf{s}_0)$ of a position vector at a point P_0 on the curve. These four vectors are all in some frame with origin O . To obtain $\mathbf{r}(\mathbf{s})$ at any point of \mathcal{C} we solve the differential equation $d\mathbf{r}/d\mathbf{s} = \hat{\mathbf{u}}(\mathbf{s})$, with initial condition $\mathbf{r}(\mathbf{s}_0)$, together with (1.3.15)–(1.3.17). Note that the curvature is always non-negative, and the torsion can be either positive or negative.

EXERCISE 1.3.11.^A For the right circular helix (1.3.13) compute the curvature, torsion, and the Frenet trihedron at every point. Show that the right circular helix is the only curve with constant curvature and constant positive torsion.

As the point P moves along the curve, the trihedron executes three rotations. These rotations about the unit tangent, principal unit normal, and unit binormal vectors are called **rolling**, **yawing**, and **pitching**, respectively. Rolling and yawing change direction of the unit binormal vector $\hat{\mathbf{b}}$, rolling and pitching change the direction of the principal unit normal vector $\hat{\mathbf{p}}$, yawing and pitching change the direction of the unit tangent vector $\hat{\mathbf{u}}$. To visualize these rotations, consider the motion of an airplane. Intuitively,

it is clear that the tangent vector $\hat{\mathbf{u}}$ points along the fuselage from the tail to the nose, and the normal vector $\hat{\mathbf{p}}$ points up perpendicular to the wings (draw a picture!) In this construction, the vector $\hat{\mathbf{b}}$ points along the wings to make $\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\mathbf{b}}$ a right-handed triple. The center of mass of the plane is the natural common origin of the three vectors. The rolling of the plane, the rotation around $\hat{\mathbf{u}}$, lifts one side of the plane relative to the other and is controlled by the *ailerons* on the back edges of the wings. Yawing, the rotation around $\hat{\mathbf{p}}$, moves the nose left and right and is controlled by the *rudder* on the vertical part of the tail. Pitching of the plane, the rotation around $\hat{\mathbf{b}}$, moves the nose up and down and is controlled by the *elevators* on the horizontal part of the tail.

Note that rolling and pitching are the main causes of motion sickness.

1.3.4 Velocity and Acceleration

Let the curve \mathcal{C} , defined by the vector function $\mathbf{r} = \mathbf{r}(t)$, be the trajectory of a point mass in some frame O . Between times t and $t + \Delta t$ the point moves through the arc length $\Delta \mathfrak{s} = \mathfrak{s}(t + \Delta t) - \mathfrak{s}(t)$, and therefore $d\mathfrak{s}(t)/dt$ is the **speed** of the point along \mathcal{C} . As we derived on page 30,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathfrak{s}}{dt} \frac{d\mathbf{r}}{d\mathfrak{s}} = \frac{d\mathfrak{s}}{dt} \hat{\mathbf{u}}(t). \quad (1.3.18)$$

Therefore, we define the **velocity** $\mathbf{v}(t)$ as

$$\mathbf{v}(t) = d\mathbf{r}/dt.$$

In particular, $\|\mathbf{v}\| = |d\mathfrak{s}/dt| = d\mathfrak{s}/dt$, that is, *the speed is the magnitude of the velocity*; recall that the arc length $\mathfrak{s} = \mathfrak{s}(t)$ is a non-decreasing function of t . This mathematical definition of velocity agrees with our physical intuition of speed in the direction of the tangent line, while making the physical concept of velocity precise, as required in a quantitative science. The definition also works well in practical problems of motion. Indeed, precise physics is mathematical physics.

Similarly, the **acceleration** $\mathbf{a}(t)$ of the point mass is, by definition,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Since $d\mathbf{v}/dt = d((d\mathfrak{s}/dt)\hat{\mathbf{u}}(t))/dt$, the product rule (1.3.4) implies

$$\frac{d\mathbf{v}}{dt} = \frac{d^2\mathfrak{s}}{dt^2} \hat{\mathbf{u}}(t) + \frac{d\mathfrak{s}}{dt} \frac{d\hat{\mathbf{u}}}{d\mathfrak{s}} \frac{d\mathfrak{s}}{dt}$$

or

$$\mathbf{a}(t) = \frac{d^2\mathfrak{s}}{dt^2} \widehat{\mathbf{u}}(t) + \left(\frac{d\mathfrak{s}}{dt}\right)^2 \frac{d\widehat{\mathbf{u}}(\mathfrak{s})}{d\mathfrak{s}}. \quad (1.3.19)$$

Equation (1.3.19) shows that the acceleration $\mathbf{a}(t)$ has two components: the **tangential acceleration** $(d^2\mathfrak{s}/dt^2) \widehat{\mathbf{u}}(t)$ and the **normal acceleration** $(d\mathfrak{s}/dt)^2 (d\widehat{\mathbf{u}}(\mathfrak{s})/d\mathfrak{s})$. By Lemma 1.1, page 26, the derivative of a unit vector is always orthogonal to the vector itself, and so the tangential and normal accelerations are mutually orthogonal. The derivation also shows that the decomposition (1.3.19) of the acceleration into the tangential and normal components does not depend on the coordinate system.

EXERCISE 1.3.12.^C In (1.3.20) below, $\mathbf{r} = \mathbf{r}(t)$ represents the position of point mass m at time t in the Cartesian coordinate system:

$$\begin{aligned} \mathbf{r}(t) &= t^2 \hat{\mathbf{i}} + 2t^2 \hat{\mathbf{j}} + t^2 \hat{\mathbf{k}}; & \mathbf{r}(t) &= 2 \cos \pi t \hat{\mathbf{i}} + 2 \sin \pi t \hat{\mathbf{j}}; \\ \mathbf{r}(t) &= 2 \cos t^2 \hat{\mathbf{i}} + 2 \sin t^2 \hat{\mathbf{j}}; & \mathbf{r}(t) &= \cos t^2 \hat{\mathbf{i}} + 2 \sin t^2 \hat{\mathbf{j}}. \end{aligned} \quad (1.3.20)$$

For each function $\mathbf{r} = \mathbf{r}(t)$,

- Sketch the corresponding trajectory;
- Compute the velocity and acceleration vectors as functions of t ;
- Draw the trajectory for $0 \leq t \leq 1$ and draw the vectors $\mathbf{r}'(1)$, $\mathbf{r}''(1)$;
- Compute the normal and tangential components of the acceleration and draw the corresponding vectors when $t = 1$;
- Verify your results using a computer algebra system.

We will now write the decomposition (1.3.19) for the CIRCULAR MOTION IN A PLANE. Let \mathcal{C} be a circle with radius R and center at the point O . Assume a point mass moves along \mathcal{C} . Choose the cartesian coordinates $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ with origin at O and $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ in the plane of the circle. Denote by $\theta(t)$ the angle between $\hat{\mathbf{i}}$ and the position vector $\mathbf{r}(t)$ of the point mass. Suppose that the function $\theta = \theta(t)$ has two continuous derivatives in t , $|\theta'(t)| > 0$, $t > 0$, and $\theta(0) = 0$. Then

$$\begin{aligned} \mathbf{r}(t) &= R \cos \theta(t) \hat{\mathbf{i}} + R \sin \theta(t) \hat{\mathbf{j}}, \\ \mathbf{v} = \mathbf{r}'(t) &= -\theta'(t) R \sin \theta(t) \hat{\mathbf{i}} + \theta'(t) R \cos \theta(t) \hat{\mathbf{j}}, \\ \|\mathbf{v}\| &= (\mathbf{r}' \cdot \mathbf{r}')^{\frac{1}{2}} = R|\theta'(t)|, \end{aligned}$$

and $\mathbf{v} \cdot \mathbf{r} = 0$. So \mathbf{v} is tangent to the circle. The acceleration \mathbf{a} is

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= -R(\theta''(t) \sin \theta(t) - (\theta'(t))^2 \cos \theta(t)) \hat{\mathbf{i}} \\ &\quad + R(\theta''(t) \cos \theta(t) - (\theta'(t))^2 \sin \theta(t)) \hat{\mathbf{j}}\end{aligned}$$

or

$$\mathbf{a} = -(\theta')^2 \mathbf{r} + (\theta''/\theta') \mathbf{v}. \quad (1.3.21)$$

Thus, the tangential component of \mathbf{a} is $(\theta''/\theta') \mathbf{v}$, and the normal component, also known as the **centripetal acceleration**, is $-(\theta')^2 \mathbf{r}$. Also, $\|\mathbf{a}\| = R\sqrt{(\theta')^4 + (\theta'')^2}$.

EXERCISE 1.3.13.^B Verify that (1.3.21) coincides with (1.3.19). Hint: First verify that $ds/dt = R\theta'(t)$ and $d\hat{\mathbf{u}}(t)/dt = -(\theta'(t)/R) \mathbf{r}(t)$.

If the rotation is uniform with constant angular speed ω , then $\theta(t) = \omega t$ and we have the familiar expressions $\|\mathbf{a}\| = \omega^2 R = \|\mathbf{v}\|^2/R$.

Note that the centripetal acceleration is in the direction of $-\mathbf{r}$, that is, in the direction *toward* the center. It is not a coincidence that the Latin verb *petere* means “to look for.”

Next, we write the decomposition (1.3.19) for the GENERAL PLANAR MOTION IN POLAR COORDINATES (r, θ) . Consider a frame with origin O and fixed cartesian coordinate system $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ so that the motion is in the $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ plane. Recall that, for a point P with position vector \mathbf{r} , $r = \|\mathbf{r}\|$, and θ is the angle from vector $\hat{\mathbf{i}}$ to \mathbf{r} . Let $\hat{\mathbf{r}} = \mathbf{r}/r$ be the unit radius vector and let $\hat{\boldsymbol{\theta}}$ be the unit vector orthogonal to \mathbf{r} so that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}}$; draw a picture or see Figure 2.1.3 on page 48 below. Then

$$\begin{cases} \hat{\mathbf{r}} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}, \\ \hat{\boldsymbol{\theta}} = -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}. \end{cases} \quad (1.3.22)$$

The vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ are functions of θ . From (1.3.22) we get

$$\begin{cases} d\hat{\mathbf{r}}/d\theta = -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}} = \hat{\boldsymbol{\theta}}, \\ d\hat{\boldsymbol{\theta}}/d\theta = -\cos \theta \hat{\mathbf{i}} - \sin \theta \hat{\mathbf{j}} = -\hat{\mathbf{r}}. \end{cases} \quad (1.3.23)$$

Let $\mathbf{r}(t)$ be the position of the point mass m at time t . In polar coordinates, $\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(\theta(t))$. The velocity of m in the frame O is $\mathbf{v} = d\mathbf{r}/dt = d(r(t)\hat{\mathbf{r}}(\theta(t)))/dt$. Using the rule (1.3.4) and the chain rule, we get $\mathbf{v} = (dr/dt) \hat{\mathbf{r}} + r(d\hat{\mathbf{r}}/d\theta)(d\theta/dt)$, or

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = \dot{r}\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}}. \quad (1.3.24)$$

The velocity \mathbf{v} is a sum of the radial velocity component $\dot{r}\hat{\mathbf{r}}$ and the angular velocity component $r\dot{\theta}\hat{\boldsymbol{\theta}}$. We call \dot{r} and $r\dot{\theta}$ the radial and angular speeds, respectively.

The acceleration \mathbf{a} in the frame O is obtained by differentiating (1.3.24) with respect to t according to the rules (1.3.3), (1.3.4):

$$\mathbf{a} = d\mathbf{v}/dt = \ddot{r}\hat{\mathbf{r}} + \dot{r}(d\hat{\mathbf{r}}/d\theta)\dot{\theta} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}(d\hat{\boldsymbol{\theta}}/d\theta)\dot{\theta},$$

or

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \quad (1.3.25)$$

The acceleration \mathbf{a} is a sum of the radial component \mathbf{a}_r and the angular component \mathbf{a}_θ , where

$$\mathbf{a}_r = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{a}_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \quad (1.3.26)$$

EXERCISE 1.3.14.^B Verify that decomposition (1.3.26) of the acceleration is a particular case of (1.3.19).

Now assume that the trajectory of the point mass is a circle with center at O and radius R . Then $r(t) = R$ for all t and $\dot{r}(t) = \ddot{r}(t) = 0$. Let $\dot{\theta}(t) = \omega(t)$. By (1.3.26),

$$\begin{cases} \mathbf{a}_r = -R\omega^2\hat{\mathbf{r}} & \text{(centripetal acceleration)} \\ \mathbf{a}_\theta = R\dot{\omega}\hat{\boldsymbol{\theta}} & \text{(angular acceleration).} \end{cases} \quad (1.3.27)$$

Also, by (1.3.24),

$$\mathbf{v} = R\omega\hat{\boldsymbol{\theta}}. \quad (1.3.28)$$

EXERCISE 1.3.15.^B Verify that formula (1.3.27) is a particular case of the decomposition (1.3.21) of the acceleration, as derived on page 34.

If we further assume that the angular speed is constant, that is, $\omega(t) = \omega_0$ for all t , then $\dot{\omega} = 0$, and, by (1.3.27),

$$\mathbf{a}_r = -R\omega_0^2\hat{\mathbf{r}}, \quad \mathbf{a}_\theta = \mathbf{0}. \quad (1.3.29)$$

EXERCISE 1.3.16.^B Verify that if the acceleration of a point mass in polar coordinates is given by (1.3.29), then the point moves around the circle of radius R with constant angular speed ω_0 . Hint: Combine (1.3.29) and (1.3.26) to get differential equations for r and θ . Solve the equations with initial conditions $r(0) = R$, $\dot{r}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = \omega_0$ to get $r(t) = R$, $\theta(t) = \omega_0 t$.

EXERCISE 1.3.17.^A Let $(r(t), \theta(t))$ be the polar coordinates of a 2-D motion of a point mass m in a fixed frame O . Let $r(t) = 3t$ and $\theta(t) = 2t$. Sketch the trajectory of the point in the frame O for $0 < t < 5$ and verify the result using a computer algebra system. Compute the velocity and acceleration vectors in the frame O in terms of the unit vectors $\hat{r}, \hat{\theta}$.