Chapter 1

Introduction

This book is devoted to strange nonchaotic attractors, which are typical to quasiperiodically forced systems. In this introductory chapter we describe what is the place of these objects in nonlinear dynamics, what is common and what is the difference of SNA compared to quasiperiodicity and chaos.

1.1 Periodicity and quasiperiodicity

It is natural to classify possible dynamical regimes observed in nonlinear systems by their complexity. The most simple nontrivial regime is a periodic one. Already the consideration of conservative systems with one degree of freedom leads to such solutions. These solutions in conservative systems appear in families and are not isolated, a small perturbation or a change of initial conditions leads to a transition to another periodic solution, in a nonlinear system the period generally changes, too.

In autonomous dissipative dynamical systems periodic solutions appear when there are mechanisms of energy supply and energy dissipation present in the system. As a result, a periodic motion with a certain amplitude appears as a structurally stable regime: it reestablishes after a perturbation; these solutions are often called self-sustained oscillations. In the phase space of a dynamical system such stable periodic motions are described by limit cycles. The minimal dimension of the phase space for a limit cycle to exist is two; moreover, in two-dimensional systems more complex structurally stable regimes are impossible.

The next complex dynamical regime is quasiperiodicity. This regime appears already in elementary linear conservative systems with two degrees of freedom, where a general solution has the form

$$x(t) = a_1 \cos(\omega_1 t + \varphi_1^0) + a_2 \cos(\omega_2 t + \varphi_2^0) \ , \tag{1.1}$$

where ω_1 and ω_2 are the natural frequencies depending on the parameters of the system. Solution (1.1) is periodic only if the ratio of the frequencies is rational, i.e. when

$$\frac{\omega_1}{\omega_2} = \frac{p}{q} \tag{1.2}$$

with integers p, q. In this case the period is $T = p\frac{2\pi}{\omega_1} = q\frac{2\pi}{\omega_2}$. By contrast, if the ratio of frequencies is an irrational number, (1.2) cannot be valid for any pair of integers p, q, and the motion is quasiperiodic. More generally, to give a definition of a quasiperiodic function of time let us introduce a function

$$g(\phi_1, \phi_2, \ldots, \phi_n) \tag{1.3}$$

of n arguments, which is 2π-periodic in each of them. These arguments (naturally called phases) are linear functions of time

$$\dot{\phi}_1 = \omega_1 \ , \quad \ldots \quad \dot{\phi}_n = \omega_n \ . \tag{1.4}$$

Next, we demand that the frequencies ω_i are linearly independent, i.e. the relation

$$k_1\omega_1 + k_2\omega_2 + \ldots + k_n\omega_n = 0 \tag{1.5}$$

cannot be satisfied for any set of integers $k_1 \ldots k_n$. Then $g(\phi_1, \phi_2, \ldots, \phi_n)$ describes a quasiperiodic motion with n incommensurate frequencies. Expression (1.1) is a particular form of (1.3).

Quasiperiodic solutions of type (1.3) naturally appear in integrable Hamiltonian systems with n degrees of freedom, where a transformation to angle-action variables is possible, and the angle variables obey (1.4). Note, however, that because the action variables may be changed under a perturbation , and the frequencies depend on these variables, condition (1.5) does not survive under a perturbation of initial conditions and/or parameters. In terms of our discrimination of periodic and quasiperiodic motions, this means that under small perturbations a quasiperiodic motion may become periodic, and vice versa. This already makes a study of structural stability of quasiperiodic regimes nontrivial, but before considering it closely we discuss the possibility to observe quasiperiodic motions in dissipative systems.

A natural way to construct a quasiperiodic regime in a dissipative dynamical system is to take a high-dimensional integrable conservative model as described above and to assume that due to supply of energy for small amplitudes (values of action variables) and to dissipation of energy for large amplitudes, a certain stable values of amplitudes is established. The dynamics then is described by the phase variables solely, according to (1.4).

There is also another way to construct a quasiperiodic state in a dissipative system. One starts here with a simple stable steady state. This state can lose stability when parameters of the system are changed, and produce a stable limit cycle via a Hopf bifurcation. Now one can assume that with a further change of parameters the periodic motion can become unstable toward the appearance of another periodic component (mode) with some other frequency, the amplitude of this mode is assumed to be stabilized at some level. As a result of this secondary Hopf bifurcation, also called Neimark-Sacker bifurcation, a quasiperiodic motion with two frequencies can appear. Assuming as a hypothesis that further secondary bifurcations can occur, one can imagine the appearance of quasiperiodic motions of higher order. Exactly this picture has been drawn by Landau in his book on Hydrodynamics [Landau and Lifshitz 1987], where he tried to imagine how a hypothetical way to turbulence via a consecutive complication of the dynamical state can occur, as a parameter (for turbulence a natural parameter is the Reynolds number) changes. Landau was not aware of the possibility of chaotic dynamics, nowadays a transition to turbulence is usually related to the appearance of chaos rather than to a high-dimensional quasiperiodic state.

1.2 Robustness of quasiperiodic motions

We have seen that quasiperiodic motion is defined by a rather subtle characteristics, namely by incommensurate frequencies. One can naturally ask, whether this property is not only mathematically, but also physically relevant, i.e. whether in a real world with inevitable noise and measurement errors one can really distinguish periodic and quasiperiodic regimes. To answer this question it is natural to introduce an observation time and to distinguish periodicity and quasiperiodicity with respect to this time. Considering relation (1.2) one can compare the possible period of the process with this observation time: if the ratio of the frequencies (1.2) is rational but p and q are so large that the period exceeds the observation time, then it is not possible to distinguish this periodic regime from a quasiperiodic one. In other words, practically a complex periodic state may be considered as a quasiperiodic one.

There is also another aspect in the relation between periodic and quasiperiodic regimes, it is related to a parameter dependence. Indeed, if the frequencies ω_1 and ω_2 depend on parameters, then their ratio depends on the parameters too and it might happen that already a small parameter variation leads to transitions periodicity $\leftrightarrow$ quasiperiodicity.

This aspect of robustness of a quasiperiodic state is a complex issue that still is not resolved in its full generality. A rather complete answer exists in the case of two frequencies (1.2) only. The main point is that one has to consider the phases ϕ_i not as independently rotating ones, but as coupled dynamical variables. For two phases one then writes

$$\dot{\phi}_1 = \omega_1 + F_1(\phi_1, \phi_2) , \qquad \dot{\phi}_2 = \omega_2 + F_2(\phi_1, \phi_2) , \tag{1.6}$$

with coupling functions F_1, F_2 which are 2π-periodic in each argument. If these functions are small, the dynamics (1.6) on the two-dimensional torus $0 \leq \phi_1, \phi_2 < 2\pi$ can be reduced to a circle map, and as a result a full characterization of periodic and quasiperiodic regimes can be achieved. We will not present the whole theory here (see [Katok and Hasselblatt 1995] for details), we only mention two important features:

(1) Periodic regimes, i.e. regimes where the observed frequencies $\Omega_{1,2} = \langle\dot{\phi}_{1,2}\rangle$ are in a rational relation like (1.2), are structurally stable: they exist for certain parameter ranges (open sets of parameters). Contrary to this, quasiperiodic states with incommensurate $\Omega_{1,2}$ are isolated. This means that e.g. if one parameter is changed, a quasiperiodic regime with a given ratio Ω_1/Ω_2 exist for one particular value of this parameter; if two parameters are varied, this regime exists on a line in the parameter plane , etc. Therefore one can say that periodic regimes are topologically stable and quasiperiodic regimes are not.

(2) For small nonlinearities in (1.6), i.e. for small $F_{1,2}$, the measure of all parameter values for which periodic regimes are observed is small while the measure of quasiperiodic states is large. One can interprete this as the abundance of quasiperiodicity: if parameters are chosen "at random", then a quasiperiodic state will be observed with high probability. For larger nonlinearities, i.e. for large $F_{1,2}$, the portion of periodic regimes grows while that of quasiperiodic regimes decreases.

These two properties mean that although being structurally unstable, quasiperiodic regimes are nevertheless physically observable. However, if one wants to have a quasiperiodic regime with a particular ratio of the frequencies, then special efforts are necessary to adjust the system parameters to the needed values.

The situation with quasiperiodicity is not so clear when the number of frequencies is larger than two. Certainly, in this case quasiperiodic states are structurally unstable as well, i.e. with a small change of parameters another regime, e.g. a periodic one, can be observed. Moreover, because the dimension of the system for phases (1.4) is large now, structurally stable regimes that are more complex than periodic ones are possible. In the famous paper by Ruelle and Takens [1971] (see also an extension in [Newhouse et al. 1978]) it was shown that also structurally stable chaotic states can occur if one adds some particular arbitrary small nonlinear terms to Eqs. (1.4). However, again, like in the simplest case of two frequencies, one can expect that from the probabilistic point of view quasiperiodic regimes are abundant while periodic and chaotic ones are rather unprobable, at least for small nonlinearities. This picture has been confirmed by numerical calculations in [Grebogi et al. 1983b, 1985], where a statistical test on how often randomly chosen functions F_i lead to quasiperiodicity has been performed.

Concluding, we can say that quasiperiodic motions, being structurally unstable, are nevertheless physically observable. However, to observe a quasiperiodic state with a particular ratio of frequencies, special efforts on adjusting the system's parameters must be made. Otherwise, it is possible to fall into periodicity, moreover, periodic states become more and more probable for large nonlinearities (large coupling between modes).

1.3 Strange nonchaotic attractors

In typical classifications of dynamical complexity chaos is considered as the next stage beyond quasiperiodicity. Chaos is usually defined as a dynamical regime with sensitive dependence on initial conditions; quantitatively this is characterized by a positive largest Lyapunov exponent. Contrary to this, in a quasiperiodic regime the largest Lyapunov exponents are equal to zero, their number is the number of phase variables in (1.4). Chaos also shares many statistical properties with noise, in particular it possesses decaying correlations. In the phase space of a dissipative system the object corresponding to chaos is a strange attractor, which is a fractal set of zero measure.

A ***strange nonchaotic attractor (SNA)*** is an object that lies in between quasiperiodicity and chaos. This concept has been introduced in a seminal paper by Grebogi, Ott, Pelikan, and Yorke [1984], and is related to a special class of dissipative systems with a quasiperiodic forcing of type (1.3). The force is characterized by n phases $\phi_1 \dots \phi_n$, and the driven system by some variables $x_1 \dots x_m$.

The term ***strange*** means that the dependence of the dynamical variables on the phases is not given by smooth relations, but constitutes some fractal. This is in contrast to a possible quasiperiodic dynamics of $x_l(t)$, where these variables are smooth functions of the phases ϕ_i and thus are described by functions of type (1.3).

The term ***nonchaotic*** means the absence of sensitive dependence on initial conditions. Quantitatively, the largest Lyapunov exponent corresponding to the variables $x_1 \dots x_m$ is negative.

A possibly complete description of strange nonchaotic attractors is the subject of this book. Numerous studies have shown that SNA is not a degenerate object existing for some special parameter values, but can be observed generally in quasiperiodically forced systems. Moreover, in such systems it is typically found in the transition region from order to chaos. One can say that in such a transition "strangeness" appears prior to "chaoticity". Moreover, there is a class of systems where chaos is simply impossible because the phase space has a too low dimension (e.g. if the dynamical variable x is just one-dimensional). In this case an SNA provides, possibly, the maximal complexity in the dynamics.

1.4 What is in the book

Strange nonchaotic attractors have been extensively studied in physical literature, with approximate analytic methods, numerically, and experimentally. However, rigorous mathematical works are rather rare. In this book we follow mainly analytic and numerical studies and try to characterize SNAs from different sides. We start in Chapter 2 with a description of different dynamical models leading to SNAs. Here we also describe experiments with SNA and discuss relations to other physical and dynamical problems. In the following chapters these models will be used to illustrate different features of SNAs: How one can describe SNA using periodic approximations to quasiperiodic forcing (Chapter 3); how one can characterize the stability of motion on SNA (Chapter 4); what are properties of correlations and spectra of SNA (Chapter 5). In Chapter 6 we analyze different transitions from regular to strange nonchaotic behavior and further to chaos. In many cases these transitions demonstrate remarkable properties of self-similarity that can be explained with a renormalization group approach (Chapter 7).