LINEARITY IN NON-LINEAR SITUATIONS

RICHARD M. ARON
Mathematics Department
Kent State University
Kent, Ohio 44242
USA
aron@math.kent.edu

We give an informal account of recent progress and questions related to the general problem: When does the subset of a vector space that has a certain (interesting) property contain an infinite dimensional vector space?

In 2001, D. García, M. Maestre, and the author published an expository article [4] in which we introduced the terms *lineable* and *spaceable.* The aim of these lectures is to give an accessible description of these notions which, we hope, will encourage others to find many more new and interesting instances of lineability and spaceability.

As the reader will soon observe, these notes will be leisurely, with either brief indications of some proofs or else no proofs at all. To highlight our casual approach, we "officially" begin with not one, but two introductions.

1. Two introductions

Introduction 1. Let's first recall three classical results.

(a). (Weierstrass (± 1856)) There is a continuous, nowhere differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(b). (Lebesgue (1904)) There is a function $f : \mathbb{R} \to \mathbb{R}$ with the property that for any $a < b$, the restriction $f|_{(a,b)} : (a, b) \to \mathbb{R}$ is surjective. (Such a function f is called *everywhere surjective.*)

(c). (Birkhoff (1929)) There is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that the set of translated functions $\{f(z), f(z + 1), f(z + 2), ...\}$ is dense in the space $\mathcal{H}(\mathbb{C})$ of entire functions, with the usual compact-open topology.

To us, the mere existence of such functions is surprising. In fact, there are very many such functions and, indeed, there are large vector spaces of such functions! To continue, it will be very helpful to have two new terms (which, until fairly recently, did not exist in the English language).

Definition 1 Let E be a Banach or Fréchet space, let $\mathcal{P}$ be a property that elements of E might have, and let $S \equiv \{x \in E \ : \ x \text{ has property } \mathcal{P}\}$. We say that $\mathcal{P}$ is *lineable* if $S \cup \{0\}$ contains an infinite dimensional vector space, and that S is *spaceable* if $S \cup \{0\}$ contains a *closed* infinite dimensional vector space.

(*i*) **Theorem** [21] The property of being continuous and nowhere differentiable is a spaceable property in the space $C([0,1])$ of continuous real-valued functions on $[0,1]$.
(ii) **Theorem** [6] The property of being everywhere surjective is lineable in the space of all functions $f : \mathbb{R} \to \mathbb{R}$. (Note that spaceability is not an issue here, since there is no natural topology on the space of all functions on $\mathbb{R}$.)
(iii) **Theorem** [14] The property of being an entire function f whose set of translates

$$\{f(z), f(z+1), ..., f(z+n), ...\}$$

is dense in $\mathcal{H}(\mathbb{C})$ is spaceable.

It is natural to ask for criteria for when lineability occurs, and when a property is found to be lineable it is natural to wonder whether it is also spaceable. By and large, these natural questions are open and in many situations they seem to be quite difficult. This leads us to our second introduction.

Introduction 2. To us, it seems that very often when one or the other of lineability and spaceability is proved, the result is surprising and almost magical. Thus, a second personal motivation for our interest in this topic, is the following: How does one get *intuition* about whether a property $\mathcal{P}$ is lineable, spaceable, or neither? The reader will note that despite the abundance of examples that are presented, we are still confounded by this problem of how to develop a sense of what is true.

Let us illustrate the problems we confront with a number of examples.

Examples 1. Let $X = \mathcal{C}[0,1]$, and let $\mathcal{P}$ be the property: $f \in X$ *attains its maximum at precisely one point of* $[0,1]$. Is $\mathcal{P}$ a lineable, or a spaceable, property? That is, does the set $\{f \in \mathcal{C}[0,1] \ : \ f$ attains its norm at precisely one point $\} \cup \{0\}$ contain an infinite dimensional vector space; does it contain a closed infinite dimensional vector space?

2. Now let $X = \{f \in \mathcal{C}(\mathbb{R}) \ : \ |f(x)| \to 0 \ \text{ as } x \to \infty\}$, which is a Banach space with the max-norm, and let $\mathcal{P}$ be the same property as in **1** above: f satisfies $\mathcal{P}$ if and only if f attains its maximum at exactly one point of $\mathbb{R}$. Is this a lineable property: is it spaceable?

3. Let $T \cong \mathbb{R}/2\pi$ be the unit circle. Let $\mathcal{P}$ be the property that the Fourier series of a function $f \in \mathcal{C}(T)$ *does not* converge at, say, the point $z = 1$. Is $\mathcal{P}$ a lineable, or a spaceable, property?

4. Let $\mathcal{P}$ be the property that a function $f \in \mathcal{C}[0,1]$ is everywhere differentiable. Is $\mathcal{P}$ lineable? Is it spaceable?

5. Fix a non-empty open subset $U \subset \mathbb{C}^n$, and let $\mathcal{H}(U)$ denote the holomorphic (complex analytic) functions on U, endowed with the standard Fréchet topology of uniform convergence on compact subsets of U. Let $\mathcal{P}$ be the property that a function $f \in \mathcal{H}(U)$ cannot be continued beyond any point of the boundary of U. (In other words, U is the *domain of existence* of f.) Lineable? Spaceable?

6. Fix an infinite dimensional Banach space E, and let $\mathcal{P}$ be the property that a continuous linear functional $\varphi \in E^*$ attains its norm; that is $||\varphi|| = \phi(x_0)$ for some $x_0 \in E$, $||x_0|| = 1$. By a classical result of R. C. James [27], E is reflexive if and only if every such φ is norm-attaining. On the other hand, in [16], E. Bishop and R. Phelps showed that for every E, the set of norm-attaining functionals in E^* is a norm-dense subset of E^*. Is $\mathcal{P}$ a lineable property; is it a spaceable property?

7. Let $T_2 : \ell_2 \to \ell_2$ be the Rolewicz continuous linear operator given by $T_2(x_1, x_2, ...) = 2(x_2, x_3, ...)$ [31]. Consider the set $\mathcal{HC}(T_2)$ of vectors $x = (x_n) \in \ell_2$ whose orbits under T is dense in ℓ_2. (In other words, $\overline{Orb(T,x)} = \overline{\{x, T_2x, T_2^2x, ..., T_2^nx, ...\}} = \ell_2$.) Does $\mathcal{HC}(T_2) \cup \{0\}$ contain an infinite dimensional subspace of ℓ_2? Does it contain a closed, infinite dimensional subspace of ℓ_2?

8. Let $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ be a differential operator acting on the space $\mathcal{H}(\mathbb{C})$ of entire functions. Let $\mathcal{P}$ be the property that the orbit of an entire function f under T is dense in $\mathcal{H}(\mathbb{C})$. Is $\mathcal{P}$ lineable; spaceable?

9. Let E be an infinite dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $P : E \to \mathbb{K}$ be a continuous polynomial. Let $\mathcal{P}$ be the property that a vector $x \in E$ is such that $P(x) = P(0)$. Is this a lineable property? In other words, does the set $\{x \in E \ : \ P(x) = P(0)\}$ contain an infinite dimensional vector space?

Note that if this set contains an infinite dimensional vector space, then it automatically contains a closed infinite dimensional vector space; in other words, in this case, lineability and spaceability are the same problem. The next section will begin by recalling the necessary background, including relevant definitions, with a review of the current status of knowledge on this and the following finite dimensional version of the same problem.

10. Consider a polynomial in n real or complex variables, $P : \mathbb{K}^n \to \mathbb{K}$ such that $P(0) = 0$. Consider the zero set $P^{-1}(0)$ of P. Does $P^{-1}(0)$ contain a vector space? If so, what can we say about the dimension of $P^{-1}(0)$?

What is especially intriguing concerning these ten problems (and many others) is the near total lack of accurate intuitive insight that we possess about them! In the rest of the paper, we will give some answers that are incomplete in many cases, in hopes that the reader will continue this very incomplete study.

2. Lineability of the zeros of polynomials

Our aim here is to examine problems 9 and 10 above. We will review some non-technical arguments that show that the situation for complex polynomials is reasonably well understood. We will briefly discuss the real case, which is considerably more challenging. Finally, we will describe some open problems for both the real and complex situations.

The Complex Case

Definition 2. Let E be a Banach space over $\mathbb{K}$, $k \in \mathbb{N}$. A continuous function $P : E \to \mathbb{K}$ is said to be a *k–homogeneous polynomial* if there exists a k–linear continuous, symmetric functional $A : E \times \cdots \times E \to \mathbb{K}$ such that $P(x) = A(x, ..., x)$ for all $x \in E$. (In this case, the associated symmetric functional A is necessarily unique.) We say that P is a *polynomial of degree k* if P is a finite sum of j–homogeneous polynomials P_j, $P = P_0 + P_1 + \cdots + P_k$.

Examples. We agree that a 0–homogeneous polynomial is a constant. A 1–homogeneous polynomial is just a linear form. A 2–homogeneous polynomial $P : \mathbb{K}^n \to \mathbb{K}$ corresponds to a *quadratic form* Q (i.e. a symmetric $n \times n$ matrix over $\mathbb{K}$), via $P(x) = x^t Q x \quad (x \in \mathbb{K}^n)$, which in turn corresponds to the symmetric bilinear form A on $\mathbb{K}^n \times \mathbb{K}^n$, $\ A(x, y) = y^t Ax$.

To simplify the statements of the results that follow, we will assume from now on all that polynomials considered will have value 0 at 0. One of the most important results to date in this area is the following theorem due to A. Plichko and A. Zagorodonyuk.

Theorem 3. [30] Let E be an infinite dimensional vector space over $\mathbb{K} = \mathbb{C}$, and let $P : E \to \mathbb{C}$ be a polynomial. Then $P^{-1}(0)$ contains an infinite dimensional subspace.

A finite dimensional version of this result may have independent interest.

Corollary 4. [10] Given $m, d \in \mathbb{N}$, there is $k = k(m, d) \in \mathbb{N}$ such that if $P : \mathbb{C}^k \to \mathbb{C}$ is a polynomial of degree d, then $P^{-1}(0)$ contains an m–dimensional subspace.

In other words, if we want to be sure that $P^{-1}(0)$ contains a vector space of (at least) a certain dimension m for all polynomials of a certain degree d, then we can find an integer k such that every polynomial of degree at most d in k complex variables will vanish on a subspace of dimension at least m. Or, described in still another way, given d, as the number k of variables tends to infinity, the dimension of the subspace contained in the zero set of a polynomial on $\mathbb{C}^k$, whose degree is at most d also tends to infinity.

We now sketch a proof of these results, following [10] (which, after all, is based on [30]). For it, we will use one well-known fact from several complex variables: *Fact:* If $f : \mathbb{C}^n \to \mathbb{C}$ is an analytic function, where $n \geq 2$, then either $f^{-1}(0) = \emptyset$ or $f^{-1}(0)$ is an unbounded set. We will apply this to the case when f is a non-constant polynomial (in fact, in the case of a polynomial in several variables, the 'fact' is easy to show), and thus we will be able to conclude that $f(z) = 0$ for some $z \neq 0$.

Proof idea for Theorem 3 and Corollary 4. The formal proof is an inductive argument on the degree of homogeneity d. The constructive idea of this can already be seen in the case $d = 1$ and $d = 2$. If $d = 1$, then P is just a continuous linear form whose kernel is a hyperplane. (In other words, if P is a linear form in k variables, the dimension of the zero set of P is $m = k - 1$.) Now, let $P : E \to \mathbb{C}$ be a 2–homogeneous polynomial with associated symmetric bilinear form A. If $\dim E \geq 2$, then by our fact, there is a non-zero vector $x_1 \in E$ such that $P(x_1) = 0$. Next, consider the set $S_1 \equiv \{x \in E \ : \ A(x, x_1) = 0\}$. This set, which in fact is a hyperplane, contains x_1. Thus S_1 can be written as $E_1 \bigoplus [x_1]$, where $\dim E_1$ is 2 less than $\dim E$. (Of course, this statement is not tremendously profound if $\dim E = \infty$.) If $\dim E_1 \geq 2$, then our fact will yield a second non-zero vector $x_2 \in E_1$ for which $P(x_2) = 0$. What this really means is that if $\dim E \geq 4$, then we can find independent vectors x_1, x_2 such that $P(x_1) = P(x_2) = 0$. Note that in addition $[x_1, x_2] \subset P^{-1}(0)$. To see this, it is enough to check that $x_1 + x_2 \in P^{-1}(0)$:

$$P(x_1+x_2) = A(x_1+x_2, x_1+x_2) = A(x_1, x_1)+A(x_1, x_2)+A(x_2, x_1)+A(x_2, x_2),$$

which since A is symmetric,

$$= P(x_1) + 2A(x_2, x_1) + P(x_2) = 0,$$

since $x_2 \in E_1$. Continuing, let $S_2 \equiv \{x \in E_1 \ : \ A(x, x_2) = 0\}$, so $S_2 = E_2 \bigoplus [x_2]$, where $\dim E_2 = \dim E_1 - 2$. Once again, if $\dim E_2 \geq 2$, then we can find $x_3 \in E_2, x_3 \neq 0$, such that $P(x_3) = 0$. The general argument proceeds by a not too difficult induction.

Two observations should be made at this point, both dealing with the method of proof and both having 'resonance' with the real case that we'll describe, very briefly, in a few paragraphs.

Remarks 5.1. The argument is essentially *constructive.* In particular, to 'build' a subspace of dimension k contained in $P^{-1}(0)$ one can use a previously constructed subspace of dimension $k - 1$ and add an appropriate vector to it.

5.2. The argument is *countable.* To us, it seems natural to expect that if $\dim E$ is uncountable, then $P^{-1}(0)$ should contain a non-separable space. Our proof merely shows that there is a separable space contained in $P^{-1}(0)$. Some interesting recent work by Plichko and Zagorodnyuk [11] has been done on this problem, which suggests that the phrase "natural to expect" in the previous sentence may not be true. On the other hand, a number of very interesting positive results have recently been obtained by M. Fernández [20] and C. Soares [34].

The Real Case

We now study the size of vector spaces contained in $P^{-1}(0)$ in the case of polynomials $P : E \to \mathbb{R}$, $P(0) = 0$. Of course, $\sum_{j=1}^{n} x_j^2 : \mathbb{R}^n \to \mathbb{R}$ shows that no extremely general result is possible. However, we can accomplish a reasonable amount. The first result is just an easy exercise in linear algebra.

Proposition 6. (case $d = 2$) Let $P : \mathbb{R}^k \to \mathbb{R}$ be a $2-$homogeneous polynomial and let Q be the associated quadratic form. Let $p =$ the number of positive eigenvalues of Q, let $n =$ the number of negative eigenvalues of Q, and let $z =$ the number of 0 eigenvalues of Q. Then $P^{-1}(0)$ contains a subspace of dimension $= \min\{p, n\} + z$.

It would be interesting to know if there is an analogous result for the case of $d = 4-$homogeneous polynomials.

For the rest of this section, we will focus on homogeneous polynomials of odd degree. Using a fairly technical argument that (apparently) works only for $d = 3$, R. Gonzalo, A. Zagorodnyuk, and the author were able to show that the analogue of Corollary 4 holds [5]. This was later extended by P. Hajek and the author (using a different technical, counting argument), as follows:

Theorem 7. [7] Given $m, d \in \mathbb{N}$ where d is odd, there is $k = k(m, d) \in \mathbb{N}$ such that if $P : \mathbb{R}^k \to \mathbb{R}$ is an odd polynomial of degree d, then $P^{-1}(0)$ contains an $m-$dimensional subspace.

Remark 8. The proofs in both [5] and [7] are *non-constructive.* Knowing the $k-1$ vectors that span a subspace of $P^{-1}(0)$ does not help us to find a k dimensional subspace. Consequently, we don't know whether the analogue of Plichko's and Zagorodnyuk's Theorem 3 holds. Specifically, suppose that $P: E \to \mathbb{R}$ is an odd-homogeneous polynomial on an infinite dimensional real Banach space E. Although we know that $P^{-1}(0)$ contains a vector space of every finite dimension, we do not know whether $P^{-1}(0)$ contain an infinite dimensional vector space.

3. Non-spaceability of differentiable functions

As was seen in §2 and will be observed in later sections, in many situations sets that might otherwise be said to have pathological, non-linear properties contain a rich linear structure. In this section, we will mention a counterpart to this situation. Namely, we will describe how poor the structure can be in the case of functions with what most people would call very good properties. Namely, we will outline the proof of V. Gurariy and W. Lusky [25] of the following theorem, due originally to Fonf, V. Gurariy, and V. Kadets [21].

Theorem 9. Let $V \subset C[0,1]$ be a closed subspace of $C[0,1]$ consisting of everywhere differentiable functions. Then $\dim V < \infty$.

Of course, there are infinite dimensional vector spaces of everywhere differentiable (in fact, C^∞) functions contained in $C[0,1]$. The point is that none of these spaces is complete.

The basic concept behind the following proof is well known; see, for example, ([32], 7.18). For the proof, we will need some terminology. First, a sequence (A_k) of subsets of $[0,1]$ is said to be *condensed* if there is some sequence (ϵ_k) of positive numbers that tends to 0 such that each A_k is an ϵ_k-net for $[0,1]$. That is, for all $t, 0 \le t \le 1$, there is $s \in A_k$ such that $|s-t| < \epsilon_k$. A sequence of functions $(f_k) \subset C[0,1]$ is said to be *condensed* if the associated sequence of zero sets, $(f_k^{-1}(0))$, is condensed.

We will omit the proof of the following slightly technical, but not difficult, lemma.

Lemma 10. Let $V \subset C[0,1]$ be an infinite dimensional space. Then V contains a condensed sequence of functions.

Proof of Theorem 9. Suppose that there is an infinite dimensional Banach space $V \subset C[0,1]$ all of whose elements are everywhere differentiable. By Lemma 10, there is a condensed sequence $(f_k) \subset V$, and we may assume that all the f_k have norm one. Let $A_k = f_k^{-1}(0)$ and let $t_k \in [0,1]$ be such that $|f_k(t_k)| = 1$. By passing to a subsequence, we may suppose that $(t_k) \longrightarrow t_0$ for some $t_0 \in [0,1]$. Let $s_k \in A_k$ be close to t_k and set

$$t_k^* \equiv \begin{cases} t_k \text{ if } |f_k(t_0)| \leq \frac{1}{2} \\ s_k \text{ if } |f_k(t_0)| > \frac{1}{2} \end{cases}.$$

Because of the condensed property, we may assume that

$$|t_k^* - t_0| < \frac{1}{32^k}.$$

Note also that $|f_k(t_k^*) - f_k(t_0)| \geq \frac{1}{2}$. Now define $f : [0,1] \longrightarrow \mathbb{R}$ by

$$f(t) = \sum_{j=1}^{\infty} \frac{\theta_j}{16^j} f_j,$$

where we have chosen $\theta_j = \pm 1$ so that $\sum_{j=1}^{k-1} \frac{\theta_j}{16^j}(f_j(t_k^*) - f_j(t_0))$ and $\sum_{j=1}^{k} \frac{\theta_j}{16^j}(f_j(t_k^*) - f_j(t_0))$ have the same sign. It is easy to see that $f \in V$. However, f is not differentiable at t_0. Indeed,

$$|f(t_k^*) - f(t_0)| \geq \frac{1}{16^k}|f_k(t_k^*) - f_k(t_0)| - \sum_{j=k+1}^{\infty} \frac{1}{2^j}|f_j(t_k^*) - f_j(t_0)| \geq \frac{1}{4 \cdot 16^k},$$

provided $k \geq 2$. Thus, $\lim_{k\to\infty} \frac{f(t_k^*)-f(t_0)}{t_k^*-t_0}$ does not exist, and we have a contradiction. Q.E.D.

4. Lineability of hypercyclic operators

Let $T : X \longrightarrow X$ be a continuous linear operator on a Banach or Fréchet space X. We say that T is *hypercyclic* if there is a vector $x \in X$ whose orbit under T, $\text{Orb}(T,x) = \{x, Tx, T^2x, ..., T^nx, ...\}$ is dense in X; the vector x is called a *hypercyclic vector* associated to T. Clearly, hypercyclic operators are much harder to come by than *cyclic* operators T, which have the defining property that *span* $\text{Orb}(T,x)$ is dense in X. Despite this, hypercyclic operators are (perhaps) surprisingly ubiquitous, and in this section we will briefly comment on this and on the fact that the set of hypercyclic vectors x that are associated to an operator T contain vector spaces that are large and not yet fully understood.

It might be fair to say that the origin of hypercyclicity dates from the short note of G. Birkhoff [15] over 75 years ago. It was Birkhoff who first proved that the translation operator $\tau_1 : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, given by $\tau_1(f)(z) = f(z+1) \quad (f \in \mathcal{H}(\mathbb{C}),\ z \in \mathbb{C})$, is hypercyclic. Around 25 years later, G. MacLane [28] proved that the derivative operator $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}),\quad D(f) = f'$, is hypercyclic. Nearly fifteen years ago, G. Godefroy and J. Shapiro [22] generalized these two results to what are called *convolution operators,* or *infinite order differential operators,* as follows:

Theorem 11. Let $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ be a continuous linear operator that satisfies the following two conditions:
(i). $T \circ D = D \circ T$;
(ii). $T \neq cI$ for any $c \in \mathbb{C}$, where $I : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is the identity operator.
Then T is hypercyclic.

However, it is only in recent years that this area of analysis has "taken off" as a research area. We will limit ourselves to reviewing three topics: (i). The dense lineability and the spaceability of the space of hypercyclic vectors; (ii). Vectors that are hypercyclic for very many hypercyclic operators; (iii). Some open problems.

P. Bourdon was perhaps the first to make the following not difficult, but important observation.

Proposition 12. [18] If $T : X \to X$ is hypercyclic, then the set $\mathcal{HC}(T)$ of hypercyclic vectors for T contains a dense vector space.

Proof for X a complex Banach space. (The argument for Fréchet spaces and for *real* spaces is somewhat more complicated.)

First, one observes that if T is hypercyclic, then $T - \lambda I$ has dense range for any complex number λ. Indeed, if this were not the case for some λ, then by the Hahn-Banach theorem there would exist a non-zero $\phi \in X^*$ such that $\phi(T - \lambda I)(x) = 0$ for all $x \in X$. As a result, we would have that $T^t(\phi) = \lambda\phi$ for some nonzero $\phi \in X^*$. Consequently, for any $x \in X, \{\phi(T^n(x)) \mid n \in \mathbb{N}\} = \{\lambda^n(\phi(x)) \mid n \in \mathbb{N}\}$, and this set is never dense in $\mathbb{C}$. Since this contradicts the hypercyclicity of T, it follows that $T - \lambda I$ always has dense range.

Note that if x is hypercyclic for T, then $\{P(T)(x) \mid P \text{ is a polynomial}\}$ is a dense vector space in X (just take $P(T) = T^n$, $n \in \mathbb{N}$). Next, we claim that any non-zero $P(T)$ has dense range. To see this factor P into linear terms, $P(z) = a(z-\lambda_1)\cdots(z-\lambda_k)$, so that $P(T) = a(T-\lambda_1 I)\cdots(T-\lambda_k I)$. Then apply the first observation, above. Finally, $P(T)(x)$ is a hypercyclic vector for T, since $\{P(T)(x), T(P(T)(x)), ..., T^n(P(T))(x), ...\} = P(T)(\{x, T(x), ..., T^n(x), ...\})$. Q.E.D.

From this, it follows that there are no hypercyclic operators on $\mathbb{C}^n$ for any $n \in \mathbb{N}$.

Now, what spaces admit hypercyclic operators? By a result proved independently by S. Ansari [2] and L. Bernal [13], every infinite dimensional Banach space admits a hypercyclic operator. In fact, this is even true for infinite dimensional Fréchet spaces, as J. Bonet and A. Peris have observed [17]. Thus, dense lineability of hypercyclic vectors occurs in every infinite dimensional space. On the other hand, the situation concerning spaceability is much more complicated and only partially understood. For instance, Bernal-González and Montes-Rodríguez [14] proved that the set of hypercyclic entire functions for the Birkhoff translation τ_1 (together with the 0 function, as usual) contains a closed hypercyclic subspace (for the compact-open topology). On the other hand, in [29] Montes shows that the set of vectors in ℓ_2 that are hypercyclic for the Rolewicz backward shift does not contain a closed infinite dimensional subspace; that is, the set of hypercyclic vectors for the weighted backward shift $(x_1, x_2, x_3, ...) \to 2(x_2, x_3, x_4, ...)$ is not spaceable. Does the same hold for $\mathcal{HC}(D)$? In a rough sense, the operator D behaves like the Rolewicz operator, and so a first guess might be that $\mathcal{HC}(D)$ does not contain a closed subspace. But guesses are rarely publishable!

This brings us to the obvious question: How can one tell, or how can one even guess, whether the set of vectors that are hypercyclic for a given hypercyclic operator is spaceable?

To us, the mere existence of a hypercyclic operator is counter-intuitive, at least at first. However, as we just noted in (i) above, given a hypercyclic operator T, the set of vectors that are hypercyclic for T is always a dense vector space. So, our intuition notwithstanding, there seem to be plenty of hypercyclic vectors for at least standard operators. In two recent articles, even more has been shown to hold.

In [1], E. Abakumov and J. Gordon prove the following remarkable result.

Theorem 13. There is a dense G_δ of vectors lying in the set

$$\cap_{\{\lambda\in\mathbb{C}\ |\ |\lambda|>1\}}\mathcal{HC}(T_\lambda),$$

where $T_\lambda : \ell_2 \to \ell_2$, $T_\lambda(x_1, x_2, ...) = \lambda(x_2, x_3, ...)$. In other words, there is a dense G_δ set of vectors in $x \in \ell_2$ such that $\{x, T_\lambda(x), T_\lambda 2(x), \cdots\}$ is dense in ℓ_2 for every $\lambda \in \mathbb{C}$, $|\lambda| > 1\}$.

In addition to giving a new proof of the result in [1], G. Costakis and M. Sambarino [23] have shown that there is a dense G_δ set of entire functions contained in the set

$$\cap_{\{b\in\mathbb{C}\ |\ b\neq 0\}}\mathcal{HC}(\tau_b),$$

where $\tau_b : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$. $f(z) \rightsquigarrow f(z+b)$. In fact, as A. Hallack observes [26], there is even a dense vector space of such functions.

This leads us to several questions. First, in light of the result of [23] it is natural to ask for even more. Namely, is the set

$$\cap\{\mathcal{HC}(T)\ |\ T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}) \text{ a convolution operator, } \mathrm{T} \neq \mathrm{cI}\}$$

non-empty? A seemingly simpler question, which would give a negative answer, is the following: Given $f \in \mathcal{H}(\mathbb{C})$, is there a convolution operator T such that $T(f) = 0$? (Clearly, if $T(f) = 0$, then f cannot be hypercyclic for T.)

Next, note that the set of hypercyclic convolution operators on $\mathcal{H}(\mathbb{C})$ form (or, really, almost form) an algebra under composition. To be specific, if S and T are two hypercyclic convolution operators on $\mathcal{H}(\mathbb{C})$, then so is their composition $S \circ T$ except in the trivial case when $S \circ T = cI$ for some $c \in \mathbb{C}$. Now, in [8] (see also [19]), the question of finding hypercyclic operators on $\mathcal{H}(\mathbb{C})$ that are not convolution operators is examined. In fact, any operator of the form $T_{\lambda,b}(f)(z) = f(\lambda z + b)$ is hypercyclic, provided $|\lambda| \geq 1$; on the other hand, $T_{\lambda,b}$ is a convolution operator if and only if $\lambda = 1$. This leads us to the question: Is the set $\{T_{\lambda,b}\ |\ \lambda, b \in \mathbb{C}, |\lambda| \geq 1\}$ of hypercyclic operators closed under composition?

Third, it is known [3] that if there are no entire functions f such that f^k is hypercyclic for the translation operator τ_1, for any $k \geq 2$. On the other hand, the set of entire functions f such that f^k is hypercyclic for D, the differentiation operator, for every $k \in \mathbb{N}$ is a dense G_δ in $\mathcal{H}(\mathbb{C})$. So, let T be a hypercyclic operator on $\mathcal{H}(\mathbb{C})$. Is there some $f \in \mathcal{HC}(T)$ such that f^k is also in $\mathcal{HC}(T)$? Characterize those T for which such a phenomenon occurs.

5. Baire category and lineability

A recurring theme throughout our exposition has been the Baire category theorem. For instance, not only is there one continuous, nowhere differentiable function on $[0,1]$ but, by use of Baire's theorem, there is a dense G_δ of such functions. Not only is there one entire function f that is hypercyclic with respect to Birkhoff's translation operator but, by Baire's theorem, there is a dense G_δ of such functions. Not only is there one continuous function on the unit circle whose Fourier series at 1 diverges but, by Baire's theorem, there is a dense G_δ of such functions [33]. (F. Bayart [12] has shown that the set of such functions is lineable, and in fact D. Pérez, J. Seoane, and the author [9] have shown that the set is *algebrable.*)

As the reader can see, examples are plentiful. However, is this always true? And, if not, how can we tell when it is true that a residual set of vectors actually contains a 'big' vector space? We don't know.

We conclude our exposition by returning to Example 1 of Introduction 2. It is here that, for the first time in this survey, one encounters a discordant note. Indeed, V. Gurariy and L. Quarta recently proved the following result:

Theorem 14. [24] Consider the set $\mathcal{M} = \{f \in \mathcal{C}[0,1] \mid f$ attains its maximum at exactly one point of $[0,1]\}$. Then no subspace $V \subset \mathcal{M}$ can have dimension ≥ 2.

However, this is by no means a counterexample to any assertion related to the Baire category theorem since the set $\mathcal{M}$ is far from a G_δ. In fact, we are grateful to L. Quarta for pointing out to the following fact to us:

Proposition 15. $\mathcal{C}[0,1]\backslash\mathcal{M}$ is a dense G_δ.

Sketch of proof of Proposition 15. For each $n \in \mathbb{N}$, let

$$U_n = \{f \in \mathcal{C}[0,1] \mid \text{ for some } x \in [0,1],\ f(x) > \max_{|t-x|\geq 1/n} f(t)\}.$$

It is not difficult to see that U_n is open, for if $f \in U_n$ and $g \sim f$, then $g(x)$ will also be $> \max_{|t-x|\geq 1/n} g(t)$. Also, if $g \in C[0,1]$ is arbitrary and $g(x_0) = \max_{t\in[0,1]} g(t)$, then by slightly increasing g near x_0 we get a function $h \sim g$, $h \in U_n$. In other words, each U_n is dense in $C[0,1]$. Finally, $\mathcal{M} = \cap_n U_n$. Indeed, suppose that $f \in U_n$ for each n, but that f attains its maximum at two points, x_0 and x_1. For large n, we will have a contradiction since $f(x_0)$ cannot be strictly greater than $\max_{|t-x_0|>1/n} f(t)$. The converse inclusion is easy. Q.E.D.

Finally, we mention that Gurariy and Quarta show [24] that if we instead let $\mathcal{M} = \{f \in \mathcal{R} \mid f$ attains its maximum at precisely one point in $\mathbb{R}\}$, then it is known that $\mathcal{M}$ contains a 2–dimensional vector space. On the other hand, it is unknown if it is lineable, or even if it contains a 3–dimensional space.

References

1. Abakumov, E.; Gordon, J., *Common hypercyclic vectors for multiples of backward shift,* J. Funct. Anal. **200** (2003), no. 2, 494-504.
2. Ansari, S., *Existence of hypercyclic operators on topological vector spaces,* J. Funct. Anal. **148** (1997), no. 2, 384-390.
3. Aron, R.; Conejero, J. A.; Peris, A.; Seoane, J. B., *Powers of hypercyclic functions for some classical hypercyclic operators,* to appear.
4. Aron, R.; García, D.; Maestre, M., *Linearity in non-linear problems,* RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **95** (2001), no. 1, 7-12.
5. Aron, R.; Gonzalo, R.; Zagorodnyuk, A., *Zeros of real polynomials,* Linear and Multilinear Algebra **48** (2000), no. 2, 107-115.
6. Aron, R.; Gurariy, V. I.; Seoane, J. B., *Lineability and spaceability of sets of functions on* ℝ, Proc. Amer. Math. Soc. **133** (2005), no. 3, 795-803.
7. Aron, R.; Hajek, P. *Zero sets of polynomials in several variables,* Archiv Math., **86** (2006), 561-568.
8. Aron, R.; Markose, D., *On universal functions.,* Satellite Conference on Infinite Dimensional Function Theory, J. Korean Math. Soc. **41** (2004), no. 1, 65-76.
9. Aron, R.; Pérez-García, D.; Seoane-Sepúlveda, J. B., *Algebrability of the set of non-convergent Fourier series,* Studia Math. **175** (1) (2006), 83-90.
10. Aron, R.; Rueda, M. P., *A problem concerning zero-subspaces of homogeneous polynomials,* Dedicated to Professor Vyacheslav Pavlovich Zahariuta. Linear Topol. Spaces Complex Anal. **3** (1997), 20-23.
11. Banakh, T.; Plichko, A.; Zagorodnyuk, A., *Zeros of quadratic functionals on non-separable spaces,* Colloq. Math. **100** (2004), no. 1, 141-147.
12. Bayart, F., *Topological and algebraic genericity of divergence and universal-*

ity, Studia Math. **167** (2005), 153-160.

13. Bernal González, L., *On hypercyclic operators on Banach spaces,* Proc. Amer. Math. Soc. **127** (1999), no. 4, 1003-1010.
14. Bernal González, L.; Montes-Rodríguez, A., *Non-finite-dimensional closed vector spaces of universal functions for composition operators,* J. Approx. Theory **82** (1995), no. 3, 375-391.
15. Birkhoff, G. D., *Démonstration d'un théorème élémentaire sur les fonctions entières,* C. R. Acad. Sci. Paris **189**, 473-475.
16. Bishop, E.; Phelps, R. R., *A proof that every Banach space is subreflexive,* Bull. Amer. Math. Soc. **67** 1961 97-98.
17. Bonet, J.; Peris, A., *Hypercyclic operators on non-normable Frchet spaces,* J. Funct. Anal. **159** (1998), no. 2, 587-595.
18. Bourdon, P., *Invariant manifolds of hypercyclic vectors,* Proc. Amer. Math. Soc. **118** (1993), no. 3, 845-847.
19. Fernández, G.; Hallack, A., *Remarks on a result about hypercyclic non-convolution operators,* J. Math. Anal. Appl. **309** (2005), no. 1, 52-55.
20. Fernández-Unzueta, M., *Zeroes of polynomials on ℓ_∞,* J. Math. Anal. Appl. **324**, 2 (2006), 1115-1124.
21. Fonf, V. P.; Gurariy, V. I.; Kadets, M. I., *An infinite dimensional subspace of $C[0,1]$ consisting of nowhere differentiable functions,* C. R. Acad. Bulgare Sci. **52** (1999), no. 11-12, 13-16.
22. Godefroy, G.; Shapiro, J., *Operators with dense, invariant, cyclic vector manifolds,* J. Funct. Anal. **98** (1991), no. 2, 229-269.
23. Costakis, G.; Sambarino, M., *Genericity of wild holomorphic functions and common hypercyclic vectors,* Adv. Math. **182** (2004), no. 2, 278-306.
24. Gurariy, V. I.; Quarta, L., *On lineability of sets of continuous functions,* J. Math. Anal. Appl. **294** (2004), no. 1, 62-72.
25. Gurariy, V. I.; Lusky, W., *Geometry of Mntz Spaces and Related Questions Series,* Lecture Notes in Mathematics, **1870**, 2005, XIII, 176 p.
26. Hallack, A., *Hypercyclicity for translations through Runge's Theorem,* to appear.
27. James, R. C., *Reflexivity and the supremum of linear functionals,* Ann. of Math. (2) **66** (1957), 159-169.
28. MacLane, G. R., *Sequences of derivatives and normal families. J. Analyse Math.,* **2**, (1952). 72-87.
29. Montes-Rodríguez, A., *Banach spaces of hypercyclic vectors,* Michigan Math. J. **43** (1996), no. 3, 419-436.
30. Plichko, A.; Zagorodnyuk, A., *On automatic continuity and three problems of The Scottish book concerning the boundedness of polynomial functionals,* J. Math. Anal. Appl. **220** (1998), no. 2, 477-494.
31. Rolewicz, S., *On orbits of elements,* Studia Math. **32** 1969 17-22.
32. Rudin, W., *Principles of Mathematical Analysis,* 3^{rd} edition, McGraw-Hill (1976).
33. Rudin, W., *Real and Complex Analysis,* 3^{rd} edition, McGraw-Hill (1987).
34. Soares, C. A., personal communication.