# Chapter 1

# Krein-Rutman Theorem and the Principal Eigenvalue

The Krein-Rutman theorem plays a very important role in nonlinear partial differential equations, as it provides the abstract basis for the proof of the existence of various principal eigenvalues, which in turn are crucial in bifurcation theory, in topological degree calculations, and in stability analysis of solutions to elliptic equations as steady-state of the corresponding parabolic equations. In this chapter, we first recall the well-known Krein-Rutman theorem and then combine it with the classical maximum principle of elliptic operators to prove the existence of principle eigenvalues for such operators.

Let $X$ be a Banach space. By a *cone* $K \subset X$ we mean a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone $K$ in $X$ induces a *partial ordering* $\leq$ by the rule: $u \leq v$ if and only if $v - u \in K$. A Banach space with such an ordering is usually called a partially ordered Banach space and the cone generating the partial ordering is called the *positive cone* of the space. If $\overline{K - K} = X$, i.e., the set $\{u - v : u, v \in K\}$ is dense in $X$, then $K$ is called a *total cone*. If $K - K = X$, $K$ is called a *reproducing cone*. If a cone has nonempty interior $K^0$, then it is called a *solid cone*. Any solid cone has the property that $K - K = X$; in particular, it is total. Indeed, choose $x_0 \in K^0$ and $r > 0$ such that the closed ball $B_r(u_0) := \{u \in X : \|u - u_0\| \leq r\}$ is contained in $K$. Then for any $u \in X \setminus \{0\}$, $v_0 := u_0 + ru/\|u\| \in K$ and hence $u = (\|u\|/r)(v_0 - u_0) \in K - K$. We write $u > v$ if $u - v \in K \setminus \{0\}$, and $u \gg v$ if $u - v \in K^0$.

Let $X^*$ denote the dual space of $X$. The set $K^* := \{l \in X^* : l(x) \geq 0 \ \forall x \in K\}$ is called the dual cone of $K$. It is easily seen that $K^*$ is closed and convex, and $\lambda K^* \subset K^*$ for any $\lambda \geq 0$. However it is not generally true that $K^* \cap (-K^*) = \{0\}$. But if $K$ is total, this last condition is satisfied and hence $K^*$ is a cone in $X^*$. Indeed, if $l \in K^* \cap (-K^*)$, then for every



ORDER STRUCTURE AND TOPOLOGICAL METHODS IN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS
Vol 1: Maximum Principles and Applications

$x \in K$, $l(x) \geq 0, -l(x) \geq 0$, and therefore $l(x) = 0$ for all $x \in K$. Since $\overline{K-K} = X$, this implies that $l(x) = 0$ for all $x \in X$, i.e., $l = 0$.

Let $\Omega$ be a bounded domain in $R^N$. It is easily seen that the set of nonnegative functions $K$ in $X = L^p(\Omega)$ is a cone satisfying $K - K = X$. However, it has empty interior. Similarly the set of nonnegative functions in $W^{1,p}(\Omega)$ gives a reproducing cone, and generally the nonnegative functions in $W^{k,p}(\Omega)$ $(k \geq 2, p > 1)$ form a total cone. On the other hand, the nonnegative functions form a solid cone in $C(\overline{\Omega})$ but only form a reproducing cone in $C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. If $\Omega$ has $C^1$ boundary $\partial\Omega$, then it is easy to see that the nonnegative functions in $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ form a solid cone; for example, any function satisfying $u(x) > 0$ in $\Omega$ and $D_\nu u(x) < 0$ on $\partial\Omega$ is in the interior of the cone, where $\nu$ denotes the outward unit normal of $\partial\Omega$.

**Theorem 1.1** (The Krein-Rutman Theorem, [Deimling(1985)] Theorem 19.2 and Ex.12) *Let $X$ be a Banach space, $K \subset X$ a total cone and $T : X \to X$ a compact linear operator that is positive (i.e., $T(K) \subset K$) with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus \{0\}$: $Tu = r(T)u$. Moreover, $r(T^*) = r(T)$ is an eigenvalue of $T^*$ with an eigenvector $u^* \in K^*$.*

Let us now use Theorem 1.1 to derive the following useful result.

**Theorem 1.2** *Let $X$ be a Banach space, $K \subset X$ a solid cone, $T : X \to X$ a compact linear operator which is strongly positive, i.e., $Tu \gg 0$ if $u > 0$. Then*

- *(a) $r(T) > 0$, and $r(T)$ is a simple eigenvalue with an eigenvector $v \in K^0$; there is no other eigenvalue with a positive eigenvector.*
- *(b) $|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*

Let us recall that $r$ is a simple eigenvalue of $T$ if there exists $v \neq 0$ such that $Tv = rv$ and $(rI - T)^n w = 0$ for some $n \geq 1$ implies $w \in span\{v\}$.

***Proof.*** **Step 1**: *There exists $v_0 > 0$ such that $Tv_0 = r(T)v_0$.*

Fix $u \in K^0$. Then $\alpha Tu \geq u$ for some $\alpha > 0$, and we can find $\sigma > 0$ such that $B_\sigma(u) \subset K$. It follows that $w \leq (\sigma)^{-1}\|w\|u$ for any $w \in X$. Let $S = \alpha T$. Then

$$u \leq S^n u \leq \sigma^{-1}\|S^n u\|u \leq \sigma^{-1}\|S^n\|\|u\|u, \ \forall n \geq 1.$$

ORDER STRUCTURE AND TOPOLOGICAL METHODS IN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS
Vol 1: Maximum Principles and Applications

Hence

$$\|S^n\| \geq \sigma/\|u\| \text{ and } r(S) = \lim_{n\to\infty} \|S^n\|^{1/n} > 0.$$

By Theorem 1.1, $r(S)$ is an eigenvalue of $S$ corresponding to a positive eigenvector $v_0 \in K \setminus \{0\}$. Clearly $r(T) = r(S)/\alpha > 0$ and $Tv_0 = r(T)v_0$.

**Step 2**: *To prove that $r(T)$ is simple, we show a more general conclusion: If $r > 0$ and $Tv = rv$ for some $v > 0$, then $r$ is a simple eigenvalue of $T$.*

Let us first show that $(rI - T)w = 0$ implies $w \in span\{v\}$. Suppose $Tw = rw$ with $w \neq 0$. Then $T(v \pm tw) = r(v \pm tw)$ for all $t > 0$. Since $T$ is strongly positive, $v \in K^0$ and the above identity implies $v \pm tw \notin \partial K$ unless $v \pm tw = 0$. But $v \pm tw \in K^0$ for small $t$ and this cannot hold for all large $t$ for otherwise $w \in K \cap (-K) = \{0\}$. Therefore there exists $t_0 \neq 0$ such that $v + t_0 w \in \partial K$ and hence $v + t_0 w = 0$. This proves $w \in span\{v\}$.

Let $(rI - T)^2 w = 0$. By what has just been proved, $rw - Tw = t_0 v$ for some $t_0 \in R^1$. If $t_0 \neq 0$, then we may assume $t_0 > 0$ (otherwise change $w$ to $-w$). Since

$$T(v + sw) = r(v + sw) - st_0 v \ll r(v + sw) \text{ for all } s > 0,$$

and $v + sw \in K^0$ for all small $s \geq 0$, we easily deduce $v + sw \in K^0$ for all $s \geq 0$. This implies that $w \in K$, and hence $w = r^{-1}(t_0 v + Tw) \in K^0$. We now have

$$w - tv \in K^0 \text{ for all small } t > 0,$$

but not for all large $t > 0$ as this would imply $v = 0$. Therefore there exists $t_1 > 0$ such that $w - t_1 v \in \partial K$. But then

$$rw - t_0 v - t_1 rv = T(w - t_1 v) \geq 0,\ w - t_1 v \geq r^{-1} t_0 v \gg 0,$$

contradicting $w - t_1 v \in \partial K$. Therefore we must have $t_0 = 0$ and hence $rw - Tw = 0$, $w \in span\{v\}$. This proves that $r$ is a simple eigenvalue.

**Step 3**: *Next we show that $T$ cannot have two positive eigenvalues $r_1 > r_2$ corresponding to positive eigenvectors*:

$$Tv_1 = r_1 v_1,\ Tv_2 = r_2 v_2.$$

Let $v(t) = v_2 - tv_1, t \geq 0$. Since $T$ is strongly positive, we have $v_1, v_2 \in K^0$. As before we have $v(t) \in K^0$ for small $t$ but not for all large $t$.

Therefore there exists $t_0 > 0$ such that $v(t_0) \in K$ but $v(t) \notin K$ for $t > t_0$. We now have

$$v_2 - t_0(r_1/r_2)v_1 = r_2^{-1}T(v_2 - t_0 v_1) \in K,$$

which implies $r_1 \leq r_2$ due to the maximality of $t_0$. This contradiction proves step 3.

**Step 4**: *If $Tw = \lambda w$ with $w \neq 0$ and $\lambda \neq r(T)$, then $|\lambda| < r(T)$.*

If $\lambda > 0$, then by Step 3, $w \notin K$. It follows that $v_0 + tw \in K$ for all small $t > 0$ but not for all large $t$. Therefore there exists $t_0 > 0$ such that $v_0 + t_0 w \in K$ and $v_0 + tw \notin K$ for $t > t_0$. It then follows that $v_0 + t_0(\lambda/r(T))w = r(T)^{-1}T(v_0 + t_0 w) \in K$. The maximality of $t_0$ implies that $\lambda \leq r(T)$ and hence $\lambda < r(T)$.

If $\lambda < 0$, then from $T^2 w = \lambda^2 w$ and $T^2 v_0 = r(T)^2 v_0$ and the above argument (applied to $T^2$) we deduce $\lambda^2 < r(T)^2$ and hence $|\lambda| < r(T)$.

Consider now the case that $\lambda = \sigma + i\tau$ with $\tau \neq 0$. Then necessarily $w = u + iv$ and

$$Tu = \sigma u - \tau v, \; Tv = \tau u + \sigma v. \tag{1.1}$$

We observe that $u$ and $v$ are linearly independent for otherwise we necessarily have $\tau = 0$. Let $X_1 := span\{u, v\}$. Then (1.1) implies that $X_1$ is an invariant subspace of $T$. We claim that $K_1 := X_1 \cap K = \{0\}$. Otherwise $K_1$ is a positive cone in $X_1$ with nonempty interior, as for any $w \in K_1 \setminus \{0\}$, $Tw \in X_1 \cap K^0 = K_1^0$. We can now apply Step 1 above to $T$ on $X_1$ to conclude that there exists $r > 0$ and $w_0 \in K_1^0$ such that $Tw_0 = rw_0$. By Steps 2 and 3, we necessarily have $r = r(T)$ and $w_0 \in span\{v_0\}$. In other words, $v_0 \in K_1$ and $v_0 = \alpha u + \beta v$ for some real numbers $\alpha$ and $\beta$. But then one can use (1.1) and $Tv_0 = r(T)v_0$ to easily derive $\alpha = \beta = 0$, a contradiction. Therefore $K_1 = \{0\}$.

From $span\{u, v\} \cap K = \{0\}$ we find that the set

$$\Sigma := \{(\xi, \eta) \in R^2 : v_0 + \xi u + \eta v \in K\}$$

is bounded and closed. Since $v_0 \in K^0$, $M := \sup\{\xi^2 + \eta^2 : (\xi, \eta) \in \Sigma\} > 0$ and is achieved at some $(\xi_0, \eta_0) \in \Sigma$. Let $z_0 = v_0 + \xi_0 u + \eta_0 v$. Then $z_0 \in K \setminus \{0\}$ and $Tz_0 \in K^0$. Therefore we can find $\alpha \in (0, r(T))$ such that $Tz_0 \geq \alpha v_0$, i.e.,

$$(r(T) - \alpha)v_0 + (\xi_1 u + \eta_1 v) \geq 0, \tag{1.2}$$

where

$$\xi_1 = \xi_0\sigma + \eta_0\tau,\ \eta_1 = \eta_0\sigma - \xi_0\tau.$$

Clearly

$$\xi_1^2 + \eta_1^2 = (\sigma^2 + \tau^2)(\xi_0^2 + \eta_0^2) = M|\lambda|^2.$$

By (1.2), we find that $(\xi_1, \eta_1)/(r(T) - \alpha) \in \Sigma$ and hence

$$\xi_1^2 + \eta_1^2 \leq M(r(T) - \alpha)^2,$$

that is,

$$|\lambda|^2 \leq (r(T) - \alpha)^2,$$

and hence $|\lambda| < r(T)$. The proof of Step 4 and hence the theorem is now complete. □

Suppose now $L$ is the elliptic operator and $\Omega$ the bounded domain as given in Theorem A.4, namely

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$$

has $C^\alpha(\overline{\Omega})$ coefficients and is strictly uniformly elliptic in the bounded domain $\Omega$ which has $C^{2,\alpha}$ boundary. Choose $\xi > 0$ large enough so that $c - \xi < 0$ in $\Omega$, and denote $L_\xi u = Lu - \xi u$. Let $K$ be the positive cone in $X := C_0^{1,\alpha}(\overline{\Omega})$ consisting of nonnegative functions. For any $v \in X$, Theorem A.1 guarantees that the problem

$$-L_\xi u = v \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

has a unique solution $u$ satisfying

$$\|u\|_{2,\alpha} \leq C\|v\|_\alpha \leq C_1\|v\|_{1,\alpha}$$

for some constant $C_1 > 0$ independent of $u$ and $v$. It follows that $T : X \to X$ defined by $Tv = u$ is a compact linear operator. Moreover, by the weak maximum principle Theorem A.34, $Tv \geq 0$ if $v \in K$. The strong maximum principle Theorem A.36 then implies that $u = Tv > 0$ in $\Omega$ if $v \in K \setminus \{0\}$, and the Hopf boundary lemma (Lemma A.35) gives further $D_\nu u < 0$ on $\partial\Omega$. This implies that $Tv \in K^0$. Therefore $T$ is strongly positive. It now follows from Theorem 1.2 that $r(T) > 0$ is a simple eigenvalue of $T$ with an eigenfunction $v \in K^0$: $Tv = r(T)v$. Thus $u = Tv$ satisfies

$$-Lu + \xi u = r(T)^{-1}u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

i.e.,

$$Lu + \lambda_1 u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

with $\lambda_1 = r(T)^{-1} - \xi$.

Generally, it is easily checked that $\mu$ is an eigenvalue of $T$ if and only if $\lambda = \mu^{-1} - \xi$ is an eigenvalue of

$$Lu + \lambda u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Theorem 1.2 now implies the following result.

**Theorem 1.3** *Under the conditions of Theorem A.4 for $L$ and $\Omega$, the eigenvalue problem (1.3) has a simple eigenvalue $\lambda_1 \in R^1$ which corresponds to a positive eigenfunction; none of the other eigenvalues corresponds to a positive eigenfunction.*

If the boundary operator is of Neumann or Robin type,

$$Bu = D_\nu u + \sigma(x)u,\ \sigma \geq 0,\ \sigma \in C^{1,\alpha}(\partial\Omega),$$

then we let $X = C^{1,\alpha}(\overline{\Omega})$ and let $K$ be the cone of nonnegative functions in this space. We define the operator $T$ analogously as in the Dirichlet case and again find that it is compact on $X$ and maps $K$ to itself, due to the weak maximum principle. Suppose now $v \in K \setminus \{0\}$. Then by the strong maximum principle, $u = Tv > 0$ in $\Omega$. Moreover, by the Hopf boundary lemma, if $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then $D_\nu u(x_0) < 0$ and hence $Bu(x_0) < 0$, contradicting the boundary condition. Therefore $u > 0$ on $\partial\Omega$. Therefore $Tv > 0$ on $\overline{\Omega}$, which implies that $Tv \in K^0$, i.e., $T$ is strongly positive. Therefore we can apply Theorem 1.2 to conclude that Theorem 1.3 holds also for the Neumann and Robin boundary conditions. The eigenvalue $\lambda_1$ in Theorem 1.3 is usually called the *principle eigenvalue.*

**Theorem 1.4** *If $\lambda \neq \lambda_1$ is an eigenvalue of (1.3) but the boundary condition is either Dirichlet, or Neumann, or Robin type, then $Re(\lambda) \geq \lambda_1$.*

***Proof.*** Suppose $w > 0$ is an eigenvector corresponding to $\lambda_1$ and $u$ is an eigenvector corresponding to $\lambda$. Set $v := u/w$. Then

$$-\lambda v = w^{-1}L(vw) = Lv - cv + 2w^{-1}a^{ij}D_j w D_i v - \lambda_1 v.$$

Writing

$$Kv := a^{ij}D_{ij}v + \tilde{b}^i D_i v,\ \tilde{b}^i := b^i + 2w^{-1}a^{ij}D_j w,$$

we obtain

$$Kv + (\lambda - \lambda_1)v = 0.$$

Take complex conjugates to yield

$$K\overline{v} + (\overline{\lambda} - \lambda_1)\overline{v} = 0.$$

Next we compute

$$K(|v|^2) = K(v\overline{v}) = \overline{v}Kv + vK\overline{v} + 2a^{ij}D_i v D_j \overline{v} \geq \overline{v}Kv + vK\overline{v},$$

since

$$a^{ij}\xi_i\overline{\xi}_j = a^{ij}(Re(\xi_i)Re(\xi_j) + Im(\xi_i)Im(\xi_j)) \geq 0$$

for any complex vector $\xi \in C^N$. We now easily obtain

$$K(|v|^2) \geq 2(Re(\lambda) - \lambda_1)|v|^2 \text{ in } \Omega.$$

Suppose now the boundary operator $B$ is either Neumann or Robin type. Then $w > 0$ over $\overline{\Omega}$ and a direct computation shows $D_\nu v = 0$ and $D_\nu |v|^2 = 0$. If $Re(\lambda) \leq \lambda_1$, then $\phi := |v|^2 \geq 0$ satisfies

$$K\phi \geq 0 \text{ in } \Omega,\ D_\nu \phi = 0 \text{ on } \partial\Omega.$$

We now apply the strong maximum principle and Hopf boundary lemma and conclude that $\phi \equiv$ constant, that is $u = cw$ and hence $\lambda = \lambda_1$, a contradiction. Therefore we must have $Re(\lambda) > \lambda_1$.

To prove the Dirichlet case, we replace $w$ by $w_\epsilon := w^{1-\epsilon}$, $0 < \epsilon < 1$, in the above discussion and obtain

$$K(|v|^2) \geq -2(Re(\lambda) + \frac{Lw_\epsilon}{w_\epsilon})|v|^2 \text{ in } \Omega.$$

Since

$$Lw_\epsilon = (1-\epsilon)w^{-\epsilon}Lw - \epsilon(1-\epsilon)w^{-1-\epsilon}a^{ij}D_i w D_j w + \epsilon c w^{1-\epsilon}$$

$$\leq (1-\epsilon)w^{-\epsilon}Lw + \epsilon c w^{1-\epsilon} \leq (\epsilon C - (1-\epsilon)\lambda_1)w_\epsilon,$$

where $C = \max_{\overline{\Omega}} c$, we deduce

$$K(|v|^2) \geq 2((1-\epsilon)\lambda_1 - \epsilon C - Re(\lambda))|v|^2 \text{ in } \Omega.$$

By the Hopf boundary lemma, we know $D_\nu w < 0$ on $\partial\Omega$. Therefore, since $u|_{\partial\Omega} = 0$,

$$\lim_{x\to x_0\in\partial\Omega} \frac{u(x)}{w(x)} = \frac{D_\nu u(x_0)}{D_\nu w(x_0)}.$$

It follows that

$$\lim_{x\to\partial\Omega} v = \lim_{x\to\partial\Omega} w(x)^\epsilon \frac{u(x)}{w(x)} = 0.$$

If $Re(\lambda) \le (1-\epsilon)\lambda_1 - \epsilon C$, then,

$$K(|v|^2) \ge 0 \text{ in } \Omega.$$

Let $\Omega_n$ be a sequence of smooth domains enlarging to $\Omega$, e.g., $\Omega_n := \{x \in \Omega : d(x,\partial\Omega) > \delta_0/n\}$ with $\delta_0 > 0$ small. We apply the maximum principle on $\Omega_n$ and deduce $\max_{\overline{\Omega}_n} |v|^2 \le \max_{\partial\Omega_n} |v|^2$. Letting $n \to \infty$, it results $|v|^2 = 0$ and hence $u = 0$, a contradiction. Therefore $Re(\lambda) > (1-\epsilon)\lambda_1 - \epsilon C$. Letting $\epsilon \to 0$ we obtain $Re(\lambda) \ge \lambda_1$. □

**Remark 1.5** The above proof shows that in the Neumann and Robin boundary conditions case, $Re(\lambda) > \lambda_1$. This is also true in the Dirichlet case; see Theorem 2.7 in the next chapter.

**Remark 1.6** Instead of (1.3), sometimes one also needs to consider the weighted eigenvalue problem

$$Lu + \lambda h(x)u = 0 \text{ in } \Omega,\ Bu = 0 \text{ on } \partial\Omega,$$

where $h(x)$ is a weight function, $B$ is the Dirichlet, Neumann or Robin boundary operator. If $h(x)$ is positive and suitably smooth, similar results to those in Theorems 1.3 and 1.4 can be analogously proved. If $h(x)$ changes sign, similar results can still be proved by considerably different techniques, see [Hess-Kato(1980)].