1. HIGHER-DIMENSIONAL PHYSICS

"There's more to this than meets the eye" (Old English saying)

1.1 Introduction

Theoretical physics is in the happy situation of being able to pluck good ideas from philosophy, work them through using the machinery of algebra, and produce something which is both stimulating and precise. It goes beyond words and equations, because when properly done it encapsulates what many people regard as reality.

We sometimes tend to forget what a stride was made when Newton realized that the force which causes an apple to fall to the ground is the same one which keeps the Moon in its orbit – and which is now known to influence the motions of even the most remote galaxies. Nowadays, gravity has to be considered in conjunction with electromagnetism plus the weak and strong forces of particle physics. Even so, it is still possible to give an account of modern physics in a few hundred pages or so. On reflection, this is remarkable. It comes about because of the enormous efficiency of mathematics as the natural language of physics, coupled with the tradition whereby physicists introduce the least number of hypotheses necessary to explain the natural world (Occam's razor of old). At present, it is commonly believed that the best way to explain all of the forces of physics is via the idea of higher dimensions.

In this regard, five-dimensional field theory is particularly useful, as it is the basic extension of the four-dimensional spacetime of Einstein gravity and is widely regarded as the low-energy limit of higher-dimensional theories which more fully address the particle interactions. This slim volume is a concise account of recent developments in 5D theory and their implications for classical and quantum physics.

1.2 Dimensions Then and Now

The idea of a "dimension" is primitive and at least partly intuitive. Recent histories of the idea are given in the books by Wesson (1999) and Halpern (2004). It was already established by the time of Newton, who realized that mass was a more fundamental concept than density, and that a proportionality between physical quantities could be converted to an equation if the latter balanced its ingredients of mass, length and time (i.e., was dimensionally homogeneous). Hence the introduction of a parameter G, which we now call Newton's constant of gravity.

The coordinates of an object (x, y, z) in ordinary space and that of local time (t) are, of course, the basic dimensions of geometry. But the concept of force, at least the gravitational kind, obliges us to introduce another dimension related to the mass of an object (m). And modern physics recognizes other such, notably the one which measures a body's electric charge (q). The role of the so-called fundamental constants of physics is primarily to transpose quantities like mass and charge into geometrical ones, principally lengths (Wesson 1999, pp. 2-11). This is illustrated most cogently by the conversion of the time to an extra coordinate $x^4 \equiv ct$ via the use of the speed of light, a ploy due to Minkowski and Einstein which forms the foundation of 4D spacetime.

The idea of a dimension is, to a certain extent, malleable. It is also important to notice that modern field theories, like general relativity, are written in terms of tensor equations which are not restricted in their dimensionality. One can speculate that had Einstein been formulating his theory of gravity today, he might have established this anonymity of dimension as a principle, on a par with the others with which we are familiar, such as that of equivalence (see Chapter 3). It is this freedom to choose the dimensionality which underlies the numerous extensions of general relativity. These include the original 5D Kaluza-Klein theory, its modern variants which are called induced-matter and membrane theory, plus the higher extensions such as 10D supersymmetry, 11D supergravity and the higher-D versions of string theory.

Kaluza initiated field theory with more than the 4 dimensions of spacetime in 1921, when he published a paper which showed how to unify gravity (as described by Einstein's equations) with electromagnetism (as described by Maxwell's equations). It is well known that Einstein kept Kaluza's paper for a couple of years before finally as referee allowing it to go forward. However, Einstein was then and remained in his later years an advocate of extra dimensions. For example, a letter to Kaluza from Einstein in 1919 stated "The formal unity of your theory is astonishing" (Halpern, 2004, p.1). Indeed, the natural way in which the 4 Maxwell equations fall out of the 15 field equations of what is a kind of general relativity in 5D, has since come to be called the Kaluza-Klein miracle. However, the mathematical basis of the unification is simple: In 5D there are 15 independent components of the metric tensor, of which one refers to a scalar field which was not at the time considered significant and was so suppressed. For similar reasons, to do with the presumed unobservability of effects to do with the extra dimension, all derivatives of the other metric coefficients with respect to the extra coordinate were set to zero (the "cylinder" condition). This left 14 metric coefficients, which could depend on the 4 coordinates of spacetime (x^{α}, $\alpha = 0,123$ for t, xyz). These 14 coefficients were determined by 14 field equations. The latter turned out to be the 10 Einstein equations and the 4 Maxwell equations. Voila: a unification of gravity and electromagnetism.

Klein pushed the 5D approach further in 1926, when he published a paper which showed how to incorporate quantum effects into the theory. He did this by the simple device of assuming that the topology of the extra dimension was not flat and open, but curved into a circle. In other words, while a local orbit in spacetime (x^{α}) would be straight, an orbit in the extra dimension (x^4) would merely go around and around. This cyclic behaviour would lead to quantum effects, *provided* the extra dimension were rolled up to a microscopic size ("compactification"). The size of the extra dimension was presumed to be related to the parameter typical of quantum phenomena, namely

Planck's constant *h*. Among other consequences of the closed topology of the extra dimension, it was shown that the cyclic momentum could be related to the charge of the electron *e*, thus explaining its quantization.

The brainwaves of Kaluza and Klein just summarized are the kind which are neat and yet powerful. They continued to be held in high regard for many years in theoretical physics, even though the latter was redirected by the algebraically simple and effective ideas on wave mechanics that were soon introduced by Schrodinger, Heisenberg and Dirac. Kaluza-Klein theory later underwent a revival, when Einstein's theory was recognized as the best basis for cosmology. But something has to be admitted: Kaluza-Klein theory in its original form is almost certainly wrong.

By this, it is not meant that an experiment was performed which in the standard but simplistic view of physics led to a disproval of the 5D theory. Rather, it means that the original Kaluza-Klein theory is now acknowledged as being at odds with a large body of modern physical lore. For example, the compactification due to Klein leads to the prediction that the world should be dominated by particles with the Planck mass of order 10^{-5} g, which is clearly not the case. (This mismatch is currently referred to as the hierarchy problem, to which we will return.) Also, the suppression of the scalar field due to Kaluza leaves little room to explain the "dark energy" currently believed to be a major component of the universe. (This is a generic form of what is commonly referred to as the cosmological-constant problem, to which attention will be given later.) Further, the cylinder condition assumed by both fathers of 5D field theory effectively rules out any way to explain matter as a geometrical effect, something which Einstein espoused and is still the goal of many physicists.

It is instructive to recall at this juncture the adage which warns us not to throw out the baby with the bath-water. In this instance, the baby is the concept of a 5 (or higher) D space; whereas the water is the smothering algebraic restrictions which were applied to the theory in its early days as a means of making progress, but which are now no longer needed. Hence modern Kaluza-Klein theory, which is algebraically rich and exists in several versions.

1.3 Higher-Dimensional Theories

These may be listed in terms of their dimensionality and physical motivation. However, all are based on Einstein's theory of general relativity. The equations for this and its canonical extension will be deferred to the next section.

Induced-matter theory is based on an unrestricted 5D manifold, where the extra dimension and derivatives with respect to the extra coordinate are used to explain the origin of 4D matter in terms of geometry. (For this reason, it is sometimes called space-time-matter theory.) As mentioned above, this goal was espoused by Einstein, who wished to transpose the "base-wood" of the right-hand side of his field equations into the "marble" of the left-hand side. That is, he wished to find an algebraic expression for what is usually called the energy-momentum tensor $\left(T_{\alpha\beta}\right)$, which was on the same footing as the purely geometrical object we nowadays refer to as the Einstein tensor $\left(G_{\alpha\beta}\right)$. That this is possible in practice was proved using an algebraic reduction of the 5D field equations by Wesson and Ponce de Leon (1992). They were, however, unaware that the technique was guaranteed in principle by a little-known theorem on local embeddings of Riemannian manifolds by Campbell (1926). We will return to the field equations and their embeddings below. Here, we note that the field equations of 5D relativity with a scalar field and dependence on the extra coordinate in general lead to 15 second-order, non-linear relations. When the field equations are set to zero to correspond to a 5D space which is apparently empty, a subset of them gives back the 10 Einstein field equations in 4D *with sources*. That is, there is an effective or induced 4D energy-momentum tensor which has the properties of what we normally call matter, but depends on the extra metric coefficients and derivatives with respect to the extra coordinate. The other 5 field equations give back a set of 4 Maxwell-like or conservation equations, plus 1 scalar relation which has the form of a wave equation. Following the demonstration that matter could be viewed as a consequence of geometry, there was a flurry of activity, resulting in several theorems and numerous exact solutions (see Wesson 1999 for a catalog). The theory has a 1-body solution which satisfies all of the classical tests of relativity in astrophysics, as well as other solutions which are relevant to particle physics.

Membrane theory is based on a 5D manifold in which there is a singular hypersurface which we call 4D spacetime. It is motivated by the wish to explain the apparently weak strength of gravity as compared to the forces of particle physics. It does this by assuming that gravity propagates freely (into the 5D bulk), whereas particle interactions are constrained to the hypersurface (the 4D brane). That this is a practical approach to unification was realized by Randall and Sundrum (1998, 1999) and by Arkani-Hamed, Dimopoulos and Dvali (1998, 1999). The original theory helped to explain the apparently small masses of elementary particles, which is also referred to as the hierarchy problem. In addition, it helped to account for the existence and size of the cosmological constant, since that parameter mediates the exponential factor in the extra coordinate which is typical of distances measured away from the brane. As with induced-matter theory, the membrane approach has evolved somewhat since its inception. Thus there has been discussion of thick branes, the existence of singular or thin branes in (4+d) dimensions or d-branes, and the possible collisions of branes as a means of explaining the big bang of traditional 4D cosmology. It should also be mentioned that the field equations of induced-matter and membrane theory have recently been shown to be equivalent by Ponce de Leon (2001; see below also). This means that the implications of these approaches for physics owes more to interpretation than algebra, and exact solutions for the former theory can be carried over to the latter.

Theories in $N > 5$ dimensions have been around for a considerable time and owe their existence to specific physical circumstances. Thus 10D supersymmetry arose from the wish to pair every integral-spin boson with a half-integral-spin fermion, and thereby cancel the enormous vacuum or zero-point fields which would ensue otherwise. The connection to ND classical field theory involves the fact that it is possible to embed any *curved* solution (with energy) of a 4D theory in a *flat* solution (without energy) of a higher-dimensional theory, provided the larger manifold has a dimension of $N \geq 10$. From the viewpoint of general relativity or a theory like it, which has 10 independent components of the metric tensor or potentials, this is hardly surprising. The main puzzle is that while supersymmetry is a property much to be desired from the perspective of theoretical particle physics, it must be very badly broken in a practical sense. The reason for this apparent conflict between theory and practice may have to do with our (perhaps unjustified) wish to reduce physics to 4D, and/or our (probably incomplete) knowledge of how to categorize the properties of particles using internal symmetry groups. The latter have, of course, to be taken into account when we attempt to estimate the "size" of the space necessary to accommodate both gravity and the particle interactions. Hence the possible unification in terms of (4+7)D or 11D supergravity. However, a different approach is to abandon completely the notion of a point – with its implied singularity – and instead model particles as strings (Szabo 2004, Gubser and Lykken 2004). The logic of this sounds compelling, and string theory offers a broad field for development. But line-like singularities are not unknown, and some of the models proposed have an unmanageably high dimensionality (e.g., N =26). One lesson which can be drawn, though, from $N > 5$D theory is that there is no holy value of N which is to be searched for as if it were a shangri-la of physics. Rather, the value of N is to be chosen on utilitarian grounds, in accordance with the physics to be studied.

1.4 Field Equations in $N \geq 4$ Dimensions

Just as Maxwell's equations provided the groundwork for Einstein's equations, so should general relativity be the foundation for field equations that use more than the 4 dimensions of spacetime.

Einstein's field equations are frequently presented as a match between a geometrical object $G_{\alpha\beta}$ and a physical object $T_{\alpha\beta}$, via a coupling constant κ, in the form $G_{\alpha\beta} = \kappa T_{\alpha\beta} (\alpha, \beta = 0{,}123)$. Here the Einstein tensor $G_{\alpha\beta} \equiv R_{\alpha\beta} - R g_{\alpha\beta} / 2$ depends on the Ricci tensor, the Ricci scalar and the metric tensor, the last defining small intervals in 4D by a quadratic line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. The energy-momentum tensor $T_{\alpha\beta}$ depends conversely on common properties of matter such as the density ρ and pressure p, together with the 4-velocities $u^\alpha \equiv dx^\alpha / ds$. However, even Einstein realized that this split between geometry and matter is subjective and artificial. One example of this concerns the cosmological constant Λ. This was originally added to the left-hand side of the field equations as a geometrical term $\Lambda g_{\alpha\beta}$, whence the curvature it causes in spacetime corresponds to a force per unit mass (or acceleration) $\Lambda rc^2/3$, where r is the distance from a suitably chosen origin of coordinates. But nowadays, it is commonly included in the right-hand side of the field equations as an effective source for the vacuum, whose equation of state is $p_v = -\rho_v c^2$, where $\rho_v = \Lambda c^2/8\pi G$ corresponds to the density of a non-material medium. (Here we take the dimensions of Λ as length^{-2} and retain physical units for the speed of light c and gravitational constant G, so the coupling constant in the field equations is $\kappa = 8\pi G/c^4$.) The question about where to put Λ is largely one of semantics. It makes little difference to the real issue, which is to obtain the $g_{\alpha\beta}$ or potentials from the field equations.

The latter in traditional form are

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \quad . \tag{1.1}$$

Taking the trace of this gives $R = 4\Lambda - \left(8\pi G/c^4\right)T$ where $T \equiv g^{\alpha\beta}T_{\alpha\beta}$. Using this to eliminate R in (1.1) makes the latter read

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}T\, g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta} \quad . \tag{1.2}$$

Here the cosmological constant is treated as a source term for the vacuum, along with the energy-momentum tensor of "ordinary" matter. If there is none of the latter then the field equations are

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad . \tag{1.3}$$

These have 10 independent components (since $g_{\alpha\beta}$ is symmetrical). They make it clear that Λ measures the mean radius of curvature of a 4D manifold that is empty of conventional sources, i.e. vacuum. If there are no sources of *any* kind – or if the ordinary matter and vacuum fields cancel as required by certain symmetries – then the field equations just read

$$R_{\alpha\beta} = 0 \quad . \tag{1.4}$$

It is these equations which give rise to the Schwarzschild and other solutions of general relativity and are verified by observations.

The field equations of 5D theory are taken by analogy with (1.4) to be given by

$$R_{AB} = 0 \ \left(A, B = 0{,}123{,}4\right) \quad . \tag{1.5}$$

Here the underlying space has coordinates $x^A = \left(t, xyz, l\right)$ where the last is a length which is commonly taken to be orthogonal to spacetime. The associated line element is $dS^2 = g_{AB} dx^A dx^B$, where the 5D metric tensor now has 15 independent components, as does (1.5). However, the theory is covariant in its five coordinates, which may be chosen for convenience. Thus a choice of coordinate frame, or gauge, may be made which reduces the number of g_{AB} to be determined from 15 to 10. This simplifies both the line element and the field equations.

The electromagnetic gauge was used extensively in earlier work on 5D relativity, since it effectively separates gravity and electromagnetism. A more modern form of this expresses the 5D line element as parts which depend on $g_{\alpha\beta}$ (akin to the Einstein gravitational potentials with associated interval $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$), Φ (a scalar field which may be related to the Higgs field by which particles acquire their masses), and A_μ (related to the Maxwell potentials of classical electromagnetism). The 5D line element then has the form

$$dS^2 = ds^2 + \varepsilon\,\Phi^2\left(dx^4 + A_\mu dx^\mu\right)^2 \quad . \tag{1.6}$$

Here $\varepsilon = \pm 1$ determines whether the extra dimension $\left(g_{44} = \varepsilon\,\Phi^2\right)$is spacelike or timelike: both are allowed by the mathematics, and we will see elsewhere that $\varepsilon = -1$ is associated with particle-like behaviour while $\varepsilon = +1$ is associated with wave-like behaviour. Most work has been done with the former choice, so we will often assume that the 5D metric has signature $(+----)$. Henceforth, we will also absorb the constants c and G by a suitable choice of units. Then the dynamics which follows from (1.6) may be investigated by minimizing the 5D interval, via $\delta\left[\int dS\right]=0$ (Wesson 1999, pp. 129-153). In general, the motion consists of the usual geodesic one found in Einstein theory, plus a Lorentz-force term of the kind found in Maxwell theory, and other effects due to the extended nature of the geometry including the scalar field. Further results on the dynamics, and the effective 4D energy-momentum tensor associated with the off-diagonal terms in line elements like (1.6), have been worked out by Ponce de Leon (2002). We eschew further discussion of metrics of this form, however, to concentrate on a more illuminating case.

The gauge for neutral matter has a line element which can be written

$$dS^2 = g_{\alpha\beta}\left(x^\gamma, l\right) dx^\alpha dx^\beta + \varepsilon\,\Phi^2\left(x^\gamma, l\right) \quad . \tag{1.7}$$

In this we have set the electromagnetic potentials ($g_{4\alpha}$) to zero, but the remaining degree of coordinate freedom has been held in reserve. (It could in principle be used to flatten the scalar potential via $\left|g_{44}\right|=1$, but while we will do this below it is instructive to see what effects follow from this field.) The components of the 5D Ricci tensor for metric (1.7) have wide applicability. They are:

$$^{5}R_{\alpha\beta} = {}^{4}R_{\alpha\beta} - \frac{\Phi_{,\alpha;\beta}}{\Phi} + \frac{\varepsilon}{2\Phi^2}\left(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}\right)$$

$$R_{4\alpha} = \Gamma\left(\frac{g^{\beta\lambda} g_{\lambda\alpha,4} - \delta^{\beta}_{\alpha} g^{\mu\nu} g_{\mu\nu,4}}{2\Gamma} \right)_{,\beta} + \frac{g^{\mu\beta} g_{\mu\beta,\lambda} g^{\lambda\sigma} g_{\sigma\alpha,4}}{4} - \frac{g^{\lambda\beta} g_{\beta\mu,\alpha} g^{\mu\sigma} g_{\sigma\lambda,4}}{4}$$

$$R_{44} = -\varepsilon\,\Phi\Box\Phi - \frac{g^{\lambda\beta}{}_{,4} g_{\lambda\beta,4}}{2} - \frac{g^{\lambda\beta} g_{\lambda\beta,44}}{2} + \frac{\Phi_{,4} g^{\lambda\beta} g_{\lambda\beta,4}}{2\Phi} - \frac{g^{\mu\beta} g^{\lambda\sigma} g_{\lambda\beta,4} g_{\mu\sigma,4}}{4} \quad . \qquad (1.8)$$

Here a comma denotes the ordinary partial derivative, a semicolon denotes the ordinary 4D covariant derivative, $\Box\Phi \equiv g^{\mu\nu}\Phi_{,\mu;\nu}$ and $\Gamma \equiv \left|\varepsilon\Phi^2\right|^{1/2}$. Superscripts are used here and below for the 5D tensors and their purely 4D parts, whenever there is a risk of confusion. When the components (1.8) are used with the 5D field equations (1.5), it is clear that we obtain tensor, vector and scalar equations which have distinct applications in physics.

The tensor components of (1.8), in conjunction with the 5D field equations $R_{AB} = 0$ (1.5), give the 10 field equations of Einstein's general relativity. The method by which this occurs is by now well known (Wesson and Ponce de Leon 1992). In summary, we form the conventional 4D Ricci tensor, and with it and the 4D Ricci scalar construct the 4D Einstein tensor $G_{\alpha\beta} \equiv {}^4R_{\alpha\beta} - {}^4R\, g_{\alpha\beta}/2$. The remaining terms in ${}^{5}R_{\alpha\beta}$ of (1.8) are then used to construct an effective or induced 4D energy-momentum tensor via $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. Several instructive results emerge during this process. For example, the 4D scalar curvature just mentioned may be shown using all of (1.8) to ge given by

$$ {}^{4}R = \frac{\varepsilon}{4\Phi^{2}} \left[g^{\mu\nu}{}_{,4} g_{\mu\nu,4} + \left(g^{\mu\nu} g_{\mu\nu,4} \right)^{2} \right] \quad . \tag{1.9} $$

This relation has been used implicitly in the literature, but explicitly as here it shows that: (a) What we call the curvature of 4D spacetime can be regarded as the result of embedding it in an x^4-dependent 5D manifold; (b) the sign of the 4D curvature depends on the signature of the 5D metric; (c) the magnitude of the 4D curvature depends strongly on the scalar field or the size of the extra dimension $\left(g_{44} = \varepsilon\Phi^{2} \right)$, so while it may be justifiable to neglect this in astrophysics (where the 4D curvature is small) it can be crucial in cosmology and particle physics. Another instructive result concerns the form of the 4D energy-momentum tensor. It is given by

$$ 8\pi T_{\alpha\beta} = \frac{\Phi_{,\alpha;\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^{2}} \left\{ \frac{\Phi_{,4} g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu} g_{\alpha\lambda,4} g_{\beta\mu,4} \right. $$

$$ \left. - \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2} + \frac{g_{\alpha\beta}}{4} \left[g^{\mu\nu}{}_{,4} g_{\mu\nu,4} + \left(g^{\mu\nu} g_{\mu\nu,4} \right)^{2} \right] \right\} \quad . \tag{1.10} $$

This relation has been used extensively in the literature, where it has been shown to give back all of the properties of ordinary matter (such as the density and pressure) for standard solutions. However, it has further implications, and shows that: (a) What we call matter in a curved 4D spacetime can be regarded as the result of the embedding in an x^4-dependent (possibly flat) 5D manifold; (b) the nature of the 4D matter depends on the signature of the 5D metric; (c) the 4D source depends on the extrinsic curvature of the embedded 4D space-time and the scalar field associated with the extra dimension, which while they are in general mixed correspond loosely to ordinary matter and the stress-energy of the vacuum. In conclusion for this paragraph, we see that a 5D manifold – which is apparently empty – contains a 4D manifold with sources, where the tensor set of the 5D field equations corresponds to the 4D Einstein equations of general relativity.

The vector components of (1.8), in conjunction with (1.5), can be couched as a set of conservation equations which resemble those found in Maxwellian electromagnetism and other field theories. They read

$$P^{\beta}_{\alpha;\beta} = 0 \quad , \tag{1.11}$$

where the 4-vector concerned is defined via

$$P^{\beta}_{\alpha} \equiv \frac{1}{2\Phi}\left(g^{\beta\sigma} g_{\sigma\alpha,4} - \delta^{\beta}_{\alpha} g^{\mu\nu} g_{\mu\nu,4}\right) \quad . \tag{1.12}$$

These are usually easy to satisfy in the continuous fluid of induced-matter theory, and are related to the stress in the surface (x^4 = 0) of membrane theory with the Z_2 symmetry (see below). It should be noted that these relations do not come from some external criterion such as the minimization of the line element, but are derived from and are an inherent part of the field equations.

The scalar or last component of (1.8), when set to zero in accordance with the field equations (1.5), yields a wave-type equation for the potential associated with the fifth dimension $\left(g_{44} = \varepsilon\Phi^2\right)$ in the metric (1.7). It is

$$\Box\Phi = -\frac{\varepsilon}{2\Phi}\left[\frac{g^{\lambda\beta}{}_{,4}g_{\lambda\beta,4}}{2} + g^{\lambda\beta}g_{\lambda\beta,44} - \frac{\Phi_{,4}g^{\lambda\beta}g_{\lambda\beta,4}}{\Phi}\right] \quad . \qquad (1.13)$$

Here as before $\Box\Phi \equiv g^{\alpha\beta}\Phi_{,\alpha;\beta}$ and some of the terms on the right-hand side are present in the energy-momentum tensor of (1.10). In fact, one can rewrite (1.13) for the static case as a Poisson-type equation with an effective source density for the Φ-field. In general (1.13) is a wave equation with a source induced by the fifth dimension.

Let us now leave the gauge for neutral matter (1.7) and focus on a special case of it, called the canonical gauge. This was the brainchild of Mashhoon, who realized that if one factorizes the 4D part of a 5D metric in a way which mimics the use of cosmic time in cosmology, significant simplification follows for both the field equations and especially the equations of motion (Mashhoon, Liu and Wesson 1994). The efficacy of this gauge is related to the fact that a quadratic factor in l on the 4D part of a 5D model has algebraic consequences similar to those of a quadratic factor in t on the 3D part of a 4D cosmological model. The latter case, in the context of 4D Friedmann-Robertson-Walker (FRW) cosmologies, is known as the Milne universe. This has several interesting properties (Rindler 1977). We will come back later to the Milne universe as a lower-dimensional example of highly-symmetric 5D manifolds. For now, we note the form of the metric and summarize its properties.

The 5D canonical metric has a line element given by

$$dS^2 = \frac{l^2}{L^2} g_{\alpha\beta}\left(x^\gamma, l\right) dx^\alpha dx^\beta - dl^2 \quad , \qquad (1.14)$$

where $x^4 = l$ is the extra coordinate and L is a constant length introduced for the consistency of physical dimensions. There is an extensive literature on (1.14), both with regard to solutions of the field equations (1.5) and the equations of motion which follow from minimizing the interval S in (1.14). Some of the consequences of (1.14) can be inferred from what we have already learned, while some will become apparent from later study. But for convenience we here summarize all of its main properties following Wesson (2002): (a) Mathematically (1.14) is general, insofar as the five available coordinate degrees of freedom have been used to set $g_{4\alpha} = 0,\ g_{44} = -1$. Physically, this removes the potentials of electromagnetic type and flattens the potential of scalar type. (b) The metric (1.14) has been extensively used in the field equations, and many solutions are known. These include solutions for the 1-body problem and cosmology which have acceptable dynamics and solutions with the opposite sign for g_{44} which describe waves. (c) When $\partial g_{\alpha\beta} / \partial l = 0$ in (1.14), the 15 field equations $R_{AB} = 0$ of (1.5) give back the Einstein equations as described above, now in the form $G_{\alpha\beta} = 3g_{\alpha\beta} / L^2$. These in general identify the scale L as the characteristic size of the 4-space. For the universe, the last-noted relations define an Einstein space with

$$\Lambda = 3 / L^2 \quad , \qquad (1.15)$$

which identifies the cosmological constant. (d) This kind of local embedding of a 4D Riemann space in a 5D Ricci-flat space can be applied to any *N*, and is guaranteed by Campbell's theorem. We will take this up in more detail below. (e) The factorization in (1.14) says in effect that the 4D part of the 5D interval is $(l / L)ds$, which defines a *momentum* space rather than a *coordinate* space if *l* is related to *m*, the rest mass of a particle. This has been discussed in the literature as a way of bridging the gap between the concepts of acceleration as used in general relativity, and force (or change of momentum) as used in quantum theory. (f) Partial confirmation of this comes from a study of the 5D geodesic and a comparison of the constants of the motion in 5D and 4D. In the Minkowski limit, the energy of a particle moving with velocity v is $E = l\left(1 - v^2\right)^{-1/2}$ in 5D, which agrees with the expression in 4D if $l = m$. (g) The five components of the geodesic equation for (1.14) split naturally into four spacetime components and an extra component. For $\partial g_{\alpha\beta}/\partial l \neq 0$, the former contain terms parallel to the 4-velocity u^{α}, which do not exist in 4D general relativity. We will look into this situation later. But we note now that for $\partial g_{\alpha\beta}/\partial l = 0$, the motion is not only geodesic in 5D but geodesic in 4D, as usual. Indeed, for $\partial g_{\alpha\beta}/\partial l = 0$, we recover the 4D Weak Equivalence Principle as a kind of symmetry of the 5D metric.

The preceding list of consequences of the canonical metric (1.14) shows that it implies departures from general relativity when its 4D part depends on the extra coordinate, but inherits many of the properties of Einstein's theory when it does not. In the latter case, the 4D cosmological constant is inherited from the 5D scaling, and has a value $\Lambda = \pm 3/L^2$ depending on the signature of the extra dimension $(\varepsilon = \mp 1)$. This is a neat result, and elucidates the use of de Sitter and anti-de Sitter spaces in approaches to cosmology and particle production, which use quantum-mechanical approaches such as tunneling. However, in general we might expect the potentials of spacetime to depend on the extra coordinate. Both for this case as in (1.14), and for the case where the scalar potential is significant as in (1.7), the vacuum will have a more complicated structure than that implied by the simple cosmological constant just noted. It was shown in (1.10) that in general the effective 4D energy-momentum tensor for neutral matter in 5D theory contains contributions from both ordinary matter and the vacuum. Ordinary matter (meaning material particles and electromagnetic fields) displays an enormous complexity of structure.

"Vacuum matter" (meaning the scalar field and virtual particles which defy Heisenberg's uncertainty relation) may display a corresponding complexity of structure. To use a cliché, 5D induced-matter theory implies that we may have only scratched the surface of "matter".

Membrane theory uses an exponential rather than the quadratic of (1.14) to factorize the 4D part of a 5D metric. Thus a generalized form of the type of metric considered by Randall and Sundrum (1998, 1999) is

$$dS^2 = e^{F(l)} g_{\alpha\beta} dx^\alpha dx^\beta - dl^2 \quad . \tag{1.16}$$

Here $F(l)$ is called the warp factor, and is commonly taken to depend on the cosmological constant Λ and the extra coordinate $x^4 = l$ in such a way as to weaken gravity away from the brane $(l = 0)$. Particle interactions, by comparison, are stronger by virtue of being confined to the brane, which is effectively the focus of spacetime. An important aspect of (at least) the early versions of brane theory is the assumption of Z_2 symmetry, which means in essence that the physics is symmetric about the hypersurface $l = 0$. This prescription is simple and effective, hence the popularity of membrane theory. However, a comparison of (1.16) and (1.7) shows that the former is merely a special case of the latter, modulo the imposition of the noted symmetry. In fact, examination shows that membrane theory and induced-matter theory are basically the same from a mathematical viewpoint, even if they differ in physical motivation. The most notable difference is that for membrane theory particles are confined to the spacetime hypersurface by the geometry, which is constructed with this in mind; whereas for induced-matter theory particles are only constrained by solutions of the 5D geodesic equation, and can wander away from spacetime at a slow rate governed by the cosmological constant or oscillate around it. That the field equations of membrane theory and space-time-matter theory are equivalent was shown by Ponce de Leon (2001). His work makes implicit use of embeddings, and we defer a discussion of these plus the connection between brane and STM worlds to the next section.

Embeddings must, however, play an important role in the extension of 5D theories to those of even higher dimension. That this is so becomes evident when we reflect on the preceding discussion. In it, we have morphed from 4D general relativity with Einstein's equations in the forms (1.1)–(1.4), to 5D relativity with the apparently empty field equations (1.5). These lead us to consider the electromagnetic gauge (1.6) and the gauge for neutral matter (1.7). The latter has associated with it the 5D Ricci components (1.8), which imply the 4D Ricci scalar (1.9) and the effective 4D energy-momentum tensor (1.10). The latter balances the Einstein equations, and leaves us with the 4 vector terms which satisfy (1.11) by virtue of (1.12), plus the 1 scalar wave equation (1.13). When the 5D metric has a 4D part which is factorized by a quadratic in the extra coordinate, we obtain the canonical metric (1.14), which leads us to view the cosmological constant (1.15) as a scale inherited from 5D. When alternatively the 5D metric has a 4D part which is factorized by an exponential in the extra coordinate, we obtain the warp metric (1.16), which leads us to view spacetime as a singular surface in 5D. All of these results are entrained – in the sense that they follow from the smooth embedding of 4D in 5D. Certain rules of differential geometry underly this embedding. The main one of these is a theorem of Campbell (1926), which was revitalized by Tavakol and coworkers, who pointed out that it also constrains the reduction from general relativity in 4D to models of gravity in 3D and 2D which may be more readily quantized (see Rippl, Romero and Tavakol 1995). It is not difficult to see how to extend the formalism outlined above for $N > 5$, so yielding theories of supersymmetry, supergravity, strings and beyond. But in so doing there is a danger of sinking into an algebraic morass. An appreciation of embedding theorems can help us avoid this and focus on the physics.

1.5 A Primer on Campbell's Theorem

Embedding theorems can be classified as local and global in nature. We are primarily concerned with the former because our field equations are local. (The distinction is relevant, because global theorems are more difficult to establish; and since they may involve boundary conditions, harder to satisfy.) There are several local embedding theorems which are pertinent to *N*D field theory, of which the main one is commonly attributed to Campbell (1926). He, however, only outlined a proof of the theorem in a pedantic if correct treatise on differential geometry. The theorem was studied and established by

Magaard (1963), resurrected as noted above by Rippl, Romero and Tavakol (1995), and applied comprehensively to gravitational theory by Seahra and Wesson (2003). The importance of Campbell's theorem is that it provides an algebraic method to proceed up or down the dimensionality ladder N of field theories like general relativity which are based on Riemannian geometry. Nowadays, it is possible to prove Campbell's theorem in short order using the lapse-and-shift technique of the ADM formalism. The latter also provides insight to the connection between different versions of 5D gravity, such as induced-matter and membrane theory. We will have reason to appeal to Campbell's theorem at different places in our studies of 5D field theory. In the present section, we wish to draw on results by Ponce de Leon (2001) and Seahra and Wesson (2003), to give an ultra-brief account of the subject.

Campbell's theorem in succinct form says: Any analytic Riemannian space $V_n(s,t)$ can be locally embedded in a Ricci-flat Riemannian space $V_{n+1}(s+1,t)$ or $V_{n+1}(s,t+1)$.

We are here using the convention that the "small" space has dimensionality n with coordinates running 0 to $n-1$, while the "large" space has dimensionality $n+1$ with coordinates running 0 to n. The total dimensionality is $N = 1 + n$, and the main focus is on $N = 5$.

To establish the veracity of this theorem (in a heuristic fashion at least), and see its relevance (particularly to the theories considered in the preceding section), consider an arbitrary manifold Σ_n in a

Ricci-flat space V_{n+1}. The embedding can be visualized by drawing a line to represent Σ_n in a surface, the normal vector n^A to it satisfying $n \cdot n \equiv n^A n_A = \varepsilon = \pm 1$. If e_α^A form an appropriate basis and the extrinsic curvature of Σ_N is $K_{\alpha\beta}$, the ADM constraints read

$$G_{AB} n^A n^B = -\frac{1}{2}\left(\varepsilon R_\alpha^\alpha + K_{\alpha\beta} K^{\alpha\beta} - K^2\right) = 0$$

$$G_{AB} e_\alpha^A n^B = K_{\alpha;\beta}^\beta - K_{,\alpha} = 0 \qquad . \qquad (1.17)$$

These relations provide 1 + n equations for the $2 \times n(n+1)/2$ quantities $g_{\alpha\beta}$, $K_{\alpha\beta}$. Given an arbitrary geometry $g_{\alpha\beta}$ for Σ_n, the constraints therefore form an under-determined system for $K_{\alpha\beta}$, so infinitely many embeddings are possible. This implies that the embedding of a system of 4D equations like (1.1)–(1.4) in a system of 5D equations like (1.5) is always possible.

This demonstration of Campbell's theorem can easily be extended to the case where V_{n+1} is a de Sitter space or anti-de Sitter space with an explicit cosmological constant, as in brane theory. Depending on the application, the remaining $n(n+1)-(n+1)=(n^2-1)$ degrees of freedom may be removed by imposing initial conditions on the geometry, physical conditions on the matter, or conditions on a boundary.

The last is relevant to membrane theory with the Z_2 symmetry. To see this, let us consider a fairly general line element with $dS^2 = g_{\alpha\beta}\left(x^\gamma, l\right)dx^\alpha dx^\beta + \varepsilon\, dl^2$ where $g_{\alpha\beta} = g_{\alpha\beta}\left(x^\gamma, +l\right)$ for $l \geq 0$ and $g_{\alpha\beta} = g_{\alpha\beta}\left(x^\gamma, -l\right)$ for $l \leq 0$ in the bulk (Ponce de Leon 2001). Non-gravitational fields are confined to the brane at $l = 0$, which is a singular surface. Let the energy-momentum in the brane be represented by $\delta(l)S_{AB}$ (where $S_{AB}n^A = 0$) and that in the bulk by T_{AB}. Then the field equations read $G_{AB} = \kappa\left[\delta(l)S_{AB} + T_{AB}\right]$ where κ is a 5D coupling constant. The extrinsic curvature discussed above changes across the brane by an amount $\Delta_{\alpha\beta} \equiv K_{\alpha\beta}\left(\Sigma_{l>0}\right) - K_{\alpha\beta}\left(\Sigma_{l<0}\right)$ which is given by the Israel junction conditions. These imply

$$\Delta_{\alpha\beta} = -\kappa\left(S_{\alpha\beta} - \frac{1}{3}Sg_{\alpha\beta}\right) \quad . \tag{1.18}$$

But the l =0 plane is symmetric, so

$$K_{\alpha\beta}\left(\Sigma_{l>0}\right) = -K_{\alpha\beta}\left(\Sigma_{l<0}\right) = -\frac{\kappa}{2}\left(S_{\alpha\beta} - \frac{1}{3}Sg_{\alpha\beta}\right) \quad . \tag{1.19}$$

This result can be used to evaluate the 4-tensor

$$P_{\alpha\beta} \equiv K_{\alpha\beta} - Kg_{\alpha\beta} = -\frac{\kappa}{2}S_{\alpha\beta} \quad . \tag{1.20}$$

However, $P_{\alpha\beta}$ is actually identical to the 4-tensor $\left(g_{\alpha\beta,4} - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu,4}\right)/2\Phi$ of induced-matter theory, which we noted above in (1.12). It obeys the field equations $P^{\beta}_{\alpha;\beta} = 0$ of (1.11), which are a subset of $R_{AB} = 0$. That is, the conserved tensor $P_{\alpha\beta}$ of induced-matter theory is essentially the same as the total energy-momentum tensor in Z_2-symmetric brane theory. Other correspondences can be established in a similar fashion.

The preceding exercise confirms the inference that induced-matter theory and membrane theory share the same algebra, and helps us understand why matter in 4D can be understood as the consequence of geometry in 5D.

1.6 Conclusion

In this chapter, we have espoused the idea that extra dimensions provide a way to better understand known physics and open a path to new physics.

The template is Einstein's general relativity, which is based on a fusion of the primitive dimensions of space and time into 4D spacetime. The feasibility of extending this approach to 5D was shown in the 1920s by Kaluza and Klein, and if we discard their restrictive conditions of cylindricity and compactification we obtain a formalism which many researchers believe can in principle offer a means of unifying gravity with the forces of particle physics (Section 1.2). 5D is not only the simplest extension of general relativity, but is also commonly regarded as the low-energy limit of higher-N theories. Most work has been done on two versions of 5D relativity which are similar mathematically but different physically. Induced-matter (or space-time-matter) theory is the older version. It views 4D mass and energy as consequences of the extra dimension, so realizing the dream of Einstein and others that matter is a manifestation of geometry. Membrane theory is the newer version of 5D relativity. It views 4D spacetime as a hypersurface or brane embedded in a 5D bulk, where gravity effectively spreads out in all directions whereas the interactions of particles are confined to the brane and so stronger, as observed. These theories are popular because they allow of detailed calculations, something which is not always the case with well-motivated but more complicated theories for $N > 5$ (Section 1.3). The field equations of all theories in $N \geq 4$ dimensions have basically the same structure, and this is why we treated them together in Section 1.4. There we concentrated again on the case $N = 5$, paying particular attention to the equations which allow us to obtain the 15 components of the metric tensor. In the classical view, these are potentials, where the 10-4-1 grouping is related to the conventional split into gravitational, electromagnetic and scalar fields. In the quantum view, the corresponding particles are the spin-2 graviton, the spin-1 photon and the spin-0 scalaron. The extension of the metric and the field equations to $N > 5$ is obvious, in which case other particles come in. However, the extension of general relativity to $N > 4$ needs to be guided by embedding theorems. The main one of these dates again from the 1920s, when it was outlined by Campbell. The plausibility of Campbell's theorem can be shown in short order using modern techniques, as can the mathematical equivalence of induced-matter theory and membrane theory (Section 1.5). In summary, the contents of this chapter provide a basis for writing down the equations for $2 < N < \infty$ and deriving a wealth of physics.

In pursuing this goal, however, some fundamental questions arise. In studying (say) 5D relativity, we introduce an extra coordinate ($x^4 = l$), and an extra metric coefficient or potential (g_{44}). The two are related, and by analogy with proper distance in the ordinary 3D space of a curved 4D manifold we can define $\left|\int g_{44}\left(x^{\alpha}, l\right) dl\right|^{1/2}$ as the "size" of the extra dimension. Even at this stage, two issues arise which need attention.

What is the nature of the fifth coordinate? Possible answers are as follows: (1) It is an algebraic abstraction. This is a conservative but sterile opinion. It implies that l figures in our calculations, but either does not appear in our final answer, or is incapable of physical interpretation once we arrive there. (2) It is related to mass. This is the view of induced-matter theory, where quantities like the density and pressure of a fluid composed of particles of rest mass m can be calculated as functions of l from the field equations. Closer inspection shows that for the special choice of gauge known as the pure-canonical metric, l and m are in fact the same thing. We will return to this possibility in later chapters, but here note that in this interpretation the scalar field of classical 5D relativity is related to the

Higgs (or mass-fixing) field of quantum theory. (3) It is a length perpendicular to a singular hypersurface. This is the view of membrane theory, where the hypersurface is spacetime. It is an acceptable opinion, and as we have remarked it automatically localizes the 4D world. But since we are made of particles and so confined to the hypersurface, our probes of the orthogonal direction have to involve quantities related to gravity, including masses.

The other issue which arises at the outset with 5D relativity concerns the size of the extra dimension, defined as above to include both the extra coordinate *and* its associated potential. This is a separate, if related, issue to what we discussed in the preceding paragraph. We should recall that even in 4D relativity, drastic physical effects can follow from the mathematical behaviour of the metric coefficients. (For example, near the horizon of an Einstein black hole in standard Schwarzschild coordinates, the time part of the metric shrinks to zero while the radial part diverges to infinity.) This issue is often presented as the question: Why do we not see the fifth dimension? Klein tried to answer this, as we have seen, by arguing that the extra dimension is compactified (or rolled up) to a microscopic size. So observing it would be like looking at a garden hose, which appears as a line from far away or as a tube from close up. Since distances are related to energies in particle experiments, we would only expect the finite size of the fifth dimension to be revealed in accelerators of powers beyond anything currently available. This is disappointing. But more cogently, and beside the fact that it leads to conflicts, many researchers view compactification in its original form as a scientific cop-out. The idea can be made more acceptable, if we assume that the universe evolves in such a way that the fifth dimension collapses as the spatial part expands. But even this is slightly suspect, and better alternatives exist. Thus for membrane theory the problem is avoided at the outset, by the construction of a 5D geometry in which the world is localized on a hypersurface. For induced-matter theory, particles are constrained with respect to the hypersurface we call spacetime by the 5D equations of motion. In the latter theory, modest excursions in the extra dimension are in fact all around us to see, in the form of matter.

Let us assume, for the purpose of going from philosophy to physics, that a fifth dimension may exist and that we wish to demonstrate it. We already know that 4D general relativity is an excellent theory, in that it is soundly based in logic and in good agreement with observation. We do not desire to tinker with the logic, but merely extend the scope of the theory. Our purpose, therefore, is to look for effects which might indicate that there is something bigger than spacetime.

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